UNIVERSITY OF COPENHAGEN
DEPARTMENT OF SCIENCE EDUCATION

PhD Thesis

Learning fractions in two schools

Case studies on teaching and learning in Japanese supplementary schools

Supervisors: Carl Winsløw, Yukiko Asami-Johansson (University of Gävle, Sweden)

Submitted on: 31 May 2024

Learning fractions in two schools
- Case studies on teaching and learning in Japanese supplementary schools -

Mayu Aoki

Supervisors: Carl Winsløw, Yukiko Asami-Johansson (University of Gävle, Sweden)

This thesis has been submitted to the PhD School of The Faculty of Science,
University of Copenhagen

Table of Contents

Acknowledgements

I am deeply grateful to everyone who has supported me, enabling me to continue my research until the end of my PhD journey.

First and foremost, I would like to express my heartfelt gratitude to my supervisors, Professor Carl Winsløw, and my co-supervisor, Associate Professor Yukiko Asami-Johansson. Meeting mentors whom I can respect deeply not only as researchers but also as individuals has been a treasure in my life. Words cannot express the depth of my gratitude. Thank you for reigniting my passion for research and making the experience enjoyable once again.

To my research group colleagues, Britta, Camilla, Derya, Jacob, Koji, Pia and Rongrong, for their support in my studies and daily life. Especially to my fellow PhD colleagues, Derya and Pia, who continued to encourage me until the end. To Rongrong, as fellow international PhD student, your presence ahead of me was a great source of strength. To Koji, who connected me with Associate Professor Yukiko Asami-Johansson when I was in a dark place during my second year of my master's program. Without your words of encouragement at that time, I am certain I would not have had the opportunity to conduct research in Denmark and meet such wonderful mentors and colleagues.

To PhD colleagues in the department, working together from morning till evening in the office, sharing lunch, and engaging in various conversations, created an environment that enabled me to persevere with my thesis writing until the very end.

To the children and teachers in the Danish and Swedish Japanese supplementary schools and to the ATD community, that guided me to carry out the research that I am passionate about.

To the friends I met in Denmark, and especially to Emil for his generous support and encouragement.

Lastly, to my lovely family in Japan, who have always supported and encouraged my journey towards my dreams. I am deeply grateful to have been born into your family and to have been able to pursue my dreams up to this point.

During my PhD journey, I have come to deeply love Denmark and rediscover the joy of research. I would like to express my joy and gratitude for this experience.

With heartfelt gratitude and love.

Abstract

In general, the mathematical knowledge and practice that occurs in an institution within a society depends on the specific institution, and more also varies between institutions established by different societies.

This PhD thesis addresses the questions: “What didactic phenomena occur when Japanese mathematics is taught and learned outside Japan?” and “What didactic phenomena arise when children are exposed in parallel to two different mathematics curricula?”

To explore these initial questions, we focus on the teaching and learning of fraction arithmetics in Japanese Supplementary Schools (JSS) attended by children in Sweden and Denmark. simultaneously. The Anthropological Theory of the Didactic (ATD), which provides the necessary institutional perspective for the elaboration and analysis of these questions, serves as the core framework for this doctoral project. Specifically, three types of knowledge within the institution are considered: knowledge to be taught, taught knowledge, and learnt knowledge; these are analysed sequentially and relationally. To achieve this, the study will analyse the Japanese mathematics curriculum related to fractions, data from lesson observations, and semi-structured interviews. Among the main results, it was found that: first, the Japanese praxeologies regarding fraction arithmetics, as defined by the Japanese *Noosphere*, are all being taught in the Japanese supplementary schools, despite different conditions and constraints, such as teachers' and pupils' background and the extent of teaching. The differences turn out to imply a lesson structure that focuses heavily on acquiring techniques. Secondly, it became evident that the context of the specific institutions, including students' parallel schooling and multilingual background, influences both the teaching of fractions and the children's acquisition of corresponding knowledge at the level of both praxis and logos. We argue that the findings of this thesis, along with the specific new methodology based on ATD, contribute not only to the understanding of mathematical practice in JSS, which has not been extensively studied yet, but also to providing new perspectives on the comparison of mathematics education practices between Japan and Western countries. Furthermore, these findings contribute to elucidate the mathematical experience of bilingual children who attend monolingual educational institutions. This includes the mathematics education of foreign children in Japan, which remains under-researched despite recent concern and increased prevalence, as well as the more general educational phenomena related to bilingual (multilingual) children who have received part of their education in one country and then moved to another one.

Resumé

I almindelighed afhænger matematisk viden og praksis i en institution af det omgivende samfund og af institutionen selv, og varierer også mellem institutioner i forskellige samfund.

Denne PhD-afhandling drejer sig om flg. spørgsmål: ”Hvilke fænomener observeres i japansk matematikundervisning udenfor Japan”, og ”Hvilke fænomener opstår når elever på én gang modtager matematikundervisning efter to forskellige læreplaner?”

For at undersøge disse overordnede spørgsmål fokuserer vi på undervisning og læring indenfor emnet brøkregning i japanske supplements-skoler (JSS) for børn der lever i Sverige og Danmark. Den antropologiske teori om det didaktiske (ATD), som leverer det nødvendige institutionelle perspektiv til at præcisere og undersøge de nævnte spørgsmål, udgør den centrale ramme for dette PhD-projekt. Vi betragter specielt flg. tre typer af viden indenfor institutionen: viden som der skal undervises i, viden som der faktisk undervises i, og viden som læres; disse analyseres sekventielt og relationelt. For at realisere dette, analyserer vi i dette studie den japanske læreplans bestemmelser vedr. brøkregning, data fra observeret undervisning, og semi-strukturerede interviews. Blandt de overordnede resultater fandt vi for det første at de prakseologier angående brøkregning, som er defineret af den japanske *noosfære*, alle indgår i undervisningen ved de japanske supplements-skoler, på trods af anderledes betingelser og begrænsninger, såsom lærernes og elevernes baggrund og omfanget af undervisningen. Det viser sig, at disse forskelle leder til en lektionsstruktur som er stærkt præget af tilegnelse af teknikker. For det andet blev det klart at de specifikke institutioners kontekst, herunder elevernes parallelle skoleforløb og deres multisproglige baggrund, øver indflydelse både på undervisningen I brøkregning og på elevernes tilegnelse af den tilsvarende viden, både på praksis- og logos-niveau. Vi argumenterer for, at afhandlingens resultater, sammen med den specifikke og nye ATD-baserede metodologi, bidrager ikke blot til forståelse af den matematiske praksis ved JSS, som ikke før har været studeret indgående, men også til nye perspektiver på sammenligning af matematikundervisning i Japan og i vestlige lande. Desuden bidrager resultaterne til at belyse tosproglige elevers matematiske udvikling i monosproglige uddannelsesinstitutioner, som også omfatter matematikundervisning af udenlandske elever i Japan, et hidtil uudforsket område, på trods af at fænomenet både er voksende og genstand for aktuel opmærksomhed. Endelig er resultaterne relevante ift mere generelle fænomener vedr. to- eller flersprogede elever, som har haft en del af deres skolegang i et land, og så flytter til et andet.

List of Papers

Paper Ⅰ

Aoki, M. (2023). Mathematics in a Japanese overseas school: Connecting classroom study and curriculum. *Asian Journal for Mathematics Education*, *2*(2), 204-219. https://doi.org/10.1177/27527263231187804

Paper Ⅱ

Aoki, M., Asami-Johansson, Y., & Winsløw, C. (accepted, modulo minor revisions). Learning to speak mathematically at the Japanese supplementary school in Sweden: critical cases of praxeological anomaly. *Educational Studies in Mathematics*.

Paper Ⅲ

Aoki, M. (submitted). Learning fractions in two schools. Submitted to: *International Journal of Mathematics Education in Science and Technology*.

Conference papers (peer reviewed, accepted)

Aoki, M., & Winsløw, C. (2022). How to map larger parts of the mathematics curriculum? The case of primary school arithmetic in Japan. In *Twelfth Congress of the European Society for Research in Mathematics Education* (*CERME12*) (No.01).

Aoki, M. (2022). Didactic transposition of fraction arithmetic in a Japanese overseas school: connecting a classroom episode to the curriculum. In *7th International Conference on the Anthropological Theory of the Didactic* (*CITAD7*) (p.17).

Aoki, M., Asami-Johansson, Y., & Winsløw, C. (2023). Learning to speak mathematically at the Japanese supplementary school in Sweden. In *Thirteenth Congress of the European Society for Research in Mathematics Education* (*CERME13*) (No.3). Alfréd Rényi Institute of Mathematics; ERME.

1 Introduction

This doctoral thesis consists of three papers that encapsulate my academic work. Each paper address different aspects of the parallel learning of mathematics within two monolingual settings, endowed with very different curricula and languages. In terms of contexts, I investigate the teaching and learning of fractional arithmetic at the so-called the Japanese supplementary schools in Denmark and Sweden.

The present initial section provides a brief overview of the context of my PhD project, including its motivations, and overarching objectives. In the following sections, I explain the background, results, overall theme and mutual links of the three papers which constitute the core of my doctoral thesis. Section 2 presents a comprehensive review of pertinent literature in fields that are drawn on and contributed to by this research project, including the (scarce) research literature concerning the functioning of Japanese supplementary schools, and the (large) are of research on bi- and multilingual mathematics education. This review serves to locate the contribution and significance of my project. Section 3 is dedicated to presenting the primary theoretical framework employed in the project: the Anthropological Theory of the Didactic (ATD), initiated by Yves Chevallard in the 1980s. I delineate how specific tools and perspectives provided by ATD were applied to the three papers. Section 4 explains the overarching research questions which are addressed in the research project, as well as certain the common methodological approaches that were utilized across the papers. In Section 5, I outline the main results which have been produced for each of the overarching research questions, based on outcome of the three papers along with a discussion on how they contribute differently to elucidate the questions. Finally, Section 6 offers proposal for future research directions.

1.1 Overall motivation

During my time as a master's student in Japan, I was part of an international research centre at the University of Hiroshima, where English was the primary language of instruction. Despite being in Japan, I was often surrounded by students and researchers from foreign countries. This environment offered the opportunity to encounter foreign languages and cultures, which brought me great joy. However, I also experienced conflicts and frustrations, particularly when engaging in academic discussions in English at the University. This sparked my curiosity about mathematics education in bilingual/multilingual contexts.

During my study, I discovered the existence of specific institutions known as Japanese supplementary schools (JSS) abroad, where mathematics is taught based on the Japanese national curriculum in Japanese language within a limited timeframe. Interestingly, children attending these schools simultaneously learn mathematics in local schools. Thus, these supplementary schools have students who learn mathematics within two parallel (non-connected) contexts, each with a monolingual setting for instruction, and with two different curricula.

Having conducted research on the Swedish mathematics curriculum during my master's program, I became aware to some extent of the significant differences between both the structure and the contents of curricula in Japan and the Nordic countries, and naturally also to the distance between the languages and culture. Therefore, I decided to examine the teaching and learning of mathematics in these institutions further, adding also the JSS in Denmark (JSS_{DK}), which admit pupils residing in both Denmark and nearby parts of Sweden, and the JSS in Sweden (JSS_{SE}).

The study of mathematics teaching in Japanese supplementary schools, could in principle be viewed as a special and somewhat peculiar case of research on mathematics education in bilingual and multilingual contexts. In such research, the teaching and learning of mathematics are investigated from a language perspective; what is "peculiar" about my context is that students are deliberately taught mathematics in two languages, not at the same school but in two parallel school systems. In my PhD project, the language perspective has some importance for the analysis, but my approach to investigate mathematics teaching and learning in these institutions does not focus exclusively on language, but rather adopts a wider institutional perspective (a detailed explanation can be found in Section 2). This perspective comes from the central theoretical framework of my study, the *Anthropological Theory of the Didactic – ATD*. ATD enables us to describe and analyse mathematical and didactical knowledge with respect to their institutional habitats, as it is considered that different forms of mathematical (didactical) knowledge exist based on the institution in ATD (a detailed explanation will be in Section 3). In addition, *didactic transposition theory*, a foundational early achievement of ATD, inspired me to analyse different types of knowledge within the institution considered: *knowledge to be taught*, *taught knowledge,* and *learnt knowledge* (detailed explanations will be provided in section 3). In fact, the three papers presented in this thesis can be roughly said to focus on one or two of these (while, naturally, not entirely ignoring the test). In the first paper, I examine knowledge to be taught at JSS (concerning fractional arithmetics), as well as corresponding the knowledge taught at JSS_{DK}. In the second paper, our focus is on taught knowledge at JSS_{SE}, and in the third paper, I concentrate on learnt knowledge at JSS_{DK}. I will provide a more detailed and nuanced explanations regarding the relationship among the three papers in Section 3.

1.2 PhD project objectives

The overarching objective of this doctoral project is both to contribute to our understanding and knowledge of how Japanese expatriate schools function in practice; there is, in particular, no research focusing on their teaching of mathematics. The analysis of teaching and learning experiences in these institutions may shed new light on a number of topics which, by themselves, have drawn considerably more scholarly attention:

- The comparison of mathematics education practices in Japan and in Western countries (e.g. Stigler and Hiebert, 1999)
- The role of language in primary school mathematics, in particular for bilingual pupils in monolingual institutions.

The choice to focus more specifically on the arithmetic of fractions is the general recognition of this domain as a central and challenging one within primary mathematics (e.g., see Siegler et al., 2012), for which extensive research exist in both the Japanese and the Western context, while the above two perspectives are only recently being investigated in this domain. It is thus hoped that our case study could contribute a new angle on both comparative studies, linguistic aspects of primary school mathematics, and the specifics of the Japanese curriculum on fractions.

2 Literature Background

A first situation of our study could, as mentioned above, be to view it as part of the research on mathematics education in bilingual and multilingual contexts. We first outline previous research on mathematics education in bilingual and multilingual contexts. We then consider the very limited literature about Japanese supplementary schools, which provides also an occasion to summarize some characteristics of the institutional context of our study. Finally, we outline certain historical and educational characteristics of the Japanese mathematical terminology related to fractional arithmetic, which are used and elaborated in the second and third papers presented in this thesis.

2.1 Mathematics education in bilingual and multilingual contexts

The body of research on the learning and teaching of mathematics in multilingual and bilingual contexts focusing on language has been established over more than half a century, as early syntheses amply demonstrate (Austin and Howson 1979; Ellerton and Clarkson 1996; Pimm 1987), and this area of study continues to expand (e.g., see. Planas et al., 2018; Barwell et al., 2021; Erath et al., 2021). Traditional themes include the language of the learner, the language of the teacher/instruction, and the language of mathematics (Planas et al., 2018), and many qualitative studies have been conducted, demonstrating the interrelatedness of these themes. Since 1990s, a prevalent agenda in this research is to regard the utilization of multiple languages as beneficial rather than problematic, while it is of course in practice often an inevitable consequence of ever-increasing patterns of migrations among different countries and linguistic regions. This agenda aims to challenge deficit perspectives that depict bilingual or multilingual learners as less capable of acquiring mathematical knowledge (Barwell, 2018). Examples of prominent scholars at the forefront of conducting rigorous research that actively contributes to this orientation include Jill Adler, who conducts research mainly in South Africa, Mamokgethi Phakeng, also based on in South Africa, Nurial Planas, working in Catalonia/Spain, and Judit Moschkovich, who operates in the US. Barwell (2018) reviews these and other studies on multilingual aspects of mathematics education, as well as their research methodologies.

It appears that there has been little or no research conducted from the institutional perspective, that is, considering the roles and status of languages as fixed by institutions, and as one among other aspects of the conditions and activity hosted by them. In schools, mathematical and didactical practices are other crucial aspects that could indeed display specific dependencies on the language or languages used in the institutions; but many studies of bilingual phenomena in mathematics education offer little or no attention to what may be specific to the mathematical practices that appear in the data, while focusing on more general patterns of language use. Nevertheless, there are a few recent studies of multilingualism – including those by Farrugia (2018), Prediger et al., (2019) and Petersson & Norén (2017) – that focus on fractions, which is also the school mathematical topic of my study. We now take a closer look on their results.

Farrugia (2018) investigated how Maltese and English languages were used and combined in three so-called *registers* (in the sense of socio-linguistics, cf. Halliday, 1989): everyday register, general school register, and technical mathematics register (these distinctions were developed by Prediger et al., 2016), the latter occurring during lessons on fractions in 4th grade. Based on the findings that the two languages fulfilled specific roles and functions within the three registers, whether used separately or in combination, she concludes that utilizing both languages is beneficial during the lessons. Furthermore, she advocates for the adoption of the translanguaging (cf. García & Wei, 2014) perspective, which views all linguistic resources as part of a single, integrated system rather than as independent entities.

Prediger and colleagues (2019) rely to some extent on the Whorfian linguistic relativity principle, which posits that language influences the way one thinks (Whorf, 1956). They investigated how the interplay of languages and conceptualisations shapes the multilingual learning processes of Turkish-German-speaking students in grade 7, concerning in particular on the part-whole relationships that are considered central to the meaning of fractions. For instance, they mention the differences which exist in the way of fractions are read in certain German and Turkish, considering that such differences may also give rise to semantic differences – that is, in the meaning which they support students to create in relation to fractions. Additionally, they present a model about different language-related nuances for the same concept in German and Turkish. As a background for this model, it is explained that in many languages, a single fraction $\frac{a}{b}$, often has three meanings: as a ratio (e.g., 3 to 5) , as a part-whole relation (3 out of 5), and as a rate (e.g., 3 Euros per 5 kg) (p.190), based on Behr et al., (1992). Here, too, this model summarizes main challenges for the students in relation to a language perspective, not from an institutional or cultural perspective.

Petersson & Norén (2017) analysed two fraction test items from two perspectives: language and mathematical knowledge. They developed two test items, and administered them to 259 ninth-grade students in Sweden. The students are categorized into four types: newly arrived second language immigrants, early arrived second language immigrants, other second language learners and first language learners (non-immigrant students). Among the research findings, it was stated that, firstly, "the proportion of students who did not associate the wording 'hälften av' (being an irregular declension of the wording 'the half of') in the test language with dividing by two, decreased with the length of time since the student immigrated" (Petersson & Norén, 2017, p.187). Secondly, "higher proportions of early arrived immigrants and other second language learners than newly arrived immigrants and first language learners had problems in correctly applying 'half of' to a fraction" (Petersson & Norén, 2017, p.187). They also state that if the students had already acquired their mathematical knowledge in their home country before coming to Sweden, and if the tests had been conducted in the language of that country, these results might not have occurred. In other words, they refer to the relationship between the language of instruction and mathematical knowledge, a perspective that aligns to some extent with the direction of this doctoral project. In particular, the authors observe the dependency of linguistic and mathematical proficiency, and consider (in part) students who are likely to have learned about fractions in two languages and in two school systems.

In summary, research on the relationship between mathematics and language, as well as the relationship between mathematics education and language in bilingual and multilingual contexts, is extensive and continues to develop. These studies often assume or recognize that mathematics instruction and learning, along with the associated language, are rooted in the social-cultural contexts of each country. However, how specific institutional contexts contribute to determine the interaction of general language and discourses specific to mathematics is much less well-understood, as are the effects of these interactions on pupils with a bilingual background; and there has been little or no analysis of how attending parallel institutions, with different curricula and language, affect learning of specific areas of mathematics. To construct the present study within the paradigm of ATD is therefore not only a first attempt to deploy this theoretical framework to research on mathematics education in multilingual contexts, or (more generally) to investigate how language impacts on mathematical instruction; it also that the study of how pupil's exposition to parallel institutions may shed new light on this larger area. We shall return in more detail to the novelty of the project and its results, in Section 3 and 5.

2.2 The Japanese supplementary schools

The focal institutions of this project, the Japanese supplementary schools, are among the overseas institutions that have been established and continually supported by Japanese government in modern times. Following World War Ⅱ, Japan underwent a paradigm shift from military expansion to industrialization from around 1955 to 1973, a period known as the high economic growth era. Major Japanese corporations embarked on international ventures, leading to the deployment of numerous expatriates and their families abroad (Kano, 2013). Such a societal paradigm shift impacted Japanese society in general and more specifically the children of expatriates. At that time, few of the concerned families intended to permanently reside overseas, and it was assumed they would return to Japan after a few years abroad. Consequently, upon their return to Japan, they would need to adapt seamlessly adapt to an ongoing education in Japanese schools. To cater for these needs, the Japanese government began establishing two types of overseas institutions: "*Nihonjingakko* (full-time Japanese schools)" and "*Holyoke* (Japanese supplementary schools: JSS)", aimed at providing Japanese education to Japanese children living abroad. Both institutions are mainly operated by local Japanese associations, and they rely both on government support and on and tuition carried by the families concerned.

As of the year 2023, there are 94 full-time schools in 49 countries and 1 region, and 237 JSS in 51 countries and 1 region, with approximately 16,000 and 20,000 children enrolled in them, respectively (Ministry of education, culture, sports, science and technology; MEXT, 2023).

In recent years, globalization has led many people to live for longer or indeterminate periods outside of their country of origin, a development driven by changing or expanding work opportunities, by an increasing participation in international exchange programs such as study abroad, by more and more families resulting from international marriages, and so on. These trends have also resulted in a diversification of the demographics of the Japanese overseas schools. The schools cater to a growing number of children from families with long-term stays, permanent residences, and international marriages (e.g., Shibano, 2014; Okumura, 2017; Japan overseas educational services (JOES), 2018; Aoki, 2023). In particular, most children attending JSS are now following two different monolingual settings concurrently, so that they have *de facto* two parallel languages and cultural identities. As a result, recent Japanese government emphases the potential of these children to become global talents (MEXT, 2016), as a new (or at least changed) way to justify the support of JSS. This also underscores the importance of conducting research on JSS (JOES, 2018; Shibano, 2019) that takes the changed role of JSS into account.

The full-time Japanese schools and JSS have the same purpose: providing Japanese education based on the Japanese national curriculum, including officially approved textbooks. However, the significant difference between the two institutions lies in that the former is functionally equivalent to full-time mainstream schools in Japan, covering all subjects. On the other hand, the JSS provides a more limited education within a shortened time framework, typically on a single day of the week such as Saturday; their pupils also attend local or international schools on weekdays. The JSS primarily focus on teaching Japanese language, but they also frequently include other subjects such as mathematics, depending on the institution. In fact, instruction in mathematics is provided as second subject in around 80 % of the JSS (Okumura, 2017).

While the official reason for providing mathematical instruction in JSS is not clearly explained by official documents, the following reasons can be considered. One likely reason – at least for the original role of JSS as catering mainly to short term expatriate children - is that mathematics is frequently considered one of the most challenging subjects that is important for the students' educational and professional success upon their return to the highly competitive contexts in

Japan. Secondly, mathematics is a highly structured discipline, and the Japanese mathematics curriculum is meticulously designed with a progressive sequence of topics and skills that children are expected to learn (Aoki, 2023), in a way that may depart considerable from regular teaching in the host country. As a result, children who have not reached a certain level of proficiency may struggle to keep up with mathematics classes upon returning to Japan. Thirdly, as mentioned in the previous section, it could be a general experience that learning mathematics is to a large extent dependent on developing a specialized use of language and terms. For instance, even if children have mastered techniques (algorithms) in another language, a lack of familiarity with specific mathematical terms and expressions in Japanese, could conceivably significantly impact their continued studies in negative ways.

The supplementary schools are also known by various alternative names such as “complementary schools”, “ heritage language schools”, “community languages schools” and “Saturday schools”, with usage varying across countries and among researchers (Farsani, 2015, p.10; 2016; Huang, 2022, p.51). For instance, some studies (e.g., Creese & Martin, 2006; Wei, 2011; Farsani, 2016) employ the term “complementary” to highlight the positive supportive function these institutions may play in relation to mainstream educational institutions. While I acknowledge this perspective, for consistency with official Japanese government documents, this study will continue to employ the term “supplementary”.

Research on general aspects of the supplementary schools worldwide has recently gained increased attention (Wei, 2006; Shibano, 2019; Huang, 2022, p.52). It focuses on generic themes such as identities, cultures, languages practices and policy (e.g., Creese et al., 2008; Francis et al., 2009, Creese et al., 2010; Francis et al., 2010; Wei 2011; Cushing et al., 2021). However, research focusing on mathematics, irrespective of the country remains notably limited (Farsani, 2015; 2016), and there is no research considering the institutional particularities of JSS (or, for that matter, full-time Japanese schools). Some research exists within similar institutional contexts. For instance, Farsani (2015; 2016) investigated verbal and non-verbal communication occurring within teaching and learning of mathematics in a complementary school in the United Kingdom, devoted to instruction in Farsi and English together. For instance, Farsani (2016) examines the case of a student who learned different techniques to carry out complex calculations with fractions in his local school and in the complementary school, and how he managed these different techniques when solving tasks. He concluded that the distinct bilingual pedagogy created a space for British-Iranian bilingual learners to integrate not only their languages but also aspects of their educational histories and practices, involving differences in how fractional arithmetics are taught in the UK and in Iran. The contexts of his research and mine align in that the children attend monolingual schools on weekdays, and the above outcome indirectly relate to the institutional perspective that is the main focus in my thesis. However, our studies differ not only in terms of the languages we focus on (Farsi-English in his study and Japanese-Danish or Swedish in mine) but also in the characteristics of the institutions themselves. This is because the institution he focuses on actually involve the use of two languages, creating a completely bilingual setting, whereas JSS is not a bilingual institution, but operates exclusively in Japanese.

2.3 Characteristics of Japanese school mathematical terminologies in relation to the history of mathematics education in Japan

Japanese utilises three systems of writing (characters): *Kanji* (漢字), *Hiragana* (ひらがな), and *Katakana* (カタカナ), which are combined to form vocabulary and sentences. Japanese people naturally distinguish between these three-writing system based on the context they serve. Kanji is a variant of Chinese ideograms that have several pronunciations, and each have their own meanings; the characters are thus semantic rather than phonetic (like Western alphabets). In Japanese, they are used mostly to represent roots of nouns or verbs. Hiragana, which is a phonetic sign system, is used to represent the supporting parts such as particles and conjugation of verbs that complement the kanji. Katakana, which is also a phonetic sign system that is fully isomorphic to hiragana, is employed for borrowed words and specialized terminology originating from foreign languages such as coffee, computer and sports. It is therefore easy to recognise words imported from foreign (mostly, Western) languages within written Japanese.

In countries where modern or Western mathematics has been "imported" relatively recently, mathematical terminology needs to be established either by devising new terms or by simply important foreign terms (like from a colonial language). This phenomenon concerns not only academic terminology, but in many cases also some or all of the mathematical terms used in school. In Japan, Western mathematics was largely unknown until the Meiji era (1868-1912), while a separate mathematical paradigm, known as *Wazan*, was commonly taught and used. During the transition to Western mathematics, mathematical terminologies were also revised and established (Yamaguchi, 1998; Date, 2011; Cousin, 2018).

Before Meiji era, that is, during Edo period (1600-1868), Japanese scholars mainly drew on Chinese texts, and mathematicians wrote textbooks on both arithmetics and geometry in this tradition, and developed several original methods (Cousin, 2018). Wasan, which was heavily influenced by Chinese mathematics, served both commercial and other societal ends, and was taught at all levels of schools from primary “temple schools” (*Terakoya*) to high-level private schools (*Shinjuku*) (Cousin, 2018). Japanese society remained more or less isolated from the surrounding world between until 1854, when the United States forced Japan to open its trade roads. The long period of isolationism was over, and Western culture was introduced massively and deliberately into Japanese society at large. A part of this societal transformation, Western mathematics became exclusively used in school mathematics, following the famous Decree of Education (*Gakusei*) in 1872. The previously dominant Wazan, which served as the mainstream school-mathematics, was practically abolished (Cousin, 2018). At that time, mathematicians who had specialized in *Wasan*, began studying mathematics abroad, such as Dairoku Kikuchi, who went to Cambridge University. As they came to possess more extensive knowledge of Western mathematics, they also began to write Japanese school mathematics textbooks based on a variety of Western mathematical textbooks (Cousin, 2018). The mathematicians of the time either used existing terms from Wazan (Yamaguchi, 1988), or created new Japanese terms for concepts and theories unique to Western mathematics (Cousin, 2018). For instance, the English word "vector" is derived from the Latin "vehere,", and in Japanese mathematics education, it is written in katakana as "ベクトル," closely approximating the English pronunciation. On the other hand, a book written by Nakahara (2000) compiles important terms in mathematics education in a dictionary format. This compilation shows that, as Yamaguchi (1988) and Cousin (2018) have noted, most mathematical terms used in schools are written in kanji, rather than simply imposing loan words; they are thus based on semantic translation. Moreover, the mathematical terminology used in Japanese school mathematics has additional distinct characteristics. As revealed in the second paper presented in this thesis, some terms that describe operations (verbs) are nominalized, and it is thus not possible to establish a one-to-one correspondence between mathematical terms in Japanese and in foreign languages.

It is of course possible to establish more involved correspondences. For instance, Table 1 illustrates mathematical terminologies and expressions that are commonly used in the teaching and learning of fractions in Japanese with English. On the left side of the table, the terms are in English, with the corresponding Japanese terms written on the right side. The readings of the kanji characters are provided in parentheses within the Japanese column.

Table 1: Examples of mathematical terminologies in English and Japanese

English	Japanese
1. Fractions	1. 分数 (Bunsuu)
2. Denominators	2. 分母 (Bunbo)
3. Numerators	3. 分子 (Bunshi)
4. Converting fractions to obtain common denominators	4. 通分 (Tsuubun)
5. Simplifying fractions	5. 約分 (Yakubun)

As we can see, terminologies differ significantly between English and Japanese. In particular, there are terms in Japanese school mathematics, where English verbal expressions correspond to nouns, such as numbers 4 and 5 in Table 1. This turns out to be a didactically relevant feature of Japanese school mathematics, which will be discussed in detail in Section 5.2.

3 Theoretical Framework – Anthropological Theory of the Didactics

This section is devoted to a coherent presentations of the most used parts from our theoretical framework, the *Anthropological Theory of the Didactic – ATD* (*Théorie anthropologique du didactique* in French, cf. Chevallard, 1999). ATD was initiated by Yves Chevallard in the 1980s, beginning with the *theory of didactic transposition*. Since then, it has advanced considerably and developed several other methodological tools, and has become an international research program driven by scholars in many countries outside of France, such as Spain, Canada, Denmark, Sweden, Japan, and so on.

In ATD, "*institution*" is a core notion, defined roughly as "any created reality of which people can be members (permanent or temporary)" (Chevallard & Bosch, 2019, p.xxxi), as a smaller entity within a larger *society*. For instance, a class, a working place, and a sports community may all be considered institutions. The teaching and learning of mathematics occurs, for instance (and perhaps primarily) in school institutions established deliberately by society. In general, the mathematical knowledge and practice that occurs in an institution within a society varies depending on the specific institution within that society, and of course between institutions established by different societies. Most if not all other analytical tools within ATD are constructed based on this institutional point of view. In the following sections, I will explain the analytical tools which became central in my studies and their used, in particular, *Didactic transposition*, *Praxeologies*, *The levels of didactic co-determination* and *Moments of didactic processes*.

3.1 Didactic transposition

The notion of *didactic transposition*, at the origin of ATD, was introduced to mathematics education in 1985 by Chevallard (1985; 1992). This notion serves as a methodological tool for didactic analysis, highlighting the existence of different piece of types of knowledge within different institutions.
Figure 1 illustrates the process of conversion or transformation of a piece of knowledge from the creative origins to the learner.

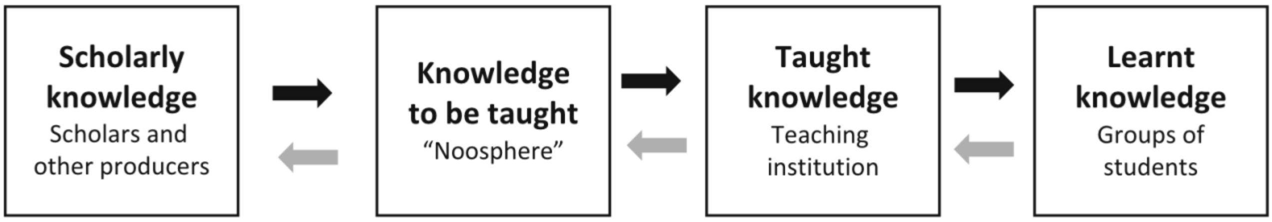

Figure 1 : The notion of didactic transposition (Bosch & Gascón, 2014, p.70)

For instance, let us consider the learning fractions in primary schools. According to this theory, the knowledge of fractions taught in this primary school is produced outside of the school and transposed into it. Initially, this piece of knowledge of fractions is produced as *scholarly knowledge* by scholars such as mathematicians and other producers. This stage is called the first transformation. The next transformation occurs when this piece of knowledge of fractions is designated as *knowledge to be taught*. Fractions, produced by mathematicians and other scholars, are evaluated and designated as knowledge to be taught in schools by the *Noosphere*, which includes those who manage the educational system, such as curriculum developers. The resulting curricula can usually be identified in official documents like syllabi and textbooks. When this piece of knowledge to be taught is actually taught in the classroom, it becomes *taught knowledge*. This transformation is carried out by teachers. After that, pupils process their confrontation with taught knowledge to produce *learnt knowledge*. The process of converting or transforming knowledge from knowledge to be taught into taught knowledge, and then into learnt knowledge is called *internal didactic transposition*, mainly occurring within didactic systems that exist within schools. Note here that in this notion, a piece of knowledge is distinguished based on the institutions to which it belongs, and there is no one-to-one correspondence between the various transpositions outlined above. The notion of didactic transposition goes beyond merely describing the process of knowledge transfer; it serves as methodology for analysing different types of knowledge and the way they arise from one another. It is also not a one-direction process, as the double arrows in the figure suggest; for instance, the process of identifying scholarly knowledge can feed back on the latter, and knowledge proposed to be taught may turn out to be difficult to teach or learn, thereby leading to modification in curricula. One may even talk about a circle, with another double arrow between learned knowledge and scholarly knowledge, since all scholarship begins from learnt knowledge, and involves in itself more or less advanced learning (particularly in university institutions, this "invisible" relationship may be quite close).

In the three papers presented in the thesis, I did not directly apply didactic transposition as a methodological tool. However, it was instrumental in formulating research questions aimed at analyzing the internal didactic transposition within the papers, and to structure the entire study, as shown in Figure 2.

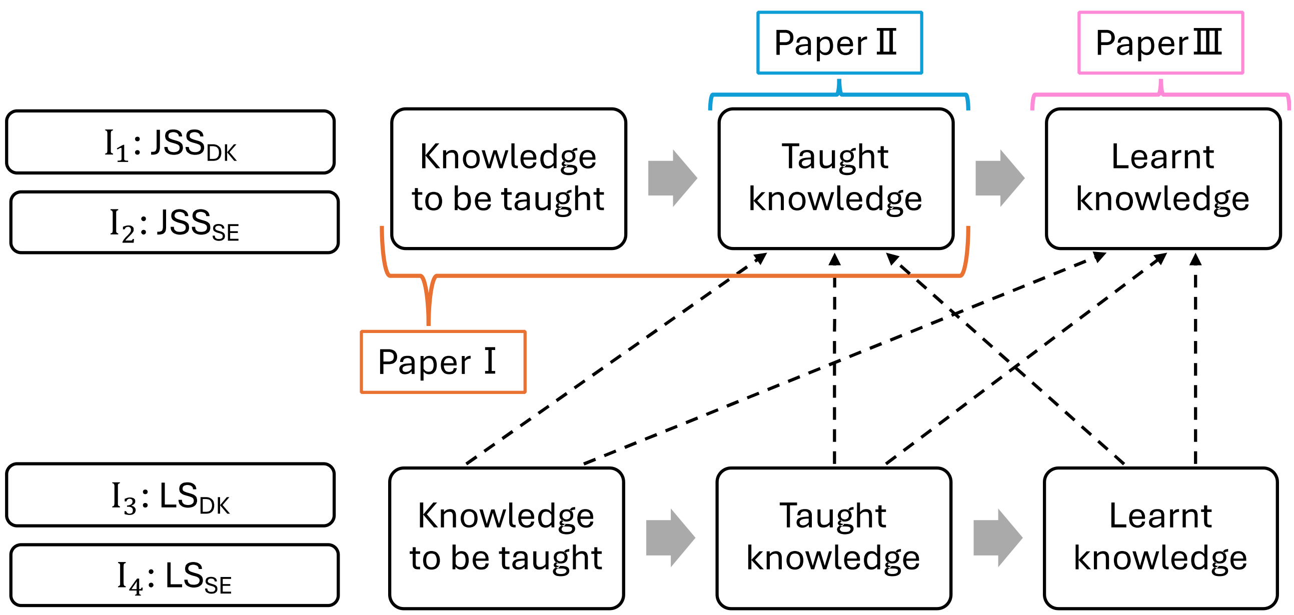

Figure 2: Map for three papers presented in the thesis

Before delving into the details of figure 2, I will explain some letters used in the Figure 2."I" signifies an institution. "JSS" and "LS" are abbreviations for Japanese Supplementary Schools and Local Schools, respectively. "DK" and "SE" represent Denmark and Sweden, the societies in which the examined institutions are localized. Figure 2 illustrates the internal didactic transposition within each institutions $I_1 \sim I_4$.

In paper I, I developed an epistemological reference model (detailed explanations will be provided in the next section), as means to represent and analyse the Japanese national program and textbooks, and to examine how fractions are taught at JSS_{DK}. Consequently, the focus of the paper was on delineating the knowledge to be taught, as well as the taught knowledge, at JSS_{DK}.

In Paper Ⅱ, my co-authors and I conducted and analysed observations of fractional arithmetics lessons in JSS_{SE}, aiming to identify differences in the knowledge to be taught, and the roles of the different languages that are used in school mathematics in Japan and Sweden. Consequently, our analysis encompassed the taught knowledge at JSS_{SE} and the knowledge to be taught in JSS_{SE} and LS_{SE}.

In paperⅢ, I conducted interviews with students from JSS_{DK} to investigate how the differences identified in paperⅡappear when analysing students' calculation processes in the mathematical context of fractions, and while explaining and justifying their calculations. Therefore, the focus in that paper was mainly on learnt knowledge, while drawing on findings from the first two papers. I will describe a comprehensive discussion of all findings, along with their status as answers to the primary research questions of the thesis, in Section 5.

3.2 Praxeologies

The notion of *praxeology* is one of the main notions within the ATD. It serves as a robust tool for describing and analysing of different types of knowledge, such as the categories related to didactic transposition. Within the ATD, it is a fundamental assumption that any human activity and knowledge can be analysed in terms of praxeologies, thereby enabling their application not only in the teaching and learning of mathematics or other subjects but also in everyday activities such as cooking rice or constructing buildings. In particular, the praxeology associated with mathematics knowledge is termed *mathematical praxeologies*, while that pertaining to didactic knowledge is referred to as *didactic praxeologies*.

A praxeology consists of two blocks: *praxis* and *logos*. This implies that any human activity comprises two dimensions —a practical one (praxis, what is done) and a theoretical one (logos, what is said or written). A praxis block contains two kinds of elements: *types of tasks* (T) and *techniques* (τ). A type of tasks T that is a collection of tasks (t) that can be solved using one or more techniques τ. The logos block is made up of two elements: *technologies* (θ) and *theories* (Θ). Technologies θ provide direct descriptions, explanations, and justification of the praxis block, particularly the techniques (logos/discourse on techniques). Theories Θ are comprehensive and unifying discourses that can generate, connect, justify and clarify given technologies. The Logos block constitutes, as a whole, the discursive environment of the praxis; the existence of such an environment is indeed characteristic of most if not all human practices. A praxeology is thus formed by a four-tuple (T, τ, θ, Θ). Consequently, we can describe different types of knowledge in mathematics activities in terms of such four-tuples, more or less related among each other by technology and theory. In fact, praxeologies appear in smaller or larger units, which are sometimes called *praxeological organisations*. A *point praxeology* is the smallest unit, consisting a single type of task (and corresponding higher levels). A *local praxeology* comprise a set of types of task unified by a common technology. A *regional praxeology* includes all pinpoint and local praxeologies that share a common theory. Finally, a *global praxeology* is the largest unit, consisting of a larger aggregation of regional praxeologies. These various granularities of praxeological organisations may also be described using the so-called *scale of levels of didactic determinacy*, which I will explain in the next section.

In all three papers presented in the thesis, praxeologies served as a primary tool. I briefly outline their application in these papers, while a more detailed explanations of the methodology will be provided in Section 4.2.

In Paper I, I made use of praxeologies to elaborate *reference models* for analysing teaching processes in JSS_{DK}. According to Bosh and Gascón (2014), researchers in didactics (of mathematics) need to an external position to analyse the process of knowledge conversion or transformation (the notion of transposition), emphasizing the importance of elaborating their own reference models (RMs). I elaborated RMs to describe knowledge to be taught focusing on domain of fractions within the Japanese national curriculum, as JSS adhere to it. These RMs were then used to analyse the development of mathematical praxeologies within teaching processes in JSS_{DK}. For this analysis, I also employed an another tool, therefore I will explain it in Section 3.4.

In Paper Ⅱ, my co-authors and I distinguished between two types of (mathematical) praxeologies, in relation to a given institution: *normal praxeologies* and *praxeologies anomalies*. Normal praxeologies refer to those praxeologies that are encouraged and expected to be taught and learned within the given institution, such as local primary schools. On the other hand, we defined praxeological anomalies as praxeologies that differ from the normal ones. During our study, we found praxeological anomalies occurring during lessons of fractions in JSS_{SE}. Subsequently, we explain these praxeological anomalies by considering the differences between natural languages and curricula (knowledge to be taught), particularly in relation to JSS_{SE} (with the same curricula and language as regular Japanese school) and Swedish local schools.

In PaperⅢ, building upon the findings of Paper Ⅱ, which identified mathematical praxeological differences between two institutions in terms of taught knowledge within JSS_{SE}, I proceeded to examine how these differences appear when analysing elements of learnt knowledge. This investigation involved conducting semi-structured interviews with students from JSS_{DK} specifically addressing instances of the students' production of fraction calculation. My analysis concentrated on the students' responses in their worksheets and their verbal explanations.

3.3 The scale of levels of didactic co-determinacy

In the previous sections, I have explained different categories of knowledge which are pertinent to analysing the teaching and learning mathematics, and emphasised that these categories can be described and analysed within the ATD by using the notion of praxeology. In this section, I introduce another important methodology from ATD: ecological analysis. Ecological analysis serves to identify the conditions and constrains that influence the teaching of a particular subject (e.g., teaching content or methods) within a given society. Ecological analysis is conducted using the scale of levels of didactic co-determinacy (Chevallard, 2019). This analysis includes both the higher, generic levels beyond the disciplines (*Humanity*, *Civilisations*, *Societies*, *Schools*, *Pedagogies* and *Didactic systems*), to lower, discipline-specific levels (*Disciplines*, *Domains*, *Sectors*, *Themes* and *Questions* (*Subjects*)) that can be modelled more directly with praxeologies (Figure 3). This notion thus provides a explicit illustration of how the generic levels influence praxeological organisations at the specific levels.

Humanity
↓↑
Civilisations
↓↑
Societies
↓↑
Schools
↓↑
Pedagogies
↓↑
Didactic systems
↓↑
Disciplines ↔ Domains ↔ Sectors ↔ Themes ↔ Questions

Figure 3: Scale of levels of didactic co-determinacy (Gascón, 2024)

In the previous section, I explained praxeological organisations as a means to describe mathematical activities and knowledge, which is indeed related to the lower level of the scale. A point praxeology corresponds to a question or a subject, like how to add two fractions. A local praxeology aligns with a theme, such as fractional arithmetics. A regional praxeology pertains to a sector, such as rational number arithmetics. Finally, a global praxeology corresponds to a domain, such as arithmetics.

In Paper Ⅰ, I employed the scale of levels of didactic co-determinacy to provide a more comprehensive view of the different granularities which occur in the reference model. The model is still limited to the lower levels of subjects, themes, and sector (fractional number arithmetics).

3.4 Moments of didactic processes

In the ATD, teaching and learning processes are analysed in terms of *moments of didactic processes*, viewing these processes as a progressive development (within a didactic system) of specific mathematical praxeologies (Barbé et al., 2005). The didactic processes are divided into six moments: the moment of the *first encounter*, the *exploratory moment*, *the technological-theoretical moment*, the *technical moment* , the *institutionalisation moment*, and the *evaluation moment*. The description of each moment is as follows:

– The moment of the first encounter: This moment occurs when pupils encounter new types of tasks (T_i) (Aoki., 2023). The primary emphasis here is on grasping the meaning of new tasks and what they require (Aoki, 2023).

- The exploratory moment: In this moment, pupils engage in activities where they attempt to solve one or more of the new task type(s) T_i and develop some initial techniques (τ_i) (Aoki, 2023). The emphasis is mainly on exploration using the knowledge pupils already possess (Aoki, 2023).
- The technological–theoretical moment: This moment occurs when explaining, examining, and discussing one or more techniques τ_i that were developed, for instance, during exploratory or technical moments. This may include naming, comparing or justifying the techniques, or developing a theory to relate and justify several technologies (Aoki, 2023). The focus here is on clarification, justifications, and establishing relationships among praxeologies (Aoki, 2023).
- The technical moment: This moment is when techniques are routinized, strengthened, and generalised (Aoki, 2023). Elaborating pupils' technical knowledge is emphasised, beyond what was accomplished through the initial exploration (Aoki, 2023). In school mathematics, a significant amount of attention in the didactical process is often devoted to the routinization, elaboration, and extension of techniques.
- The institutionalisation moment: These are elements of the didactical process where the teacher summarises or introduces knowledge—whether technical or theoretical—that the pupils are expected to have learned or will learn. The focus here is on classroom activities to official objectives.
- The evaluation moment: This part of the didactical process is dedicated to assessing the praxeologies that the pupils are supposed to have learned, determining their quality and pupils' mastery of the target knowledge. For instance, pupils might take a test, or they might evaluate the extent to which the work completed so far leaves open questions for future study.

In Paper I, I deployed ecological analysis to analyse teaching and learning processes in JSS_{DK}. Specifically, I described how mathematical praxeologies related to knowledge to be taught are developed and shared during these processes in JSS_{DK}, and also identified which moments are emphasized, and by consequence, which are given less priority, in part due to the shorter time available for the didactical processes, in relation to regular Japanese school.

4 Research Questions and Methodology

Based on the discussion above, this section presents the general research questions (RQs) that guide and connect the various parts of this project. Following that, I will explain methodologies for addressing these RQs and their relevance to the three papers presented in the thesis.

4.1 Research questions

As stated in Section 1.2, the overarching objectives of this doctoral project is both to contribute to our understanding and knowledge of how Japanese expatriate schools function, and to contribute to more general *problematiques* concerning the role of institutions, curricula and language in primary mathematics education. This has led to the following general research questions (RQs):

RQ1. In what ways do mathematical praxeologies regarding fractions, established as knowledge to be taught in Japan, develop in teaching processes in JSS?

RQ2. What didactic phenomena can be observed from fraction lessons in JSS involving children who are simultaneously learning two different praxeologies? In addition, what are the underlying causes of these phenomena?

RQ3. What didactic phenomena can be observed when children, who are simultaneously learning two different praxeologies, solve specific fraction-related tasks?

These research questions are addressed in the three papers presented in the thesis. Paper Ⅰ is related to RQ1, and Paper Ⅱ and Ⅲ address RQ2 and RQ3.

4.2 Context, Data and Methodology

In order to investigate the above research questions (RQs), I collected data from the Japanese supplementary schools in Denmark and Sweden. Before delving into the methodology used to address these RQs, I will provide a brief overview of each institution.

4.2.1 Context: Japanese supplementary schools in Denmark and Sweden

The Japanese supplementary schools in Denmark (JSS$_{DK}$) and Sweden (JSS$_{SE}$) are located in rented facilities and offer classes from grades 1 to 9 on Saturdays. During weekdays, children attend local or international schools. Children living in Southern Sweden enroll in JSS$_{DK}$, so that JSS$_{DK}$ has children who attend local schools in either Denmark and Southern Sweden. The academic year at all JSS begins in April and ends in March, aligning with the academic calendar in Japan. This differs from Danish and Swedish local schools, whose school years start in August and end in June. Due to this difference of academic calendar, some discrepancies in grade levels occur. In addition, at JSS$_{DK}$, it is possible to repeat the same grade, further contributing to discrepancies between the grade levels JSS$_{DK}$ and the local schools. Approximately 80 pupils are enrolled in grade 1-6 at JSS$_{DK}$, while around 120 pupils are enrolled at JSS$_{SE}$. Of these pupils, 90% at JSS$_{DK}$ and 80% at JSS$_{SE}$ have dual cultural backgrounds, meaning that most pupils are bilingual or multilingual from home. Japanese proficiency varies greatly among the children, depending in part on whether their parents' background.

Recruitment of teachers is conducted locally, and involves no requirement for a teacher's certificate or teaching experiences. Due to the inherent difficulty in recruiting teachers, imposing such conditions would exacerbate the challenge. This issue is common to supplementary schools in other countries as well. There is a system for dispatching teachers to overseas educational institutions, and through this system, a person with principal experience at Japanese local schools is sent to JSS$_{SE}$ every two years, as JSS$_{SE}$ is the large chool among the Nordic countries.
Both institutions provide Japanese languages and mathematics. The mathematics lessons are offered based on the Japanese national curriculum (MEXT, 2017) and both schools use the textbooks published by TOKYO SHYOSEKI. This textbook series is one of several series authorised by the government, and is among the most widely used, alongside other prominent publishers such as KEIRINKAN and DAINIPPONTOSHO (Nihonkyouzaisyuppan, 2024). At JSS$_{DK}$, each subject is allotted 90 minutes (45 minutes × 2 sessions) per week, while at JSS$_{SE}$, Japanese language classes last for 135 minutes (45 minutes × 3 sessions) and mathematics classes for 90 minutes (45 minutes × 2 sessions). Notably, both institutions cover all mathematical content taught in Japanese local schools (JLS), albeit with only 54-57 hours per year, compared to the 136-175 hours per year (depending on grade level) offered in JLS (MEXT, 2021). This in itself imply special constraints whose implications are to be investigated as part of RQ1. Children and teachers are consistently encouraged to use the Japanese language during mathematics lessons.

4.2.2 Methodology for RQ1

To address RQ1, I first examined what mathematical praxeologies regarding fractions are set as knowledge to be taught for grades 1-6 in local Japanese primary schools. This involved elaborating a regional praxeological reference model (RPRM).

When defining logos blocks, I primarily referred to the Japanese national program, particularly a comprehensive summary table which appears there (MEXT, 2017, pp.12-15). We note that MEXT issues two levels of national programs: a general course of study for primary schools (SHOGAKKO GAKUSHU SHIDO YORYO) and a primary schools teaching guide for the general course of study in mathematics (SHOGAKKO GAKUSHU SHIDO YORYO KAISETSU SANSU-HEN). While the former outlines broad educational objectives and a cursory overview of subject content, the latter furnishes detailed instructions specifying principal components of to individual subjects, such as mathematics. For establishing the reference model for the teaching of fractions, the latter document serves as the primary reference and is referred to as "the program" in this paper. The table mentioned above succinctly summarises the mathematical knowledge to be taught in each domain and grade level. The table is divided into five domains: a) Numbers and Calculations, b) Geometric Figures, c) Measurements (grades 1 to 3) or Variation (grades 4 to 6) and d) Data handling. For instance, in the Geometric Figures (b) domain, triangles and squares are introduced in grade 2, while isosceles triangles, equilateral triangles, angles, circles and spheres are covered in grade 3. This table, therefore, describes the overall architecture of the mathematical knowledge to be taught at each grade level, with details specified elsewhere . In particular, to identifying praxis blocks related to the logos blocks, I mainly utilised textbooks to determine the exact types of tasks and techniques that are described in less extensive detail in the program.

The process of elaborating the RPRM concerning fractional arithmetics is as follows. I used the same method described by Aoki (2023).

1. Browse through the comprehensive summary table for the relevant domain: *numbers and calculations*, dividing it into two sectors, fractions as object (theory 1) and operations with fractions (theory 2) (Aoki, 2023).
2. Pick up mathematical contents regarding fractions in the table and categorize these (within each of the two theories) according to technologies (Aoki, 2023).
3. Determine the grade in which pupils learn these technologies (Aoki, 2023).

4. After identifying the logos blocks, browse through relevant textbook chapters, and analyse all examples and exercises to define types of tasks (Aoki, 2023). We do not describe all techniques corresponding to various types of tasks here; instead, we describe the techniques related to the lessons discussed later (Aoki, 2023).
5. Whenever a task is encountered that does not belong to any previously identified category, a new type is added into the he model (Aoki, 2023).
6. Determine whether these types of tasks are positioned in the right logos blocks, and confirm all titles of technologies, particularly if the titles of technologies are consistent (Aoki, 2023). If necessary, modify the title or reposition the types of tasks (Aoki, 2023).

Secondly, I collected data from two lessons in JSS_{DK} in 2021 in the 5^{th} grade class (with three pupils). Both lessons lasted a total of 90 minutes (2 sessions of 45 minutes each), with a break in between. The main topic in the lessons was about addition and subtraction of fractions with different denominators. The lessons were voice-recorded from beginning to end, and I took field notes during the observations to support the later analysis of transcripts.

I then transcribed them as they were (that is, in Japanese) and translated the transcripts into English. Based on the RPRM associated with these lessons, and the notion of moments of didactic processes, I analysed how these mathematical praxeologies develop during lessons and illustrated this in a table (the results are described in the following section). Didactic moments are defined in relation to praxeologies developed in the episode; therefore, the duration of the episode (in time) depends on when new types of tasks, techniques, and so on, are introduced (Aoki, 2023). Types of tasks and techniques were initially described based on the established RPRM, with additional techniques developed by the pupils also included (Aoki, 2023).

4.2.3 Methodology for RQ2

To answer RQ2, lesson observations were conducted. I collected data from four lessons in JSS_{SE} in 2022 in the 5^{th} grade class (with 19 pupils). These lessons lasted a total of 180 minutes, with breaks every 45 minutes. The entire lessons, from the beginning to the end, were video-recorded. These lessons were also about addition and subtraction of fractions with different denominators. These data were analysed in the following three steps. I used the same method described by Aoki et al., (accepted, modulo minor revisions).

1. The first author analysed recorded video interactions between the teacher and pupils, focusing particularly on a) episodes where pupils made utterances, which do not correspond to the teacher's expectations, and b) moments where the teacher explicitly highlighted pupils' utterances.
2. The first author transcribed all the episodes in Japanese and Swedish (using the language in which statements were originally uttered), and then translated into English. However, some parts were retained in Japanese and Swedish to enable the analysis of critical linguistic phenomena.
3. All authors analysed the episodes extracted in Step 1, identifying praxeological anomalies, and determining hypotheses for why these anomalies occurred. In this context, anomalies could arise at the level of logos (e.g., inappropriate use of Japanese to describe specific aspects of the mathematical praxis) as well as within the praxis itself.

4.2.4 Methodology for RQ3

To address RQ3, I conducted semi-structured interviews with three 7th-grade students at JSS_{DK} in the course of 2023. I focused on students enrolled in local schools in Denmark and south Sweden during weekdays. Two of the students were from Sweden, and one was from Denmark. The interviews were conducted individually. At the beginning of each interview, I distributed a worksheet containing the following tasks to the students. The tasks, originally written in Japanese, has been translated into English here.

1. Solve $\frac{6}{5}+\frac{7}{5}$ and write the answer in a worksheet.
2. Explain how to calculate $\frac{6}{5}+\frac{7}{5}$ to the interviewer.
3. Solve $\frac{9}{8}-\frac{5}{6}$ and write the answer in a worksheet.
4. Explain how to calculate $\frac{9}{8}-\frac{5}{6}$ to the interviewer.

I initially instructed the students to write their answers on their worksheets for Task 1. Afterward, I directed them to orally explain their written solutions to me (this relates to Task 2). Tasks 3 and 4 were devolved in the same manner. The interviews were repeated twice, with the second interview occurring five months after the first. Following these interviews (Tasks 1 to 4), I asked the students two additional questions: "What do you think about learning mathematics in two different schools?", and "How do you manage parallel learning of mathematics?" All interviews were recorded from the beginning to the end, and the interviewer took on-the-fly notes throughout. The recorded interviews were analysed in two steps. First, all recordings were transcribed in Japanese (the original language). Second, I analysed the students' written solutions for Tasks 1 and 3 and the transcriptions based on the six anomalies previously identified by my co-authors and me.

5 Main Results

This section presents the main outline of the results for RQ1, RQ2 and RQ3.

5.1 Main results for RQ1

The first research question was stated as follows:

RQ1. In what ways do mathematical praxeologies regarding fractions, established as knowledge to be taught in Japan, develop in teaching processes in Japanese supplementary schools?

This research question was addressed in Paper I presented in this thesis. The result are illustrated in Table 2. For details on the mathematical praxeologies mentioned in Table 2, please refer to Table 3. Table 2 outlines the flow of the lessons from beginning to end. As seen in Table 2, the lessons began with an institutionalisation moment and ended with an evaluation moment, with several moments occurring in between.

Table 2: Teaching processes in the lessons (Aoki, 2023)

Types of tasks (T)	Episode	Didactic moments
T_1, T_2, T_3, T_4	1 (2m8s)	Institutionalisation moment (with well-known praxeologies being what is institutionalised)
T_5	2 (13s)	(supposed) First encounter
	2.1 (3m43s)	Moment of technical work (τ_{5-1}, τ_5)
	2.2(18s)	Institutionalisation moment

T_6		3 (14s)	First encounter
		3.1 (1m39s)	Exploratory moment (τ_6 , τ_{6-3}*)
		3.2 (1m46s)	Moment of technical work (τ_6)
		3.3 (11s)	Institutionalisation moment
		3.4 (42s)	Moment of technical work (τ_6)
	T_2	3.5 (1m11s)	Institutionalisation moment (with well-known praxeologies being what is institutionalised)
		3.6 (2m)	Moment of technical work (τ_{6-1})
		3.7 (1m8s)	Institutionalisation moment (τ_6 , τ_{6-1})
T_5		4 (10s)	Institutionalisation moment (τ_6)
		4.1 (52s)	Moment of technical work (τ_{5-2})
		4.2 (12s)	Institutionalisation moment (τ_{5-2})
		4.3 (54s)	Moment of technical work (τ_{5-1})
		4.4 (1m13s)	Technological-theoretical moment
		4.5 (19s)	Institutionalisation moment
		4.6 (1m36s)	Moments of technical work (τ_{5-2})
T_7		5 (28s)	Institutionalisation moment (τ_6)
		5.1 (1m41s)	Moment of technical work (τ_7)
T_5		6 (1m20s)	Institutionalisation moment (τ_{6-1})
		6.1 (3m53s)	Technological-theoretical moment
		6.2 (2m24s)	Institutionalisation moment (τ_{6-1} , τ_{6-2})
T_6		7 (7s)	Institutionalisation moment (τ_{6-1})
		7.1 (1m)	Moment of technical work (τ_{6-1} , τ_{6-2})
		7.2 (1m25s)	Technological-theoretical moment
T_5		8 (2m44s)	Moment of technical work (τ_{5-2} , τ_{5-5})

	8.1 (21s)	Technological-theoretical moment
	9	Evaluation moment

Table 3: Excerpt from the regional praxeological reference model related to the lessons (Aoki, 2023)

Sector$_1$: Fractions as objects
Theme.$_{1\text{-}3}$: Ordering fractions
T_7: Order fractions with different denominators. τ_7: Find the (least) common multiple of the denominators and rewrite the fractions with the same denominators, then order the fractions.
Theme$_{1\text{-}4}$: Relating integers, fractions, and decimals
T_1: Rewrite fractions as divisions. τ_1: Replace an expression $\frac{a}{b}$ by $a \div b$. T_2: Rewrite fractions as decimals (or whole numbers). τ_2: Division algorithm. T_3: Rewrite a (finite) decimal as a fraction. τ_3: Use the appropriate power of 10 as denominator and the digits as the numerator.
Theme$_{1\text{-}5}$: Equivalence of fractions
T_6: Given a fraction, find other fractions that are equivalent to the given fraction. τ_6: Multiplying the numerator and the denominator by the same integer. τ_{6-1}: Dividing the numerator and the denominator by the same integer. τ_{6-2} : Dividing the numerator and the denominator by the greatest common divisor.
Sector$_2$: Operations with fractions
Theme$_{2\text{-}1}$: Addition and subtraction of fractions
T_4: Addition and subtraction of fractions with the same denominators ($\frac{a}{b} \pm \frac{c}{b}$).

$\tau_4: \frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$

T_5: Addition and subtraction of fractions with different denominators.

τ_5: Rewrite two fractions to have common denominators using number lines, then use τ_4 to add them.

τ_{5-1}: Find the common multiple of the denominators and rewrite the fractions with the same denominators, then use τ_4 to add them.

τ_{5-2}: Find the least common multiple of the denominators and rewrite the fractions with the same denominators, then use τ_4 to add them.

τ_{5-3}: Use τ_{5-1} and then use τ_{6-1}.

τ_{5-4}: Use τ_{5-2} and then use τ_{6-1}.

τ_{5-5}: Use τ_{5-1} and then use τ_{6-2}.

τ_{5-6}: Use τ_{5-2} and then use τ_{6-2}.

Based on the analysis of the lessons, utilising the regional praxeological reference model and the identified moments of didactic processes, two main outcome have emerged (Aoki, 2023).

- First, it was evident that not much time was allocated to each moment, and the fusion of the moment of technical work and the moment of institutionalisation occurred frequently. The short time allocated to each moment inevitably led to a lesson structure focused mainly on acquiring techniques. For example, the lesson structure did not facilitate comparison and justification of various techniques through examination and discussion. I hypothesis this situation arise in part from teachers' efforts to faithfully teach praxeologies set according to the Japanese context in the Japanese supplementary school, where the context, including time constraints, differs significantly from that of regular schools in Japan. Note here that this outcome is not necessarily due to a lack of mathematical or didactic knowledge on the part of the teacher. The purpose of the JSS itself is to teach Japanese praxeologies, and the teacher merely attempts to adhere to this faithfully. This may, indeed, be considered an inevitable outcome, as mathematical knowledge determined by the Japanese noosphere for institutions within Japanese society is being taught outside of Japan, in contexts that differ significantly from Japanese society.

- Secondly, the intended gradual build-up of techniques proposed by the textbook authors was to some extent disrupted, during the lessons, under the influence of the experiences of pupils who attend regular (Danish, Swedish, or international) schools on weekdays. For instance, in the textbook, pupils are expected to develop a certain technique τ_{5-1} (find the common multiple of the denominators and rewrite the fractions with the same denominators, to produce $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$) based on their external encounter with an alternative technique τ_6 (multiplying the numerator and the denominator by the same integer). In fact, pupils are expected to learn τ_5, then τ_6, and subsequently τ_{5-1}. However, during the lesson, τ_{5-1} was introduced by a pupil first, followed by the teacher referring to τ_5, and then τ_6. This sequence indicates, as in the first outcome, that the teachers are faithfully teaching Japanese praxeologies to the pupils.

In summary, the mathematical knowledge valued by the Japanese noosphere for institutions within Japanese society is not being fully transposed to the different context of the supplementary school, but is being taught faithfully.

5.2 Main results for RQ2

The second research question was stated as follows:

RQ2. What didactic phenomena can be observed from fraction lessons in JSS involving children who are simultaneously learning two different praxeologies? In addition, what are the underlying causes of these phenomena?

This research questioned was addressed in Paper Ⅱ. The results identified the following five phenomena. In the subsequent text, Japanese words are presented in italics, followed by their English translation in parentheses.

- The first phenomenon involves the amalgamation and interchange of specific mathematical terminologies from Japanese and Swedish. Specifically, a pupil confused the Japanese terms for “denominators” and “numerators”, and terms such as “times” “multiply” and “denominators” were occasionally uttered in Swedish by some pupils.
- The second phenomenon pertains to the pronunciation of fractions and the writing order of fractions. A pupil reads out fractions in Japanese while utilising the syntactical

structure of Swedish. Specifically, he articulated them in the order of the numerator to the denominator, whereas Japanese syntax typically follows the order from the denominator to the numerator. In addition, a pupil wrote fractions from top to bottom, indicating that he followed the Swedish order in the process of writing fractions.

- The third phenomenon concerns the representation of the multiplication symbol. A pupil utilised the notation "·" commonly employed in Swedish local schools, instead of "×", which is used in Japanese local schools at the primary level.
- The fourth phenomenon is that a pupil reads a fraction as division according to Swedish praxeologies. Specifically, when the teacher asked a pupil to read the fraction $\frac{3}{4}$ by pointing to it and asking, "How do you read this fraction?", the pupil responded with "*san waru yon*" (3 divided by 4) following the Swedish praxeology, whereas the expected response according to Japanese praxeology would be "*yon bun no san*" (3 over 4). Thus, the expected distinction of division and fraction is not realised.
- The fifth phenomenon is that the mathematical term "*tsubun*" (converting fractions to equivalent ones with common denominators), which does not exist in Swedish school mathematical vocabulary, but is important within Japanese school mathematics, may or may not be used by pupils, which has considerable influence on practice (and specifically, the use of techniques). Concretely, in Japanese school mathematics, when converting fractions to equivalent ones with common denominators, two techniques are employed. The first technique accomplishes this by using the product of the original denominators as a common denominator (τ_1). The second one involves determining the least common multiple of the original denominators (τ_2). Therefore, the term "*tsubun*" can be considered a technology comprising two techniques (τ_1 and τ_2). Consequently, the concept observed in Japanese textbooks and the episode associating "tsubun" with τ_2 in the Japanese context led to the conclusion that phenomena at the level of logos may substantially influence the praxis (use of the techniques) in the context of adding and subtracting fractions with different denominators.

In summary, the above phenomena arose due to the differences in the curricula between JSS_{SE} and local schools in Sweden, including different mathematical logos. These phenomena can be attributed to the fact that pupils are learning (and retaining) praxeologies from two different intuitions in parallel. The fifth phenomenon is also related to the concept of institution and is due to cultural differences between institutions, ultimately stemming from the cultural contexts to which these institutions belong.

5.3 Main results for RQ3

The third research question was stated as follows:

RQ3. What didactic phenomena can be observed when children, who are simultaneously learning two different praxeologies, solve specific fraction-related tasks?

This research question was addressed in Paper Ⅲ. Overall we did observe moments where students exhibit parallel praxeologies to varying degrees, both in term of praxis (use of the techniques) and logos (explaining and justifying techniques) when solving specific fraction-related tasks. In particular, parallel praxeologies occurred in the following episodes, where we refer to the three students as Pupil A, Pupil B and Pupil C. On weekdays Pupil A and C both attend Swedish local schools, and Pupil B attends a Danish local school.

- The process of writing fractions: When completing tasks: $\frac{6}{5}+\frac{7}{5}$ (t_1) and $\frac{9}{8}-\frac{5}{6}$ (t_2) on their worksheets, Pupil A consistently wrote from top to the bottom, Pupil B consistently wrote from bottom to the top, and Pupil C employed both manners at different times. Thus the second phenomenon discussed in Section 5.2, which is anomalies in the way fractions are written, was also observed here. In addition, regarding t_2, when converting fractions to equivalent ones with common denominators, there are two techniques: τ_1 and τ_2 (defined in section 5.2). For instance, Pupil A, who mainly followed the Swedish praxeologies, used τ_1, and pupil B, who followed the Japanese praxeologies, along with Pupil C, who followed parallel praxeologies, used τ_2. Pupil C expressed the calculation process as $\frac{27}{24}-\frac{20}{24}$, with this part ($\frac{27}{24}$ and $\frac{20}{24}$) written in the Japanese way. In fact, employing τ_2 may naturally lead to writing from the numerator first (following the Japanese praxeologies). Therefore, the techniques used may potentially influence the process of writing fractions.
- The discourse related to the order of explaining calculation of numerator and denominator: In Paper Ⅲ, the discourse explaining the calculation processes of Pupil A for t_1 and t_2 was presented, with an analysis focusing on whether the explanation of their calculation steps began with the numerator or the denominator. Similar analyses were

performed on the logos of the other two students. As a result, it was observed that there was a certain degree of parallel praxeology in the order of explaining the calculation processes, suggesting that parallel praxeology may influence the order in which these processes are explained.

- Techniques for converting fractions to equivalent ones with common denominators: In Section 5.2 (following Paper II), we demonstrated that there are two techniques, τ_1 and τ_2, for converting fractions to equivalent ones with common denominators. In the Japanese context, pupils are ultimately trained to use τ_2 as they gradually grasp the advantages of τ_2 over τ_1. This technique becomes habitual and almost automatic through consistent practice. By contrast, in the Swedish and Danish contexts, τ_1 is commonly utilised in schools, while τ_2 is not prioritised and may not even be taught, at least it does not appear in commonly used textbooks. Considering this, I analysed the pupil's answers to t_2 during the two interviews. As the second interview was conducted five months after the first, one could assume that the students answers reflect intermediate development. The results showed that while Pupil C seems to have adopted the praxeological norm of the Japan (consistently employing τ_2), the other two students displayed more inconsistency or lack of adherence to the norms of one the two schools they attend (employing both techniques, τ_1 and τ_2,). Therefore, we observed parallel praxeological norms in praxis (use of techniques).

One can speculate that conducting interviews with more students and including a wider array of tasks would be necessary to estimate the actual *extent* of the above phenomena.

In addition, during the session of additional interviews after the interviews regarding $\mathrm{t}_1$and t_2, students also demonstrated some awareness of differences in praxeological norms between the two institutions (JSSSE (Japan) and Demark or Sweden), both at the level of praxis and logos. For instance, differences in the form of long multiplication and long division, as well as the distinct reading order of fractions (the second phenomenon described in Section 5.2) and a fraction as division (the fourth phenomenon described in Section 5.2), were pointed out spontaneously by the students. Therefore, I concluded that students experiencing two systems of praxeological norms, while attending two schools in parallel, at least to some extent become aware of the institutional relativity of such norms for mathematical praxeologies, unlike students who do not have such an experience.

Learning about fractions in two monolingual environments simultaneously, may at first sight appear as independent processes, since students are expected to learn the specific mathematical praxeologies and languages associated with their respective institutions. Furthermore, their learning of Japanese mathematics in supplementary schools, conducted in Japanese, is relatively limited compared to their mathematics learning in local schools on weekdays. Despite this, it can be observed that they possess parallel praxeologies at both the praxis and logos levels. Moreover, in some instances, the fraction praxeologies of the two educational institutions influence each other.

6 Conclusions and Perspectives for Future Research

In my doctoral project, I have defined a main objective and developed three overall research questions to elucidate them (Section 4.1). The main objective was both to contribute to our understanding and knowledge of how Japanese expatriate schools function, in particular, the Japanese supplementary schools, and to contribute to more general *problematiques* concerning the role of institutions, curricula and language in primary mathematics education.

To address the three RQs, the Japanese supplementary schools in Denmark (JSS_{DK}) and Sweden (JSS_{SE}) were chosen as cases. The RQs are explored in the three papers presented in the thesis. RQ1 was answered in PaperI, and PaperII and III addressed RQ2 and RQ3, respectively.

This doctoral project makes three main contributions. First, it sheds light on concrete phenomena occurring in the teaching of fractions in Japanese supplementary schools that have not been previously explored. Secondly, by developing new methodologies based on ATD, it reveals how the context of specific institutions determines the interaction between general language and mathematics-specific discourse, and how these interactions combine to influence the learning of pupils with bilingual backgrounds. Third, it elucidates some differences and characteristics of the praxeologies related to fractions in Japanese, Swedish and Danish school mathematics.

In this study, I focused on the Japanese supplementary schools among the two educational institutions attended by the children. However, I believe that conducting further research focusing on the children's experiences in local schools would further strengthen the hypotheses developed in this study. Additionally, since the curriculum used is similar to that of local schools in Japan, comparative studies with Japanese regular school are also necessary. Finally, while this study only dealt with internal didactic transposition, it is also necessary to the scholarly knowledge behind the external didactic transposition, and the functions and structures of the noosphere in the three national contexts (and similar ones). Indeed, with still more pupils being subject to sequential or parallel exposures to mathematics in more or less different transpositions, these contribution of such extensions of research perspective is not limited to classical comparative studies of mathematics education in more or less disjoint institutions, but are also relevant to the increasingly international perspectives of pupils, as well as to us as didactic researchers.

References

Aoki, M. (2023). Mathematics in a Japanese overseas school: Connecting classroom study and curriculum. *Asian Journal for Mathematics Education*, *2*(2), 204-219. https://doi.org/10.1177/27527263231187804

Austin, J.L., & Howson, A.G. (1979). Language and mathematical education. *Educational Studies in Mathematics*, *10*, 161–197. https://doi.org/10.1007/BF00230986

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice: The case of limits of functions in Spanish high schools. *Educational Studies in Mathematics*, *59*, 235-268.

Barwell, R., Clarkson, P., Halai, A., Kazima, M., Moschkovich, J., Planas, N., Setati, M., Valero, P., & Villavicencio, M. (2016). Introduction: An ICMI study on language diversity in mathematics education. In Barwell B, Clarkson P, Halai A, et al. (Eds) *Mathematics Education and Language Diversity: The 21st ICMI Study*. Cham, Switzerland: Springer International, pp.1-22. https://doi.org/10.1007/978-3-319-14511-2_1

Bosch, M., & Gascón, J. (2014). Introduction to the anthropological theory of the didactic (ATD). In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education, advances in mathematics education*. Switzerland: Springer International Publishing.

Behr, M. J., Harel, G., Post, T. R., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296–332). New York: Macmilian.

Cousin, M. (2018). The revolution in mathematics education during the Meiji Era (1868-1912): A study of the textbooks used to teach computation, geometry and algebra. In F. Furinghetti & A. Karp (Eds.), *Researching the history of mathematics education* (pp. 45–60). Cham, Switzerland: Springer. https://doi.org/10.1007/978-3-319-68294-5_3

Chevallard, Y., & Johsua, M. A. (1985). *La transposition didactique: du savoir savant au savoir enseigné*. La Pensée Sauvage.

Chevallard, Y. (1992). Fundamental concepts in didactics: Perspectives provided by an anthropological approach. *Research in didactique of mathematics: Selected papers*, 131-168.

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic an attempt at a principled approach. *Hiroshima journal of mathematics education*, *12*, 71-114. https://doi.org/10.24529/hjme.1205

Chevallard, Y., & Bosch, M. (2019). A short (and somewhat subjective) glossary of the ATD. In M. Bosch, Y. Chevallard, F. J. García, & J. Monaghan (Eds.), *Working with the anthropological theory of the didactic in mathematics education*. A comprehensive casebook (pp. xviii–xxx). Routledge.

Creese, A., Bhatt, A., Bhojani, N., & Martin, P. (2006). Multicultural, heritage and learner identities in complementary schools. *Language and Education*, *20*(1), 23–43. https://doi.org/10.1080/09500780608668708

Cushing, I., Georgiou, A., & Karatsareas, P. (2021). Where two worlds meet: language policing in mainstream and complementary schools in England. *International Journal of Bilingual Education and Bilingualism*, 1–17. https://doi.org/10.1080/13670050.2021.1933894

Date, B. (2011). *Sugaku kyoiku ni okeru bunka-teki kachi ni kansuru kenkyu* [A study on cultural values in mathematics education]. *Journal of the Japan Society of Mathematical Education, 92*(R95), 5-14.

Ellerton, N.F., & Clarkson, P.C. (1996). Language factors in mathematics teaching and learning. In Bishop, A.J., Clements, K., Keitel, C., Kilpatrick, J., Laborde, C. (Eds.), *International Handbook of Mathematics Education. Kluwer International Handbooks of Education*, 987-1033. Dordrecht: Kluwer. https://doi.org/10.1007/978-94-009-1465-0_27

Erath, K., Ingram, J., Moschkovich, J., & Prediger, S. (2021). Designing and enacting instruction that enhances language for mathematics learning: a review of the state of development and research. *ZDM Mathematics Education* 53, 245–262. https://doi.org/10.1007/s11858-020-01213-2

Farrugia, M.T. (2018). Learning fractions through two languages in an elementary classroom: the interrelation of maltese and english with the mathematics register(s). In Romanowski, P., Jedynak, M. (Eds) *Current Research in Bilingualism and Bilingual Education*. *Multilingual Education*, *26*. Springer International Publishing. https://doi.org/10.1007/978-3-319-92396-3_6

Farsani, D. (2015). *Making multi-modal mathematical meaning in multilingual classrooms* [Doctoral dissertation, University of Birmingham]. https://etheses.bham.ac.uk//id/eprint/5752/1/Farsani15PhD.pdf

Farsani, D. (2016). Complementary functions of learning mathematics in complementary schools. In P. C. Clarkson & A. Halai (Eds.) *Teaching & learning mathematics in multilingual classrooms: Issues for policy, practice and teacher education* (pp.227–247). Rotterdam: Sense Publishers.

Francis, B., Archer, L., & Mau, A. (2009). Language as capital, or language as identity? Chinese complementary school pupils' perspectives on the purposes and benefits of complementary schools. *British Educational Research Journal*, *35*(4), 519–538. https://doi.org/10.1080/01411920802044586

Francis, B., Archer, L., & Mau, A. (2010). Parents' and teachers' constructions of the purposes of Chinese complementary schooling: 'culture', identity and power. *Race Ethnicity and Education*, *13*(1), 101–117. https://doi.org/10.1080/13613320903550089

García, O., & Wei, L. (2014). *Translanguaging: Language, Bilingualism and education*. Basingstoke: Palgrave Macmillan

Gascón, J. (2024). Contributions of the anthropological theory of the didactic to the epistemological programme of research in mathematics education. *ZDM–Mathematics Education*, 1-12. https://doi.org/10.1007/s11858-024-01563-1

Halliday, M. A. K. (1989). *Spoken and written language*. Oxford: Oxford University Press.

Huang, C. (2022). *Diversity and Imagined Communities at a Japanese Supplementary School in New Zealand* [Doctoral dissertation, University of Auckland]. https://researchspace.auckland.ac.nz/bitstream/handle/2292/58448/Huang-2022-thesis.pdf?sequence=1&isAllowed=y

Japan Overseas Educational Services [JOES]. (2018). *Hoshu jugyoko jido seito no gakushu jotai chosa-to hokokusho* [Report on the learning situations in Japanese supplementary schools]. https://ag-5.jp/cms/ag5/common/pdf/theme4/Gakushujokyochosa.pdf

Kano, N. (2013). Japanese community schools: New pedagogy for a changing population. In O. García, Z. Zakharia, & B. Otcu (Eds.), *Bilingual community education and multilingualism: Beyond heritage languages in a global city* (pp.99–112). Clevedon: Multilingual Matters.

Li, W. (2006). Complementary schools, past, present and future. *Language and Education*, *20*(1), 76–83. https://doi.org/10.1080/09500780608668711

Li, W. (2011). Multilinguality, multimodality, and multicompetence: Code-and modeswitching by minority ethnic children in complementary schools. *The Modern Language Journal*, *95*(3), 370-384. https://doi.org/10.1111/j.1540-4781.2011.01209.x

Ministry of Education, Culture, Sports, Science and Technology-Japan [MEXT] . (2016). *Zaigai kyoiku shisetsu gurobaru jinzai ikusei kyoka senryaku* [Global human resource development enhancement strategy for overseas educational institutions]. https://www.mext.go.jp/a_menu/shotou/clarinet/002/__icsFiles/afieldfile/2016/09/08/1376422_2.pdf

MEXT. (2017). *Syougakko gakusyu shido yoryo kaisetsu, sansu-hen* [Primary school teaching guide for the Japanese course of study: Mathematics]. Tokyo: Nihon Bunkyou Shuppan.

MEXT. (2021). *Hyoujun jugyou jisuu ni tsuite* [Regarding standard class hours]. https://www.mext.go.jp/content/20210629-mxt_kyoiku01-000016453_4.pdf

MEXT. (2023). *Zaigai kyoiku shisetsu no gaiyo* [Overview of overseas educational institutions]. https://www.mext.go.jp/a_menu/shotou/clarinet/002/002.htm

Nakahara, T. (2000). *Sansu/sugakuka juyo yogo 300 no kiso chishiki (juyo yogo 300 no kiso chishiki 5)* [Basic knowledge of 300 important terms in mathematics for primary to secondary schools]. Tokyo: Meijitosho.

Nihonkyouzaisyuppan. (2014). 2024 nendo kyoukasyo saitaku hyou [Textbook adoption list in 2024]. https://www.nihonkyouzai.jp/kyozai/list/11089

Okumura, S., & Obara, Y. (2017). Integrating language and content: challenges in a japanese supplementary school in Victoria. *Online Submission*, *51*(2), 24-29.

Petersson, J., & Norén, E. (2017). To halve a fraction: An issue for second language learners. *Education Inquiry*, *8*(3), 173–191. https://doi.org/10.1080/20004508.2016.1275187

Planas, N., Morgan, C., & Schütte, M. (2018). Mathematics education and language. Lessons from two decades of research. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger, & K. Ruthven (Eds.), *Developing research in mathematics education. Twenty years of communication, cooperation and collaboration in Europe* (pp. 196– 210). London: Routledge.

Prediger, S., Clarkson, P., & Boses, A. (2016). Purposefully relating multilingual registers: Building theory and teaching strategies for bilingual learners based on an integration of three traditions. *Mathematics education and language diversity: The 21st ICMI study*, 193-215. https://doi.org/10.1007/978-3-319-14511-2_11

Prediger, S., Kuzu, T., Schüler-Meyer, A., & Wagner, J. (2019). One mind, two languages– separate conceptualisations? A case study of students' bilingual modes for dealing with language-related conceptualisations of fractions. *Research in Mathematics Education*, *21*(2), 188-207. https://doi.org/10.1080/14794802.2019.1602561

Shibano, J. (2014). *Nihonjin gakko kyoin no 'Nihonrashisa' o meguru jissen to katto: Toransunashonaruka suru zaigai kyoiku shisetsu o jirei ni* [Practices and conflicts surrounding "japaneseness" among teachers at Japanese schools: a case study of transnational overseas educational institutions]. *Japanese Journal of Educational Sociology*, *95,* 111-130.

Shibano, J. (2019). Kaigai kikokusei: kyoiku mondai no hensen to aratana doko [Children returning from abroad: changes in educational issues and new trends]. In Nukaga, M., Shibano, J., & Miura, A. (Eds.)., *Rethinking Education from Migrants* (pp. 47-60). Kyoto: Nakanishiya syupan.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*(7), 691-697. https://doi.org/10.1177/0956797612440101

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving in the classroom*. New York: The Free Press.

Whorf, B.L. (1956). *Language, thought and reality*. In J.B.Carroll (Eds.), *Selected Writings of Benjamin Lee Whorf* (134-159). MIT Press.

Yamaguchi, K. (1998). *Fujisawa rikitaro "suugaku ni mochiiru jino eiwa taiyaku jisho" ni tsuite: "sugaku yougo yakugo kai" no yougo tono hikaku* [Vocabulary of mathematical terms in English and Japanese (1889) by R. Fujisawa : a comparison with "translation of mathematical terms" by translation committee of Tokyo mathematical society]. *Kyushu Sangyo University Institutional Repository*, *11*, 115-134.

Paper I

MATHEMATICS IN A JAPANESE OVERSEAS SCHOOL: CONNECTING CLASSROOM STUDY AND CURRICULUM

Mayu Aoki

Department of Science Education, University of Copenhagen, Denmark;
mayu13ak@gmail.com

ABSTRACT

The aim of this paper is to investigate how the contents related to the arithmetic of fractions, as described in the official curriculum issued by the Japanese Ministry of Education, Culture, Sports, Science, and Technology (MEXT), are implemented in actual teaching at the so-called Japanese supplementary school in Denmark. Additionally, at a methodological level, this case study demonstrates how so-called praxeological reference models can be developed at different levels of detail to describe classroom teaching at a particular grade level and the overall curriculum of primary school in a connected way. This study is based on the notions of praxeological reference models and moments of didactic processes from the Anthropological Theory of the Didactic. The findings reveal that (1) not much time is allocated to each moment and (2) the flow of gradually building up techniques, as intended by the textbook authors is to some extent disrupted in the lesson.

KEYWORDS

Teaching fractions, The Japanese curriculum, The Japanese supplementary school, The Anthropological Theory of Didactic, Praxeology, Didactic moments

1 INTRODUCTION

In mathematics education research, there is a tendency to study classroom episodes almost independently from studies of curricula (or official programs). In particular, classroom studies typically feature the curriculum as (at most) a background part of the context. In this paper, we attempt to demonstrate, through a case study, how modelling and analysing the curriculum can be done in a way that informs and is tightly connected to the analysis of observations from a classroom conducted with this curriculum. This is mainly a methodological point of the paper. It draws heavily on elements from the anthropological theory of the didactics, which we outline in Section 3.

The case we consider is also interesting in its own right, as it relates to the phenomenon of expatriate schools (here, a so-called Japanese Supplementary School in Denmark [JSS_{DK}]). This school uses the same curriculum as ordinary Japanese schools. However, mathematics is taught once a week for 90 min (we will say more about the specific case of Japanese supplementary schools [JSS] in Section 2). How is mathematics taught in the school? How does the Japanese mathematics curriculum transpose to contexts outside of Japan? We would imagine that different institutional conditions and constraints may lead the curriculum to be "implemented" (or rather transposed) in very different ways, but it has not been theoretically clarified. Based on classroom observations, we investigate how a particular topic (addition and subtraction of fractions) is realized in actual teaching in the JSS_{DK}. In this paper, we have not observed lessons in Japanese ordinary schools, so it is not possible to explicitly discuss the differences between the didactic processes in these and in JSS_{DK}. However, we briefly develop some hypothetical differences based on a previous study by Stigler and Hiebert (2009). As Stigler and Perry (2009) stress the importance of cross-cultural comparison for an explicit understanding of teaching mathematics, comparing the didactic processes in those two institutions will provide a better understanding of the didactic processes in JSS_{DK}.

The paper is structured as follows. After presenting the theoretical framework, the context, and our research questions, we first present a meticulous analysis of the Japanese curriculum on the arithmetic of fractions and subsequently show how the resulting model can be used to analyze an episode of fraction teaching in JSS_{DK}. We finally reflect on the scope or generalizability of the proposed method for studying curricula and classroom teaching with the same model.

2 CONTEXT AND BACKGROUND

Globalization has caused an increasing number of people to live in (for them) foreign countries for extended but limited periods of time, for instance, to work in an overseas branch of their company. In particular, around 83,000 Japanese children in grades 1 to 9 live abroad as of 2017 (Ministry of Education, Culture, Sports, Science, and Technology [MEXT], 2021). To ensure continuous schooling of children, several countries, such as France, the United Kingdom, and Germany have established overseas school systems. As for Japan, the government established Japanese educational institutions abroad for Japanese children as early as the 1950s (Shibano, 2020), including the so-called Japanese Supplementary Schools (JSS). These schools were initially established for children who were supposed to go back to Japan; however, in more recent years, there has been an increasing number of children in these schools who live permanently (or at least for an indeterminate period of time) in the foreign country (Shibano, 2020).

The schools receive financial support from Japan's MEXT, in addition to the tuition paid by the parents of children attending the school. The institutions function only on Saturdays or after school hours, as children are supposed to attend a national or international school, for which the JSS then serves as a supplement, focused mainly on the Japanese language. However, the schools also teach other subjects, in fact, 80% of the schools provide Japanese as the primary subject and math- ematics as another subject, based on the Japanese national program and Japanese textbooks (Okumura & Obara, 2017). All subjects are taught in Japanese. Okumura and Obara (2017) point out that the reason mathematics is taught in addition to Japanese is because mathematics is provided in local schools as well. Teaching in both the local school and JSS can then be applied to each other and help to improve the Japanese language skills of the children (Okumura & Obara, 2017).

The JSS_{DK}, the context of the case study presented in this paper, operates every Saturday morning and provides teaching of Japanese and Mathematics. Children attend local or international schools in Denmark or southern Sweden. The first language of most of the children at JSS_{DK} is Danish or Swedish, so in terms of Japanese proficiency, not all are at the level of children of their age in Japan. One lesson lasts 45 min, and on each Saturday, there are two lessons for each of the two subjects taught. In particular, the time devoted to mathematics is less than half of what is common in regular Japanese schools. In fact, the school covers all the mathematical content that children in Japan learn, but with only 55.5 h per year, unlike the 136–175 h per year that are available in regular Japanese schools. We note that all grades (1–9) are taught separately, and due to the modest number of children (80 in total), it is not uncommon for a class to have just a few children, unlike regular schools in Japan where classes usually consist of 35 children (MEXT, 2021). As ensuring a sufficient and stable number of teachers is a challenge for all supplementary schools, the teachers at this school, like those in other supplementary schools, are hired locally regardless of whether they hold a teaching license or have teaching experience, and they have other professions as their main job. Like teachers at other supplementary schools, they have limited time to develop lesson plans, and there are few opportunities for in-school or outside-school training to improve their teaching skills. Consequently, teachers conduct lessons based on their own teaching experience at the JSS and teaching guide.

3 THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Our study is based on the Anthropological Theory of Didactic (ATD), specifically focusing on the notions of didactic transposition, praxeology, levels of didactic codetermination, and moments of didactic processes. All of these are needed to formulate and motivate our specific research questions.

The theory of didactic transposition was elaborated for the case of mathematics education by Yves Chevallard, in the late 1970s. It considers that knowledge (including both practical and theoretical knowledge) must be understood as residing in institutions, and focuses on the way knowledge is *transposed* between institutions, which is not merely "transportation" but also requires adaptation to the conditions of the receiving institution. The main transpositions related to education comprises four stages in the transposition process: *Scholarly knowledge* (in institutions outside of school, like universities), *Knowledge to be taught* (at schools, but formulated elsewhere like in ministries), *Taught knowledge,* and *Learnt knowledge* (both within the school). In particular, the transposition from knowledge to be taught to learnt knowledge is called *internal didactic transposition* (Chevallard, 1985). Within this theory, the knowledge that pupils acquire in schools is understood in terms of the three previous stages. The term "transpositions" here is used in a metaphorical sense: such as changing from one key to another key to play a piece of music, knowledge is successively adapted to fit official programs for schools, the classroom, and finally the individual pupil (Chevallard, 1999). In this paper, we study the internal didactic transposition taking place at JSSDK: How teachers implement the official curriculum in mathematics to the classroom within this institution, in view of the special conditions outlined above.

To conduct this study, we first need to elaborate an explicit reference model to describe the knowledge to be taught. In ATD, explicit reference models play a crucial role for researchers to keep distance from transposition processes (Bosch & Gascón, 2006), and the role of the model depends on the specific research purpose—in particular, the research questions. Here, the model is used to clarify what mathematical knowledge is officially set to be taught in Japanese primary schools (comprising grades 1–6). In order to develop this model, we need the notion of praxeology and the levels of didactic codetermination.

In fact, another fundamental assumption of ATD is that any human activity and knowledge can be modeled in terms of praxeologies. A praxeology consists of two interrelated parts. The first part is called the practical block. It consists of a type of task (T) and techniques (τ) that can be used to solve tasks of the type T. The second part is the logos block which is composed of a technology (θ) to explain the techniques and a theory (Θ) to justify and unify several technologies. A praxeology regarding mathematical knowledge is called a mathematical praxeology or a mathematical organiza- tion (Barbé et al., 2005). Praxeological models can describe all four levels in more or less detail, depending on the research aims: As in any other empirical discipline, didactic researchers need to construct explicit models of central phenomena to be studied, such as mathematical practice and knowledge.

Praxeologies are closely related to the first three levels in the theory of didactic codetermination. These levels are used to specify knowledge and practice in school institutions at different levels of institutional granularity, within and above the school itself. While a total of 10 different levels may be considered (Artigue & Winsløw, 2010), we shall focus here primarily on the three lowest levels:

- The first level, called Subject, corresponds to a point praxeology (T, τ, θ, Θ) to be taught and is often referred to by the symbol T. For instance, addition and subtraction of fractions with a common denominator $[(\frac{a}{b}) \pm (\frac{c}{b})]$ are situated at this level. Subjects are then to be developed in actual teaching, and official descriptions of the subject may comprise specific indications of the techniques as well.

- The second level, called Theme, is an organization of several subjects to be taught, and teachers usually enjoy some freedom also at this level; still, the knowledge to be taught usually comprises indications about the theory blocks that unify and organize the theme. For instance, a theme labeled “Addition and Subtraction of fractions” could determine a progression from the case of fractions with like denominators, toward the general case, while also drawing on a theory of equivalent fractions, as we shall see later. The praxeologies involved (T_i, τ_i, θ, Θ) are unified by a tech- nology to describe and justify the different techniques involved.
- The third level, called Sector, is unified by a theory and is thus constituted by praxeologies (T_i, τ_i, θi, Θ) where the central object is the unifying theory. For instance, the sector “Operation with fractions” will be characterized by a theory on the arithmetic of fractions, in which the four oper- ations are related theoretically (for instance, some operations are inverses of each other, and there is a distributive law to relate addition and multiplication, and so on).

The subject level thus corresponds to more concrete and “small” parts of the knowledge to be taught, and can be associated with specific tasks or exercises that may sometimes be found in a simple section of the textbook, to be studied in short periods of time such as a few lessons; a sector may extend over several school years and will typically be intertwined with, and draw, on other sectors.

These levels together with the more detailed description of praxeologies are used, in this paper, as a tool for modeling the mathematical knowledge to be taught regarding fractions in the Japanese primary school. We develop the detailed model in Section 5.

As we mentioned before, our focus is how knowledge to be taught transposes to teach knowledge in JSS_{DK}. Therefore, we need to investigate the didactic processes that occur in classrooms, extend in time and form the teachers’ way to organize a subject, theme or even sector in time. When analyzing a lesson, the notion of didactic processes allows us to describe the didactic flow in terms of the prax- eological reference model (PRM), and in particular to classify episodes in the teaching in terms of what elements of the praxeologies are being worked on.

Didactic processes are considered to be built by six kinds of episodes or moments: Moments of first encounter, exploratory moments, technological–theoretical moments, technical moments, institu- tionalization moments, and evaluation moments (Barbé et al., 2005). Considering a classroom protocol, we concretely identify (not necessarily in this order):

- First encounters occur when pupils meet new types of tasks (T_i); here the focus is on simply understanding what the new tasks mean and ask for.
- Exploratory moments consist of activities in which pupils try to solve one or more tasks of the new type(s) T_i and develop some first techniques (τ_i); here the focus is on the exploration based on what pupils already know.
- Technological–theoretical moment occurs when one or more techniques τ_i (developed, for instance, in an explorative or technical moment) are examined and discussed, for instance, to name, compare or justify the techniques, or to develop a theory that could

relate and justify several technologies. The focus here is thus on explication, justifications, and the relation of praxeologies.

- Moments of technical work occur when techniques are introduced, strengthened, and general- ized, and the focus is thus on elaborating pupils' technical knowledge, beyond what can be obtained by a first exploration. In mathematics, the routinization of techniques, as well as elaboration and extension of techniques, may take up rather considerable parts of the didactical process.
- Institutionalization moments are elements of the didactical process where the teacher summa- rizes or introduces knowledge—technical or theoretical—which the students are supposed to have learned or should subsequently work on learning; the focus here is on relating the work in the classroom to official aims.
- Finally, evaluation moments are those parts of the didactical processes which are devoted to assessing some praxeologies which the students are supposed to have learned, in view of deter- mining their qualities: For instance, students may be submitted to a test, or they may be engaged in evaluating the extent to which the work carried out so far leaves open questions that could be studied in the future.

The six moments are used to analyze the didactic processes in the JSS_{DK}, not only to divide them into episodes with the said characteristics, but also to identify moments that are emphasized and to which more time is devoted, possibly at the expense of others.

We can now formulate the research questions of this paper:

RQ1: What mathematical praxeologies regarding the arithmetic of fractions are set as knowledge to be taught in grades 1–6 in Japanese primary school? How can the goals be described at different levels of didactical codetermination?

RQ2: How do these mathematical praxeologies develop in the didactic processes in JSS_{DK}, particularly at the level of a theme relating to several subjects? What moments are emphasized?

4 METHODOLOGY

4.1 METHODOLOGY FOR RQ1

In order to answer RQ1, we elaborated a PRM of the sectors on fractions (hereafter, we call this the regional PRM), stretching over five grades (2–6). We mainly used the program and textbooks as data to base this model on. The program was used for defining the levels of sectors and themes, and the latter was referred to for identifying the subject levels.

There are two levels of national programs issued by the MEXT in Japan: A general course of study for primary school (SHOGAKKO GAKUSHU SHIDO YORYO) and a primary school teaching guide for the Japanese course of study in mathematics (SHOGAKKKO GAKUSHU SHIDO YORYO KAISETSU SANSU-HEN). The former specifies the basic act of education, general edu- cational aims, and an outline of contents for teaching in each discipline. By contrast, the latter is published for each discipline, such as mathematics, and contains more detail than the course of study. In our study, we primarily used the latter document and referred to it as the program. In the program, there is a very useful table (MEXT, 2017, pp. 12–15), which summarizes the structure of the content of mathematics in grades 1 to 6. The content to be taught is shown by five domains: “A. Numbers and Calculations,” “B. Geometric Figures,” C. “Measurements” (grades 1 to 3) or “Variation” (grades 4 to 6), and D. “Data handling.” This table then specified more concise labels for each domain, for instance, “teaching Addition and Subtraction of simple fractions” occurs in grade 3, in the domain of A. We mainly used this table when we defined the sectors and themes, although it does not suffice to describe all themes precisely. For instance, “teaching simple fractions such as $\frac{1}{2}$ and $\frac{1}{3}$” is mentioned in the grade 2 in the domain A, but it leaves somewhat open what and how to teach. We therefore also refer to the later pages in the program and to concrete tasks in the textbook, where a more detailed description is given.

The textbooks are authorized by MEXT and are published by commercial textbook companies in Japan. Hence, the two levels of national programs and textbooks are interconnected and tightly aligned. The schools select the textbook company and distribute it to pupils. Here, we choose text- books published by TOKYO SHYOSEKI publishing company (one of the textbook series authorized by MEXT) as it is used in JSS_{DK}. Besides that, textbooks are widely used in ordinary schools in Japan. Here, we used the textbooks to define subject levels in terms of types of tasks and techniques since explicit tasks and techniques are stated in the textbook, more than in the program.

The process of elaborating the regional PRM is then as follows:

1. Browse through the table for the domain of A: Numbers and calculations, dividing it into two sectors, fractions (Sector1) and operations with fractions (Sector2).
2. Pick up mathematical contents regarding fractions in the table and categorize these (within each of the two sectors) as a theme.
3. Identify which grade these themes are taught in.
4. After identifying the sectors and themes, browse through relevant textbook chapters, and analyze all examples and exercises to define types of tasks. We do not describe all techniques relative to types of tasks here, but we describe the techniques related to the lesson presented later.
5. Whenever a task is encountered, which does not belong to a type of task already identified, a new type is added to the model.
6. Ensure whether these types of tasks are positioned in the right themes, and confirm all titles of themes, particularly if the titles of themes are consistent. If necessary, modify the title or reposition the types of tasks.

Some of the tasks in the textbooks are not independent. In this analysis, subordinate tasks were con- sidered techniques to solve main tasks. We do not explain for instance, why the sector is divided into two parts, and how we modify the title of themes. However, we will answer these questions in the next section as it is easier to explain in the context of our regional PRM.

4.2 Methodology for RQ2

In order to investigate RQ2, we collected data (specified later) from actual lessons observed in the JSS_{DK}. We first present an outline of the lesson, and then explain how to analyze it in terms of didactical moments, corresponding to concrete themes from the regional PRM.

The two lessons investigated here were taught on 30 October 2021 in a 5th grade with three pupils, and lasted in total for 90 min, with a break in the middle. The teacher we observed is Japanese and has experience teaching in this school for 7 years. She has licenses to teach Japanese languages in Japanese lower and upper secondary schools and a certificate to teach in special schools for children with special needs. The two of the three pupils go to the Swedish primary school on weekdays, while one attends Danish primary school. The lessons were given in Japanese, as always. During the lessons, the teacher used a teaching guide provided by the publisher of the textbook they used. The teacher—as is also common—had prepared slides in advance to show on the smart board, mainly consisting of pages from the textbook, with some space where she could add handwritten notes during the lesson. In addition, she put the title of the subject and definitions of the mathematical terms, summarized in her own words.

From the beginning, the teacher let pupils open the textbook, and the pupils kept the textbook open during the lessons. Pupils also had personal notebooks, but they wrote answers to tasks directly in the textbook. Only once did they take notes in the notebook, at the demand of the teacher, regarding the definition of Reduction to a common denominator and Reduce fractions. The lessons were based on a part of chapter 10 in the 5B textbook: Let's Extend Addition and Subtraction of Fractions (Fujii & Majima, 2021, pp. 2–19), and the teacher conducted the lesson based on slides showing pages 2 up to 12. We took field notes during the observation, as well as pictures of smart board and pupils' productions. The lesson was voice-recorded and transcribed in Japanese. After that we translated the lesson transcript into English. The analysis of these data is presented in a table showing how the lesson can be subdivided into episodes, each corresponding to didactic moments. Didactic moments are defined relative to praxeologies that are developed in the episode; so, the extent of the episode (in time) depends on when new types of tasks, techniques, and so on are introduced. We described types of tasks and techniques based on our reference model established to answer RQ1, while additional techniques developed by pupils were also noted.

This way, the flow of the lesson was analyzed in terms of how pupils established praxeologies are drawn on, and how new ones develop, through the identified moments. The primary data for the anal- ysis is the transcript, but as always for analyzing mathematics lessons, captures of written represen- tations such as photos are indispensable to interpret what is being talked about. Field notes help to relate the transcript and the pictures.

5 RESULTS

In this section, we present the results for each research question separately. Note that the presentation of results related to RQ2 relies heavily on the model developed to answer RQ1.

5.1 Praxeological reference model

We present our regional PRM of fractions in Table 1; reading the content clearly requires some explanation. First, we note that we have found it useful to model the parts of the arithmetic domain that focuses on fractions, as two sectors: *Fractions as objects* (sector_1) and *operations with fractions* (sector_2). The main reason why we categorized it as two sectors is that one of the five domains is called "Numbers and Calculations" in the Japanese program, so that it corresponds to the official way in which the knowledge to be taught is categorized. In sector_1, five themes are shown:

- *Semantics of fractions, including visualisations* ($\text{theme}_{1\text{-}1}$), means parts of the teaching focused on explaining the meaning of fractions, viewed as individual quantities. For instance, pupils learn how simple fractions are used to represent the size of parts that arise when an object is cut into pieces, such as $\frac{1}{2}$ in the case of two pieces. Fractions representing quantity, such as $\frac{2}{3}m$ and $\frac{4}{5}l$ are included in this theme. Pupils need to be able to read out simple fractions, like "one half". In addition, pupils should acquire terminologies related to fractions (e.g., the numbers 2 and 1 in $\frac{1}{2}$ are called the denominator and numerator, respectively).
- *Fractions in relation to multiplication and division of natural numbers* ($\text{theme}_{1\text{-}2}$) are about how fractions can be interpreted as operators involving arithmetic of natural numbers: Division, multiplication, or both. Here, we refer to examples mentioned in the Japanese program (MEXT, 2017, p.107). Figure 1 shows 12 marbles (on the left in the figure). To the right we see two groups of 12 marbles. This figure represents that 12 marbles are twice as many as 6 marbles, but also that $\frac{1}{2}$ times 12 marbles gives six marbles.

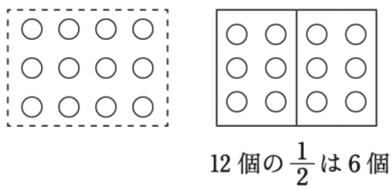

Figure 1: A diagram from the program (MEXT, 2017, p.107)

- *Ordering fractions* ($\text{theme}_{1\text{-}3}$), mean pupils learn how to compare the magnitude of two or more fractions. For instance, this theme relates to a task such as "Arrange $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{1}{4}$ in ascending order" (Fujii & Majima, 2021, p. 9).

- *Relating integers, fractions, and decimals* (theme$_{1\text{-}4}$) concerns the passages between these three kinds of number representations. For instance, pupils will be able to represent the result of a division as a fraction (e.g., $7 \div 4 = \frac{4}{7}$) and rewrite a fraction to a division (e.g., $\frac{2}{9} = 9 \div 2$). In addition, pupils learn that integers n can be represented as $\frac{n}{1}$, for instance, $4 = 4 \div 1 = \frac{4}{1}$ (Fujii & Majima, 2021, p. 117).
- *Equivalence of fractions* (theme$_{1\text{-}5}$) includes activities such as producing several fractions that are equivalent to a given fraction and simplifying fractions, for instance, "Reduce the following fractions: $\frac{8}{12}$, $2\frac{18}{24}$ and $\frac{90}{15}$ " (Fujii & Majima, 2021, p. 12).

In sector$_2$, we distinguished four themes: *Addition and subtraction of fractions*, *multiplication of fractions*, *division of fractions and mixed calculations (with >1 operation)*. We defined addition and subtraction of fractions as the same theme because both operations rely on the same techniques. We do not think detailed explanations of the first three themes are needed here. Mixed calculations means that pupils learn to handle expressions in which multiple operations are included such as $\frac{3}{5} \div \frac{3}{4} \times \frac{5}{4}$ (Fujii & Majima, 2021, p.68) and $(\frac{1}{6} + \frac{1}{3}) \times \frac{4}{5}$ (Fujii & Majima, 2021, p.75). In addition, pupils learn that the commutative law, the associative law, and the distributive law can be applied to fractions, such as to rewrite $\frac{3}{4} \times 5 + \frac{3}{4} \times 7$ as one product (Fujii & Majima, 2021, p.49).

All themes were named and categorized in a way which is strongly influenced by the table in the program (MEXT, 2017, pp. 12-15). We directly used the name shown in the table as titles of most of the themes in our model (e.g., "equivalence of fractions"). Note that we simply chose to include the mathematical contents that fall under the label "fractions" in the program (MEXT,2017, pp. 12-15); therefore, mathematical contents indirectly related to fractions (e.g., ratio or probability) was not included in our model. Our regional PRM indicates what mathematical praxeologies regarding fractions are set as knowledge to be taught from grades 2 to 6 in Japanese primary schools. Notice that fractions are not taught before grade 2. As with any model, our regional PRM leaves out many details, like how the individual themes are related to each other. Our model does show in what grade(s) the theme—or some part of it—is actually taught, according to the program and textbooks; but it does not show what is taught first or next within a given grade, and this may in fact vary accord- ing to the choices of teachers and textbooks.

Table 1: The regional PRM of fractions

Sector$_1$: Fractions as objects	Grades				
	G2	G3	G4	G5	G6
Theme$_{1\text{-}1}$: Semantics of fractions, including visualisations					
Theme$_{1\text{-}2}$: Fractions in relation to multiplication and division of natural numbers					
Theme$_{1\text{-}3}$: Ordering fractions					

Theme$_{1\text{-}4}$: Relating integers, fractions, and decimals					
Theme$_{1\text{-}5}$: Equivalence of fractions					
Sector$_2$: Operation with fractions	G2	G3	G4	G5	G6
Theme$_{2\text{-}1}$: Addition and Subtraction of fractions					
Theme$_{2\text{-}2}$: Multiplication of fractions					
Theme$_{2\text{-}3}$: Division of fractions					
Theme$_{2\text{-}4}$: Mixed calculations (with >1 operation)					

The subject level, corresponding to types of tasks (T) and techniques (t), is not shown in Table 1, but of course, each theme includes several subjects. But if needed, we can "zoom in" on any theme, as well will now exemplify. Concretely, in Table 2, we present the local PRM of the themes, theme$_{1\text{-}3}$, theme$_{1\text{-}4}$, theme$_{1\text{-}5}$ and theme$_{2\text{-}1}$ that appear in the lesson we observed, as we used this PRM later for investigating RQ2 with the same lesson as case.

Seven types of tasks with corresponding techniques are shown in our model. All types of tasks and techniques are identified based on our analysis of the pages of the textbook corresponding to the lesson (Fujii & Majima, 2021, pp. 2–19); in other words, these types of tasks and techniques are expected to be worked on during the lesson. The Japanese textbook breaks several techniques into small steps. For instance, in order to acquire the technique (τ_{5-1}): Find common multiply of the denominator and rewrite the fractions with the same denominators, and then use τ_4 to add them, the technique (τ_5): Rewrite two fractions to have common denominators using number lines and then use τ_4 to add them introduced first, then the technique (τ_6): Multiplying the numerator and the denominator by the same integer is presented.

Table 2: The local PRM regarding theme related to the lesson which we observed

Sector$_1$: Fractions as objects
Theme.$_{1\text{-}3}$: Ordering fractions
T_7: Order fractions with different denominators. τ_7: Find the (least) common multiple of the denominators and rewrite the fractions with the same denominators, then order the fractions.
Theme$_{1\text{-}4}$: Relating integers, fractions, and decimals
T_1: Rewrite fractions as divisions. τ_1: Replace an expression $\frac{a}{b}$ by $a \div b$.

T_2: Rewrite fractions as decimals (or whole numbers). τ_2: Division algorithm. T_3: Rewrite a (finite) decimal as a fraction. τ_3: Use the appropriate power of 10 as denominator and the digits as the numerator.
Theme$_{1\text{-}5}$: Equivalence of fractions
T_6: Given a fraction, find other fractions that are equivalent to the given fraction. τ_6: Multiplying the numerator and the denominator by the same integer. τ_{6-1}: Dividing the numerator and the denominator by the same integer. τ_{6-2} : Dividing the numerator and the denominator by the greatest common divisor.
Sector$_2$: Operations with fractions
Theme$_{2\text{-}1}$: Addition and subtraction of fractions
T_4: Addition and subtraction of fractions with the same denominators ($\frac{a}{b} \pm \frac{c}{b}$). τ_4: $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$ T_5: Addition and subtraction of fractions with different denominators. τ_5: Rewrite two fractions to have common denominators using number lines, then use τ_4 to add them. τ_{5-1}: Find the common multiple of the denominators and rewrite the fractions with the same denominators, then use τ_4 to add them. τ_{5-2}: Find the least common multiple of the denominators and rewrite the fractions with the same denominators, then use τ_4 to add them. τ_{5-3}: Use τ_{5-1} and then use τ_{6-1}. τ_{5-4}: Use τ_{5-2} and then use τ_{6-1}. τ_{5-5}: Use τ_{5-1} and then use τ_{6-2}. τ_{5-6}: Use τ_{5-2} and then use τ_{6-2}.

Thus, in short, Tables 1 and 2 constitute our answers RQ1: What mathematical praxeologies regard- ing the arithmetic of fractions are set as knowledge to be taught in grades 1–6 in the primary school in Japan? We notice how the levels of codetermination (subject, theme, and sector) and praxeologies allowed us to provide these answers at different scales, which can serve different purposes. While the regional PRM can be used to discuss links across grades, the local PRM could be used to situate and study the details of content taught in a lesson or developed in a textbook chapter.

5.2 Praxeological analysis of the lesson

In this section, we analysed a grade 5 lesson in JSS_{DK} based on the models developed to answer RQ1, and the theoretical framework of six moments. In particular, we investigated how these mathematical praxeologies develop during *didactic processes* as these – not the matter to be taught – may differ from JSS_{DK}. We first describe the first few episodes briefly to show the flow of the lesson, as it appears in the transcript (all of the lessons are whole class teaching). Then, we present the table with a summary of the didactic processes in the whole lesson. When we describe pupils' utterances, we assign them names as P_1, P_2 and P_3.

5.2.1 Description of the first 3 episodes

1. Episode 1: Reviewing what pupils have learned about fractions

The teacher presented the following tasks: $\frac{3}{4} = \square \div \square$, $0.3 = \frac{\square}{\square}$, $\frac{3}{5} + \frac{4}{5} = \frac{\square}{5}$ and $\frac{5}{6} - \frac{2}{6} = \frac{\square}{6}$ (Fujii & Majima, 2021, p. 3). She asked the pupils to solve these tasks, which they can easily do as the corresponding techniques have been taught to the pupils in grades 3, 4 and 5.

2. Episode 2-2.2: Addition of fractions with different denominators

The teacher presented the following task: *There are* $\frac{1}{2}\ell$ *of milk and* $\frac{1}{3}\ell$ *of milk. How much milk is there altogether* (Fujii & Majima, 2021, p.2), and asked the pupils to transform this into a mathematical expression (expected answer: $\frac{1}{2} + \frac{1}{3}$). However, P_1 immediately formulated a technique to calculate the sum of fractions, in his own words "rewrite 3 and 2 to a common denominator". The teacher agreed with P_1's suggestion but again asked the pupils to provide the mathematical expression. P_3 suggested the mathematical expression $\frac{1}{2} + \frac{1}{3}$, and P_1 suggested $\frac{1}{2}$ can be rewritten as $\frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6}$. Except for what is visible on the screen, this technique was not mentioned in the textbook yet; however, the teacher followed P_1's suggestion and continued asking P_3 how to rewrite $\frac{1}{3}$. They finally confirmed $\frac{1}{2} + \frac{1}{3}$ can be rewritten $\frac{3}{6} + \frac{2}{6}$, so that the answer is $\frac{5}{6}\ell$. After that, the teacher introduced the following task from the textbook: *Use number lines to find some fractions that are of the same magnitude as* $\frac{1}{2}$ *and other fractions that are the same magnitude as* $\frac{1}{3}$*. Then, find fractions that have the same denominators* (Fujii & Majima, 2021, pp. 3-4). Two pupils did so quickly, while the third needed some help from the teacher. Next, a task: $\frac{1}{2} + \frac{1}{3} = \frac{\square}{6} + \frac{\square}{6} = \frac{\square}{6}$ (Fujii & Majima, 2021, p.4) was presented. The teacher and the pupils found the answers for those two tasks together, and the teacher summarised that one could calculate the sum of two fractions with different denominators by converting the two fractions so that they have the same denominator. But we note that the questions about equivalent fractions were not so clearly motivated by the addition task, as one pupil had already provided the technique for this task.

3. Episode 3-3.7: Equivalent fractions

The teacher presented the following task: *Find other fractions, besides* $\frac{6}{8}$ *and* $\frac{9}{12}$ *,which are of the same magnitude as* $\frac{3}{4}$ (Fujii & Majima, 2021, p.5). P_1 and P_2 immediately proposed $\frac{12}{16}$ and

$\frac{15}{20}$, respectively. However, P_3 was confused about how to find these other fractions, so P_3 asked "Are $\frac{12}{16}$ and $\frac{15}{20}$ of the same magnitude as $\frac{3}{4}$?". P_2 said that " $\frac{15}{20}$ is the same magnitude as $\frac{3}{4}$ because both the numerator and the denominator are multiplied by 5". The teacher agreed with P_2's opinion. P_3 asked the teacher, "should I add 4 to 20 and 3 to 15?" The teacher answered that "it is not addition," but P_2 claimed that P_3 was right. The teacher and pupils confirmed that if we add 4 to 20 in the denominator and 3 to 15 in the numerator, the answer will be $\frac{18}{24}$ and that $\frac{18}{24}$ is the of same magnitude as $\frac{3}{4}$. Everyone then agreed that the technique developed by P_3 can work for this task.

After those interactions, the teacher pointed to a diagram in the textbook (see Fig. 2) and confirmed the answer that $\frac{3}{4}$ must be expanded by 2, 3 and 4 to yield, respectively, $\frac{6}{8}$, $\frac{9}{12}$ and $\frac{12}{16}$.

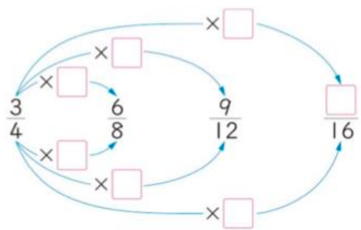

Figure 2: A diagram from the textbook (Fujii & Majima, 2021, p.5)

Based on this technique, the teacher confirmed that $\frac{15}{20}$, $\frac{24}{18}$ and $\frac{75}{100}$, found by the pupils, were all of the same magnitude as $\frac{3}{4}$. Finally, the teacher summarised: Fractions with the same magnitude can be found by multiplying the denominator and numerator by the same number.

After that, the teacher pointed to a diagram in the textbook presents a task: *Determine whether* $\frac{12}{16}$ *and* $\frac{3}{4}$ *are of the same magnitude by rewriting them as decimal numbers* (Fujii & Majima, 2021, p.6) and everyone confirmed that $\frac{12}{16}$ and $\frac{3}{4}$ can be rewritten as 0.75, so that $\frac{12}{16}$ and $\frac{3}{4}$ are of same magnitude. The teacher pointed to a diagram in the textbook (see Fig. 3) and asked how to convert $\frac{6}{8}$, $\frac{9}{12}$, and $\frac{12}{16}$ into $\frac{3}{4}$.

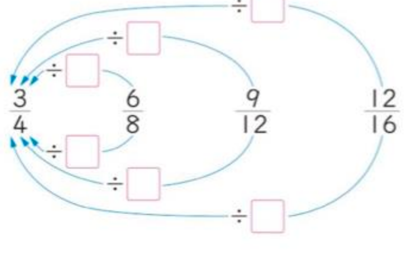

Figure 3: A diagram from the textbook (Fujii & Majima, 2021, p.5)

P_2 said "previously, we multiplied, so now we divide by the same numbers". The teacher confirmed that when one divided the numerator and denominator of $\frac{12}{16}$ by 4, one gets $\frac{3}{4}$, and dividing similarly by 3 in $\frac{9}{12}$ leads to $\frac{3}{4}$, and dividing by 2 in $\frac{6}{8}$ produces $\frac{3}{4}$. Finally, the teacher summarised that if we multiply or divide the denominator and the numerator by the same number, the magnitude of a fraction does not change. However, P_3 could not grasp what the teacher said, so the teacher explained it again, referring to diagram 1 and 2. Then, the teacher moved on to the next task.

5.2.2 Analysis of the above episodes and the whole lesson

Table 3 was developed based on analysing the teacher and pupil's utterances and writing in terms of how correspond to different moments of the didactic process, in relation the praxeologies that are being developed (also indicated, using the notation from our reference model, shown in Table 2).

As we described previously, episode 1 began by reviewing some of what pupils had learned about fractions earlier, by posing tasks of the following types: *Rewrite fractions as divisions* (T_1), *Rewrite fractions to decimals (or whole numbers)* (T_2), *Rewrite a (finite) decimal as a fraction* (T_3) and *Addition and subtraction of fractions with the same denominators* (T_4). This moment is an *Institutionalisation moment* (with well-known praxeologies being what is institutionalised).

Episode 2-2.2 was divided into 3 moments: *(supposed) first encounter*, *moment of technical work* and *institutionalization moment*. First, the teacher presented the "new" task (t_1) which is of type T_5: *Addition and subtraction of fractions with different denominator*. Here, t_1 is intended to be a new task in the textbook; however, one pupil immediately presents the technique: *Find the common multiple of the denominators and rewrite the fractions with the same denominators, then use τ_4 to add them* (τ_{5-1}) without referring to the context of t_1. Therefore, we call it a *"supposed" first encounter* – it turns out that at least one pupil has met the type of task before, presumably in "ordinary" school. After that, the teacher and pupils worked examine again the technique τ_{5-1} and the answer for t_1, and through filling out blanks in the textbook. The pupils met τ_5: *Rewrite two fractions to have common denominators using number lines, then use τ_4 to add them.* At this moment, the actual work was more like confirming or routinising techniques, rather than exploring new task, so we considered this a *moment of technical work*. In the end, the teacher developed a general technology, using mathematical terminologies such as "we can calculate fractions with different denominators when we rewrite the different denominators to the like denominator". This is again an *institutionalization moment*.

Episode 3-3.7 was divided into 8 moments. The class encountered the task (t_2): *Find other fractions besides $\frac{6}{8}$ and $\frac{9}{12}$ that are the same magnitude as $\frac{3}{4}$* (Fujii & Majima, 2021, p.5) which is of type T_6: *Given a fraction, find other fractions that are equivalent to the given fraction*. The pupils explored the two techniques: *Multiplying the numerator and the denominator by the same integer* (τ_6) and *adding the numerator by the same integer and the denominator by the same integer* (τ_{6-3}). As we saw in the description of the episode, the latter technique (τ_{6-3})

was suggested by one pupil. In the lesson, despite the teachers' initial rejection, they together confirmed τ_{6-3} can work for t_2, but they did not discuss whether these techniques are correct in general, and the teacher led the pupils to use τ_6 instead. We thus successively see a *moment of first encounter*, an *exploratory moment,* and a *moment of technical work*. The class developed the answers for t_2 based on τ_6, and the teacher summarised that fractions with the same magnitude can be found by multiplying the same number in the denominator and numerator. After that, they continued confirming the left of the answers based on τ_6. We thus have, successively, an *institutionalization moment* and a *moment of technical work*. Next, they worked on another task: *Determine whether* $\frac{12}{16}$ *and* $\frac{3}{4}$ *are the same magnitude by rewriting them as decimal numbers* (Fujii & Majima, 2021, p.6), which are of type T_2: *Rewrite fractions as decimals (or whole numbers)*, and then they moved to another task: *How to change* $\frac{6}{8}, \frac{9}{12},$ *and* $\frac{12}{16}$ *into* $\frac{3}{4}$ (Fujii & Majima, 2021, p.6). The teacher and pupils developed the following technique: *Dividing the numerator and the denominator by the same integer* (τ_{6-1}). Finally, the teacher summarised that if we multiply or divide the denominator and the numerator by the same number, the magnitude of a fraction does not change. Hence, we have an *institutionalisation moment* (with well-known praxeologies being what is institutionalized), *a moment of technical work* and finally an *institutionalisation moment*.

In this way, we analysed the whole lesson and produced Table 3 to answer RQ2. It is divided into 30 episodes that can each be characterised as a moment of the didactic process, numbered in Table 3 with two levels, to reflect how some moments refer to the same praxeologies (for instance, episodes 2-2.2 all relate to T_5). The techniques were also described in the order in which they were developed, for instance, in the episode 2.1, τ_{5-1} occurred before τ_5.

Table 3: Didactic processes in the lesson

Types of tasks (T)	Episode	Didactic moments
T_1, T_2, T_3, T_4	1 (2m8s)	Institutionalization moment (with well-known praxeologies being what is institutionalised)
T_5	2 (13s)	(Supposed) first encounter
	2.1 (3m43s)	Moment of technical work (τ_{5-1}, τ_5)
	2.2(18s)	Institutionalisation moment
T_6	3 (14s)	First encounter
	3.1 (1m39s)	Exploratory moment (τ_6 , τ_{6-3}*)
	3.2 (1m46s)	Moment of technical work (τ_6)
	3.3 (11s)	Institutionalization moment
	3.4 (42s)	Moment of technical work (τ_6)

	T_2	3.5 (1m11s)	Institutionalization moment (with well-known praxeologies being what is institutionalised)
		3.6 (2m)	Moment of technical work (τ_{6-1})
		3.7 (1m8s)	Institutionalization moment (τ_6 , τ_{6-1})
T_5		4 (10s)	Institutionalization moment (τ_6)
		4.1 (52s)	Moment of technical work (τ_{5-2})
		4.2 (12s)	Institutionalization moment (τ_{5-2})
		4.3 (54s)	Moment of technical work (τ_{5-1})
		4.4 (1m13s)	Technological-theoretical moment
		4.5 (19s)	Institutionalization moment
		4.6 (1m36s)	Moments of technical work (τ_{5-2})
T_7		5 (28s)	Institutionalization moment (τ_6)
		5.1 (1m41s)	Moment of technical work (τ_7)
T_5		6 (1m20s)	Institutionalization moment (τ_{6-1})
		6.1 (3m53s)	Technological-theoretical moment
		6.2 (2m24s)	Institutionalization moment (τ_{6-1} , τ_{6-2})
T_6		7 (7s)	Institutionalization moment (τ_{6-1})
		7.1 (1m)	Moment of technical work (τ_{6-1} , τ_{6-2})
		7.2 (1m25s)	Technological-theoretical moment
T_5		8 (2m44s)	Moment of technical work (τ_{5-2} , τ_{5-5})
		8.1 (21s)	Technological-theoretical moment
		9	Evaluation moment

As we can see from Table 3, the lesson began with *an institutionalisation moment* (with well-known praxeologies being what is institutionalized), as the teacher started with a review of what pupils had previously learned about fractions, and ended with an *evaluation moment.* Between those moments, several moments occurred. *Technological-theoretical moments* developed several times in the latter half of the lessons. At this moment, utterances were identified where the teacher and pupils did not simply confirm the technique but where the teacher justified the technique. For instance, the teacher mentioned a technology: *tsubun* (finding a common denominator, in Japanese), to justify τ_{5-1}. It is obvious that not much time was allocated to each moment as several techniques were presented within 90 minutes. Also, we can see that *exploratory moments* were almost absent in the lesson. In terms of praxeologies, as we saw in the section 5.1, the textbook is written so that pupils may develop new techniques step by step. Concretely, for instance, τ_{5-1}: *Find the common multiple of the denominators and rewrite the fractions with the same denominators, then use* τ_4 expect to develop based on τ_6: *Multiplying the numerator and the denominator by the same integer.* But in the lesson, τ_{5-1} was developed before encountering τ_6. Therefore, the flow of building up techniques gradually, intended by the textbook authors, was to some extent broken in the lesson, mainly because some pupils knew techniques that were supposed to be new. This is a common phenomenon observed also in other lessons and can be explained by the pupils' experience from attending a regular (Danish, Swedish or international) school on weekdays. Moreover, the limited time as well as the teachers' choice to work directly with projected excerpts of textbook in class, seems to favour a more rapid pace, and to hinder a reason exploration in which various techniques are developed and compared.

6 DISCUSSION AND CONCLUSIONS

Based on the praxeological analysis of the lesson in the JSS_{DK}, we have seen how two specific themes related to fractions, and defined by MEXT, are taught in JSS_{DK}. We then described the characteristics of the didactic processes. We concluded that not much time is allocated to each moment, as several techniques are presented, and *exploratory moments* are almost absent. Here, we briefly discuss some hypothetical differences in JSS_{DK} and Japanese ordinary schools, based on a previous study by Stigler & Hiebert (1999).

Stigler & Hiebert (1999) outline a pattern of lessons in Japanese ordinary schools, consisting of a sequence of five activities: 1. *Reviewing the previous lesson* 2. *Presenting the problem of the day* 3. *Students working individually or in groups* 4. *Discussing solution methods* 5. *Highlighting and summarising the major points*. In the first activity, the teacher briefly confirms what students have retained from previous lessons. In the second phase, a goal and core problem for the day are presented by the teacher. In the third phase, students initially work on the problem individually, often for five to ten minutes, and then discuss their solutions (ideas) with neighbors or in small groups. In the fourth phase, the teacher lets students present one or more ideas. After that, they discuss the differences between each solution and summarise which method is more efficient. In the fifth phase, the teacher tries to summarise what students learned during in the current lesson. These five phases can be characterised as moments of the didactic processes of the lesson. The first and second phases constitute a moment of *institutionalisation (*with well-known praxeologies being what is institutionalized) and a *first encounter* with a new type of task. The third phase is an e*xploratory moment*. The fourth phase constitutes one or more *technological-theoretical moments* and *moments of technical work*. The fifth phase involves moments of *institutionalization* and sometimes also *evaluation.* In other words, the "script" of ordinary lessons in Japan often follows an order which is similar to that proposed by the theory of didactic moments.

In the observed lesson, the pupils were not given much time to think individually or in groups about how to solve the tasks proposed by the teacher. Based on Stigler & Hiebert (1999) (as well as my own experiences), this is quite different from ordinary schools in Japan. In addition, not enough time was allocated to each moment, which means pupils have to acquire several techniques at a relatively high pace in the JSS_{DK}. The conditions and constraints of supplementary schools may easily explain these differences, both the relatively low number of pupils in the classroom, and limited time for teaching. Besides these material reasons, one can also speculate that the relative rarity of exploratory moments is related to the practice of "direct teaching from the textbook" in the JSS$_{DK}$. Organising exploratory moments, where several (in part, perhaps inadequate) techniques appear and are compared, requires both resources in terms of time and teacher preparation which are not available in supplementary schools. At the same time, as we saw in the lesson we analysed, pupils may know intended techniques from their mathematics teaching in a regular school.

REFERENCES

Artigue, M., & Winsløw, C. (2010). International comparative studies on mathematics education: A viewpoint from the anthropological theory of didactics. *Recherches en didactique des mathématiques*, *30*(1), 47-82.

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice: The case of limits of functions in Spanish high schools. *Education Studies in Mathematics, 59,* 235-268. https://doi.org/10.1007/s10649-005-5889-z

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI bulletin*, *58*(58), 51-65.

Chevallard, Y. (1985). *La transposition didactique*, Grenoble, Lapensée sauvage.

Chevallard, Y. (1999). Didactique? Is it a plaisanterie? You must be joking! A critical comment on terminology. *Instructional Science*, *27*, 5-7.

Fujii, T., & Majima, H. (2021). *Atarashi sansu 5* [*New mathematics for elementary school for grade 5*]. Tokyo, Japan: Tokyo Syoseki Shuppan.

Ministry of Education, Culture, Sports, Science and Technology [MEXT]. (2017). *Syougakko gakusyu shido yoryo kaisetsu, sansu-hen* [Primary school teaching guide for the Japanese course of study: Mathematics]. Tokyo, Japan: Nihon Bunkyou Shuppan.

MEXT. (2021). *Kaigaide manabu nihonno kodomotachi* [Japanese children studying abroad]. https://www.mext.go.jp/a_menu/shotou/clarinet/002/001.htm

Okumura, S., & Obara, Y. (2017). Integrating language and content: Challenges in a Japanese supplementary school in Victoria. *Babel, 51*(2)*,* 20-25.

Shibano, J. (2020). Kaigai Kikokusei: Kyouiku Mondai no Hensen to Aratana Doukou [Returning children from overseas: The changes in education problems and new tends]. In M, Fukumoto (Ed.). Imin kara Kyoiku wo Kangaeru [Rethinking education from migrants] (2nd ed., pp.47-60). Kyoto, Japan: Nakanishi Press.

Stigler, J. W., & Hiebert, J. (2009). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York, NY: Free Press.

Stigler, J., & Perry, M. (1988). Cross-cultural studies of mathematics teaching and learning: Recent findings and new directions. In D. Grouws & T. Cooney (Eds.), Perspectives on research on effective mathematics teach- ing (Vol. 1). Reston, VA: National Council of Teachers of Mathematics.

Paper II

Learning to speak mathematically at the Japanese supplementary school in Sweden: critical cases of praxeological anomaly

Mayu Aoki[1], Yukiko Asami-Johansson[2], Carl Winsløw[3]

[1]University of Copenhagen, Department of Science Education, Copenhagen, Denmark; mayu13ak@gmail.com

[2]University of Gävle, Department of Electrical Engineering, Mathematics and Science, Gävle, Sweden; yukiko.asamijohansson@hig.se

[3]University of Copenhagen, Department of Science Education, Copenhagen, Denmark; winslow@ind.ku.dk

Abstract

Natural language is known to play a crucial and specific role for childrens' learning in school mathematics. Not only does it carry special vocabulary, but subtle differences between natural languages may lead to surprising challenges, for instance for learners who are not taught mathematics in their mother tongue. In this paper, the anthropological theory of didactic (ATD) is used as a main framework and we analyse some praxeological anomalies from the teaching of fractions at Japanese schools abroad, while they attend, at the same time a regular school in some other language (in this case, the local language, Swedish). Our findings indicate that these praxeological anomalies arise not only from linguistic disparities related to specialised vocabulary and syntax for elementary mathematics but also from institutional and curricular differences. This study gives new insights on these language challenges related to mathematics as taught at expatriate schools, particularly in the case of Japanese.

Keywords: Teaching fractions, Language, Japanese and Swedish curriculum, Japanese supplementary schools abroad, Anthropological Theory of Didactic, Praxeological anomalies

1 Introduction

A common prejudice is that mathematics is the least language-dependent subject in school, but at least five decades of research has demonstrated that language and mathematics interact in complex ways that are essential to mathematics education (see e.g. Austin & Howson, 1979; Pimm, 1987; Morgan et al., 2014;Planas & Pimm, 2023). Indeed, the questions related to the subtle ways in which learners' natural language backgrounds affect their learning in school mathematics is by now a classic and highly developed area of research. This is especially true when considering the case of learners whose out-of-school language (or home language) is not the same as the language of instruction at school; such learners are particularly prevalent in countries with several languages, such as in large regions of Africa, as well as in Western countries with large migrant communities (Barwell, et al., 2016; Planas, et al, 2018).

In this paper we look at a quite different situation, namely that of children attending mathematics teaching simultaneously in two different schools, with different curricula, and using two languages which are both, to some extent, actively used by these pupils in their private lives. Concretely this is the case for children attending so-called Japanese supplementary schools (JSS) in countries outside of Japan, and at the same time going to a local school in that country. JSS are typically operating on Saturdays, to enable full participation in the other school. These children are typically bi- or trilingual, their first language would typically be Japanese, but in other – increasingly common – cases, they have equal or superior mastery and daily practice in the language of the country in which they live (especially if one of their parents originates from there). This particular situation offers a different context to study how language and mathematical content interact, as the children are in fact offered parallel opportunities for learning mathematics in two languages and according to two relatively different curricula. In this paper we focus on the effects of this parallel situation in the setting of the teaching of fractions at a JSS in Sweden.

2 Bi- or multi-lingual classrooms and learning fractions

The significance of pupils' acquisition of academic language and its impact on mathematics learning has been emphasised (e.g., Cuevas, 1984; Clarkson, 1992; Prediger et al., 2013). More recently, attention has also been given to actual or potential opportunities which pupils could have from being able to work with mathematics in two languages (e.g., Clarkson, 2007; Planas, 2014; Prediger et al., 2019). Both hypotheses rely to some extent on the Whorfian linguistic relativity principle that language influences the way one thinks (Whorf, 1956, p. 214): the deficit perspective focuses on the shortcomings of the children in the official language of instruction, when this is the only acknowledged avenue being offered for mathematical reasoning), while the opportunity perspective emphasises the potential of combining and connecting different ways of thinking which arise from the two languages, at least if use of the pupils' first language is to some extent facilitated at school (Prediger & Wessel, 2011).

There has been active research on the micro perspective, investigating the intricacies of handling multiple languages in the teaching and learning of specific mathematical topics (e.g., Setati &Adler, 2000; Farrugia, 2022), and their interplay with mathematical conceptualisation (e.g., Prediger and Wessel, 2011; Prediger et al., 2019). Regarding the learning of the basic notions related to fractions, Petersson and Norén (2017) highlight a novel perspective on research in multilingual mathematics education in the Nordic context, and conducted fractions tests on immigrant students in Sweden. The findings revealed distinctive challenges among students using Swedish as their second language, contingent upon their experience with the instructional language, and effectively illuminated the intricate relationship between language proficiency and the mastery of fractions. Prediger et al., (2019) examined four nuances of conceptualisation for the part-whole concept of fractions for Turkish-German speaking students. In their study, the majority of students employed *bilingual complementarity mode* (e.g. Moschkovich, 2007), and students referred to language-related nuances of conceptualisations in both languages, and combined aspects of different nuances. These modes were found to provide opportunities for deepening conceptual understanding of fractions within the part-whole concept. The authors emphasised the significance of analysing how the interaction between different languages and conceptualisations shapes the learning processes for multilingual students. In this regard, we believe that our study contributes, to some extent, to this perspective, in terms of identifying both potentials and obstacles related to how fraction learning of students using two different curricula in two different languages is affected by attending these different institutions.

3 The context of our study: The Japanese supplementary school in Sweden

The context of our study is *the Japanese Supplementary Schools* (JSS), which were established by the Japanese Ministry of Education, Culture, Sports, Science and Technology (MEXT). These schools were originally founded for children expected to return to Japan; nevertheless, in recent years, there has been a growing population of children who attend these schools while residing permanently (or, at the very least, for an indefinite duration) in the host country. These schools provide mainly Japanese language instructions and, many of the them also offer additional subjects such as mathematics. The instructional approach across all subjects is aligned with the Japanese national curriculum and Japanese textbooks. Due to their part-time nature, reductions in the curriculum delivery is common. JSS are not ordinary educational institutions, as the pupils attend local schools on weekdays.

In our study, we choose a JSS in Sweden (JSS_{SE}) as mathematics is offered as an additional subject. There is no official statement on why JSS_{SE} have opted for mathematics. The first author, being a teacher at the JSS in Denmark (JSS_{DK}), had ample opportunities for interaction with parents and children. Parents, of course, enroll their children in JSS_{DK} not solely for the improvement of mathematical abilities but mainly for the enhancement of Japanese language proficiency. Nevertheless, both parents and children feel that learning mathematics at JSS_{DK} positively contributes to the local pupils' mathematical education.

4 Theoretical framework and Research Questions

Our study is grounded in the Anthropological Theory of the Didactic (ATD) proposed by Yves Chevallard since the early 1980s (Bosch & Gascòn, 2006). This theory is utilised to explain how mathematical knowledge and practice develop and is shared within an institution, through the notion of praxeology. Praxeologies consist of two parts: *praxis* and *logos*. The praxis is made up of a type of tasks (T) and techniques (τ), and the logos consists of technologies (θ) and theories (Θ). The set of several tasks (t) becomes a type of tasks (T), typically a family of tasks that can be solved by some specific technique (τ). The technique (τ) is explained by a technology (θ), that is unified and justified by the more abstract discourse of a theory (Θ). Concretely, for instance, converting two fractions with different denominators into forms with the same denominators constitute a type of tasks (T), and could be achieved by at least two techniques (τ_1 and τ_2). The first technique accomplishes this by using the product of the original denominators as a common denominator (τ_1). The second one involves determining the least common multiple of the original denominators (τ_2). The discourse on these techniques constitutes a technology (θ), and the theory (Θ) that justifies this technology is multiplication. Praxis and logos are tightly interactive and as we will elaborate later, the technology can be strongly reliant on specific mathematical terms that are common in school mathematics but absent from everyday language. ATD thus proposes a theoretical tool to connect concrete discourse (logos) to the mathematical praxis within a specific school institution. It is crucial to note that language is a central component in praxeologies, however, the scope of praxeologies extends beyond language. As the main assumption of ATD, praxeologies are highly depend on the institutions in which they occur, and furthermore, the praxeologies taught in the classroom are considered to be transposed from those intended to be taught. In a given institution, such as local primary schools, certain praxeologies (including mathematical ones) are encouraged and expected – we call these *normal praxeologies*, in the sense of conforming to the norms of the institution. We define *praxeological anomalies* as deviations of praxeologies from the normal praxeologies, in our case, when the mathematical praxeologies of pupils in a school institution differ significantly from the expectations of the institution. We emphasise that a praxeological anomaly is institution-specific; what may be considered an anomaly in one institution can easily be normal in another. The term anomaly is therefore not utilised in a negative sense in this study. Pupils will inevitably encounter relative praxeological anomalies when learning mathematics in two different institutions simultaneously, particularly when these operate in different languages and with different curricula.

The main institutions considered here are the Japanese supplementary school in Sweden (JSS_{SE}), and regular Swedish schools. While observing the mathematics lessons at JSS_{SE}, we may identify traces of praxeologies that the pupils have acquired in their Swedish school. This affects not only mathematical logos (where the language difference is evidently of importance) but, as we shall see, also praxis. In any case, praxeological anomalies in mathematics may arise not only from insufficient learning at the institution we consider, but also from similar – but more or less different – praxeologies that were acquired in another institution. We can now formulate the research questions of this paper as follows.

RQ1: What specific praxeological anomalies are observed during lessons on fractions at JSS_{SE}? To what extent do they reflect differences between the natural languages (Japanese, Swedish) and the way they refer to fractions?

RQ2: To what extent are the praxeological anomalies observed in RQ1 related to differences between the Japanese curriculum on fractions (taught in JSS_{SE}) and the Swedish curriculum (met by pupils in their regular school) that are not directly related to the two natural languages?

It is a separate and more general problem to identify the "remnant anomalies" referred to in RQ2, a problem that could of course arise more generally when studying the conditions of pupils who attend, consecutively or in parallel, two different school systems with different languages of instruction.

5 Context, Data and Methodology

The episodes studied here took place during four lessons (180 minutes) in a 5th grade class with 19 pupils. In JSS_{SE}, there is only one class for 5th grade. The teacher we observed was Japanese and had experience to teach at JSS_{SE} for two and a half years. On weekdays, she worked as a teaching assistant at a local Swedish primary school. She had also taught mathematics in local Japanese schools for 9 years. The lessons were given in Japanese, but some pupils occesionally expresses themselves in Swedish. The primary focus was on addition and subtraction of fractions with different denominators, which were covered in four lessons.The teacher mainly used the whiteboard, but the projector was sometimes used for showing excerpts from the textbook, and pupils' written productions. The teacher interacted with the pupils to solve problems from the textbook. During the lesson, the pupils took notes and did not open the textbook. For the lesson design, the teacher usually follows the *Teacher's Guide* provided by the publisher of the textbook. The Teachers' Guide provides suggestions for the flow of the lessons and for main tasks that are considered to be "appropriate" from the perspective of the Japanese *noosphere* (Chevallard, 2019), by which we mean roughly curriculum developers in Japan.

We based this study on the praxeological analysis of the parts of the national Course of Study of Japan that deals with the teaching of fractions, which was presented by Aoki and Winsløw (2022). The model specifies in particular both larger and smaller collections of praxeologies that structure the teaching of fractions from grade 2 through 6. We analysed the technology related to the praxis of addition and subtraction of fractions with different denominators, linking that praxis also to other themes like equivalent fractions.

The first author positioned a video camera at the back of the classroom and recorded the entire lessons from the beginning to the end. The second author also observed the lesson with the first author. Three episodes were selected, and the corresponding video was analysed in three steps.

1. The first author extracted interactions between a teacher and pupils from recorded videos, focusing specifically on a) episodes where pupils made utterances, which do not correspond to the teacher's expectations, and b) instances where the teacher explicitly emphasises pupils' utterances.
2. All the episodes were transcribed in the original in Japanese and Swedish (the language in which they were originally uttered), and then translated into English by the first author. However, some parts were retained in Japanese and Swedish to facilitate the analysis of language phenomena.
3. All authors to discussed episodes where praxeological anomalies related to language and other institutional conditions occur. In this context, anomalies could occur at the level of logos, such as inadequate use of Japanese to describe specific aspects of the mathematical praxis, as well as within the praxis itself.

The first author is fluent in Japanese and has experience teaching in a Japanese supplementary school in Denmark, as well as observing lessons several times in a primary school in Japan. The second author is fluent in both Japanese and Swedish and is an experienced teacher educator and mathematics education researcher in Sweden. The third author knows both Swedish and Japanese to some extent and has considerable experience in international comparative studies of school mathematics, including the Japanese context. These various backgrounds helped to generate and validate multiple perspectives on the data. English functioned as a neutral working language, in which we formulated and discussed the delicate observations regarding the two languages which appear in the data. Naturally we are very aware that Swedish is much closer to English than Japanese, which has been a challenge also in writing this paper, as one could easily leave the reader to consider e.g. Swedish syntax as "natural" and the Japanese one as "deviating". We strived to single out objective differences that matter to the pupils' mathematical praxeologies.

6 Findings

We present three episodes that illustrate praxeological anomalies associated with language and other institutional conditions. As a result, we discovered six anomalies, which were extracted from the pupils' utterances. Additionally, they were derived from instances where the teacher intentionally or spontaneously emphasises certain words in her interaction with pupils. Before presenting the episodes, we explain some conventions used in the extracts of transcription from the class. The pupils in the episodes are all given pseudonyms. The descriptions of gestures and other actions observed are shown in square brackets. Most of the dialogue is in Japanese, the default language of the school, and it is presented in English translation. However, in some cases the pupils use one or more Swedish words, which we then state in **bold fonts** followed by the English translation in parenthesis. In other some cases, it is important for the analysis to state one or more Japanese words as originally uttered, which we then do in *italics* (followed by English translation in parenthesis). This is done when the Japanese term is important to the analysis, like a special mathematical term which is central to the episode.

Episode 1: Anomalies related to terminology and syntax related to fractions.

The first episode exposes some fundamental praxeological anomalies associated with natural language, and the representation of the multiplication symbol in writing.

When fractions are read out in Japanese, one reads the denominator first, and then the numerator: *yon bunno san* means literally "four (*yon*) parts (*bunno*) three (*san*)", and mathematically it means $\frac{3}{4}$. This is different from the order used in Swedish and English (as first the numerator, then the denominator is read out). Therefore, there is a clear syntax difference between Japanese and Swedish at this point. However, the meaning (semantics) of *yon bunno san* is the same as the Swedish **tre fjärdedelar,** which means literally "three four-parts". Both in Japanese and in Swedish, therefore, the reading of $\frac{3}{4}$ conveys the meaning "three four-parts", similar to the English "three fourths". This is analogue to "three cars", consisting of a cardinal number (three) and an object (cars). The syntax difference arises from the general possibility in Japanese, to put the cardinal number after the object: *kuruma sandai* means three (*sandai*) cars (*kuruma*). Not only are there differences in how fractions are read, but there are also variations in the process of writing fractions. In Japanese, fractions are written from bottom to top, while in contrast, they are written from top to bottom in Swedish. Hence, there are congruent differences in syntax and the process of writing between Japanese and Swedish. In the Japanese local primary school mathematics, the symbol "×" is commonly employed as the multiplication symbol, and the symbol "·" is only introduced in secondary school. By contrast, the symbol "·" is commonly used in Swedish primary schools.

The praxeological anomaly emerges in the following episode, wherein a pupil uses an incorrect order in Japanese. Another anomaly arises from the confusion surrounding Japanese terms such as denominator, numerator, fraction, times and multiply, which pupils are likely to encounter at the supplementary school, and not in their daily life practice of Japanese. Furthermore, an anomaly related to the multiplication symbol are evident in the pupil's handwritten characters on the whiteboard (Fig 1). This episode involves the teacher and two pupils, one (Alex) is at the whiteboard and Victor intervenes in the dialogue from his seat. The situation (Figure 1) is about how to solve the task (t_1): "Let us think how to calculate $\frac{1}{2}+\frac{1}{3}$", taken from the textbook by Fujii and Majima (2021a, p.3).

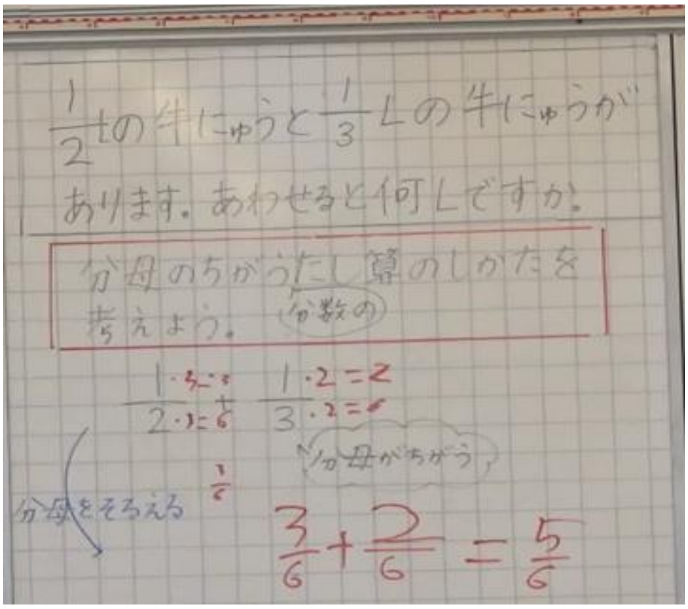

Fig. 1 Handwriting by the teacher and Alex on the whiteboard

Alex explained his solution to the teacher and the other pupils, in front of the whiteboard.

1 Alex:	[Pointing at 2 in $\frac{1}{2}$ written at the board] First of all, this is called *bunbo* (the denominator) right? We do the same denominators. If they are the same, it is easy, isn't it. And then, [pointing at 1 and 2 in $\frac{1}{2}$ at the board] the top and the bottom are… well... **gånger** (times)
2 Teacher:	If you multiply…
3 Alex:	If you multiply, if you multiply by the same, and it is the same number, right. So… [writes 1] *ichi* [writes fraction bar and 2] *bun* (parts) right, multiply [writes ·and×] something? Well… [delete ×] I… [pointing at 2 and 3 in $\frac{1}{2}+\frac{1}{3}$ at the board] if you multiply 2 by 3, well… both are 6. **Multiplicera** (multiply).
4 Teacher:	Multiply.
5 Alex:	Because it is multiplied, and 2 times 3 is…well… *bun…bunsu* (fractions) ...well…
6 Teacher:	*Bunbo*? *Bunshi* (the numerator)?
7 Alex:	*Bunbo*! *Bunbo*! Eh… 2 times 3 is 6, right? [Writes $\frac{1\cdot 3=3}{2\cdot 3=6}$] 3 and 6. So… [writes $\frac{3}{6}$ from top to bottom] *san* (three) *bunno* (parts)…no! *roku bunno san* (3 over 6) So, [pointing at $\frac{3}{6}$ and $\frac{1}{2}$ at the board] *roku bunno san* (3 over 6) is the same as *hanbun* (a half).
8 Alex:	Then, [pointing at $\frac{1}{3}$] this one also, [pointing at $\frac{3}{6}$ at the board] well…**nämnaren** (the denominator)
9 Teacher:	*Bunshi*.
10 Alex:	*Bunshi*. [pointing at 6 in $\frac{3}{6}$ at the board]
11 Teacher:	Ah, *bunbo*!
12 Alex:	*Bunbo*! If we do *bunbo*, we can do it easier. So, this one [points at $\frac{1}{3}$], 3 times something is 6. And it is 2, isn't it? So, [writes $\frac{1}{3\cdot 2=6}$] 3 times 2 is 6. And, because the top and the bottom is the same, [writes $\frac{1\cdot 2=2}{3}$] 1 times 2 is 2. Okay?
13 Victor:	Yes. It is correct.
14 Alex:	So, *ni* (two)… **nej** (no), *roku bunno ni* (2 over 6). And [looks at $\frac{3}{6}+\frac{2}{6}$ at the board] *san bunno roku tasu* (6 over 3 plus) well… *ni bunno roku* (6 over 2) ah! [correcting himself] *roku bunno ni* (2 over 6). So, this one [pointing at $\frac{3}{6}+\frac{2}{6}$] is the same as this [pointing at $\frac{1}{2}+\frac{1}{3}$]. So, [writing at "$=\frac{5}{6}$" from top to bottom, next to $\frac{3}{6}+\frac{2}{6}$ at the board].

In the default technology of the school, Alex attempts to explain how to calculate $\frac{1}{2}+\frac{1}{3}$ in Japanese; however, he switches some parts to Swedish. In the aforementioned episodes, three praxeological anomalies can be discerned. The first anomaly is observed when, in turn 3, 7 and 14, Alex utilises Swedish syntax while reading out fractions in Japanese. In turn 1 and 12, he also mentions "top and bottom", which can be interpreted as the denominator and numerator. Moreover, he writes fractions from top to bottom, which indicates that he follows the Swedish method in the process of writing fractions. It is unlikely that such anomalies would be made by a fifth-grade pupils in Japanese local schools. It is important to notice that these anomalies arise arbitrarily; they reflect a confusion of Japanese and Swedish syntax, which are parts of the technological norms in the two school institutions. This anomaly has been observed not only in this episode but also in several other cases, suggesting that it is not limited to how individuals perceive the parts and wholes of fractions. In addition, Bartolini Bussi et al. (2014) even suggests that the Asian system, which reads from the whole before the part, might enhance the conceptual understanding of fractions. Similar observations might be inferred from Alex's phenomenon, as described above. The second anomaly is that Alex struggles to use specific mathematical terminologies in Japanese, such as *bunbo* (denominators) and *bunshi* (numerators) in turn 1, 5 and 8, and sometimes he switches to Swedish like **gånger** (times), **multiplicera** (multiply) and **nämnaren** (denominators) in turn 1, 3 and 8. The third anomaly pertains to the representation of the multiplication symbol. In the particular instance, Alex used "·" on the whiteboard as the multiplication symbol (refer to Fig.1). As previously noted, in the Japanese context, "×" is conventionally utilized as the multiplication symbol. The teacher did not rectify this anomaly to align with the normal praxeology. In the aforementioned episode, praxeological anomalies occurred because Alex followed the Swedish school praxeology. If this were to occur in a Swedish local school, there would be no anomaly. Alex's proficiency in Japanese, particularly in using appropriate grammar and vocabulary to adequately explain his techniques, appears less developed when compared to local 5th grades in Japan. However, interpreting Alex's statements idiomatically in turns 1 and 12, he goes beyond merely explaining the techniques (algorithm) straightforwardly. Instead, he provides an explanation and justification for aligning the denominators to the same number, stating, "to make the calculation possible". He thus shows useful capacity to explain and justify one's techniques through technological discourse. Especially in JSS_{SE}, it is required that pupils articulate their ideas in Japanese, highlighting the importance of adhering to and learning Japanese praxeology rather than the Swedish one. Therefore, in episode 1, the primary objective is not simply to learn the technique but to acquire the technology in Japanese. But just as importantly, pupils gain insights into the differences between the normal mathematical technologies of the two institutions (Japanese and Swedish local schools).

Episode2: Fractions as numbers or as (related to) division.

This second episode concerns a praxeological anomaly caused by differences in the way fractions are taught in Japanese and Swedish schools.

In the Japanese school mathematics context, the normal (expected) way to read out the symbol $\frac{3}{4}$ is "*yon bunno san*", which is roughly similar to three over four in English. By contrast, in the Swedish context, there are two normal ways: one is "**tre över fyra**" (literally "three over four"), and another one is "**tre delat med fyra**" (literally, three divided by four). The second option corresponds, then, to read the fraction as a division, and this would be highly unlikely among Japanese pupils, and indeed it would be a praxeological anomaly. In Japanese local school mathematics, when pupils learn division, the special notation "÷" is used to designate the operation, like $3 \div 4$. Therefore, "three divided by four" corresponds, in the Japanese school praxeology, uniquely to $3 \div 4$, and not to $\frac{3}{4}$, which refers to a number (not a division). In fifth grade, pupils learn (as mathematical results) about identities like $\frac{3}{4} = 3 \div 4$; but the number (fraction) and the operation are still distinguished. In Sweden, the symbol ÷ is not used in school mathematics, except to explain how to divide using a calculator.

This episode occurred at the beginning of lesson on fractions, where the teacher displayed a page from the textbook (refer to Fig. 2) using a projector.

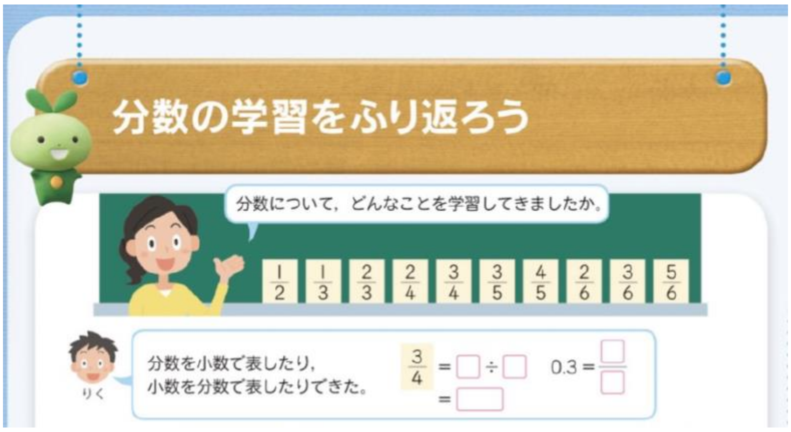

Fig. 2 Textbook projected using a projector.

1 Teacher:	Fractions, this one [points at "$\frac{3}{4}$"], do you remember how to read this out?
2 Pupils:	Yes, we remember.
3 Teacher:	Jonas.
4 Jonas:	Hm?
5 Teacher:	How to read out of this one [points at "$\frac{3}{4}$"]?

6 Jonas: *Bunsu…* (fractions).

7 Teacher: As a *bunsu*.

8 Jonas: *San waru yon* (3 divided by 4).

9 Teacher: *San waru…* yes. [Points at " □ ÷ □ " in the textbook, where it is an exercise for pupils to fill the boxes in the expression " $\frac{3}{4}$ =□ ÷ □"] … *San waru yon*, right. *San waru yon.*

10 Jonas: Or *yon bunno san* (3 over 4).

11 Teacher: Yes. [Points at 4 and then 3 in $\frac{3}{4}$] *Yon bunno san*. The way of reading out is *yon bunno san*. [Points at " □ ÷ □ " in the textbook, where it is an exercise for pupils to fill the boxes in the expression " $\frac{3}{4}$ =□ ÷ □"] and it could also be rewritten as *san waru yon*.

The actual task written on the textbook aimed at considering what pupils had learned so far, allowing them to express fractions as decimals and vice versa. They were specifically asked to write the answer for $\frac{3}{4} = 3 \div 4 = 0.75$. However, in turn 1 and 7, the teacher presented Jonas with an original task: "How do you read out $\frac{3}{4}$ as a fraction?". This task seems to be promoted by the teacher's comprehension of the differences in the syntax of reading fractions in Japanese and Swedish. In Japanese local schools, it is customary to pronounce $\frac{3}{4}$ as "*Yon bunno san* (3 over 4)", following a practice ingrained from the second grades that involves reading from the denominator first. Consequently, it is improbable for fifth-grade pupils in Japanese local schools to encounter a task reading out of fractions. So, this task is exclusive to $\mathrm{JSS}_{\mathrm{SE}}$, highlighting that only children adept in both praxeologies, specifically the Japanese and Swedish one, would face such a task. In turn 8, Jonas' initial answer to the teacher's question, specifically "*san waru yon* (3 divided by 4)" is evident. It is uncommon in Japanese local schools to answer a question about how to read a fraction by providing a division-based response. It is conceivable that Jonas might have been influenced by the task " $\frac{3}{4}$ =□ ÷ □ = □" in the textbook. We interpret this as an anomaly caused by the differences in the way fractions are taught in Japanese and Swedish local schools. In turn 9, the teacher neither directly corrects Jonas' answer nor affirms it as accurate. Subsequently, when Jonas offers the normal (expected) reading "or *yon bunno san* " in turn 10, the teacher explicitly confirms in turn 11 that "The way of reading out is *yon bunno san*" and goes on to acknowledge Jonas' initial erroneous answer: "and it could also be rewritten as *san waru yon*". It is common for teachers not to correct an erroneous answer right away, but to encourage pupils to find the right answer and then, while strongly acknowledging this answer, implicitly correct the first one. Probably the teacher (also employed in a Swedish local school) is aware of the different praxeological norms on this specific point, and takes the opportunity to help pupils make the distinction which is expected in Japanese.

Episode 3: The role of terms to reify processes into objects.

The third episode considers certain praxeological anomalies at the level of logos, where the differences in elements of technology and theory substantially influence the praxis (use of the techniques) in the context of adding and subtracting of fractions with different denominators.

When converting fractions to equivalent ones with common denominators (T), two techniques are employed as explained before. In the Japanese context, τ_1 is introduced initially, but later on τ_2 is designated as the preferred technique to be used when solving tasks of type T. Note here that in Japanese school mathematics, this procedure of converting fractions to obtain common denominators is given a specific term that pupils encounter solely within the school context: "*tsubun"*. The term "*tsubun*" is a noun which designates that procedure. In the Japanese national program, it is emphasised that "when one does *tsuubun* for two fractions, it is expressed succinctly by utilising *saishokobaisu* (the least common multiple) of the two denominators" (MEXT, p.245). The Japanese Teacher's Guide provides a special section entitled "*tsubun* and *saishokobaisu*", which states: "when ones does *tsubun*, it is efficient to utilise the *saishokobaisu* of the denominators… Subsequently, the final goal is that pupils should practice this technique in excercises while utilising *saishokobaisu* (Fujii & Majima, 2021b, p.18). In many Japanese textbooks, one finds important points formulated in boxed text providing advice or tips for pupils, as in Figure 3. The text says: "If you find *saishokobaisu*, you can do *tsuubun* using the smallest possible common denominator for the fractions" (Fujii & Majima, 2021a, p.8).

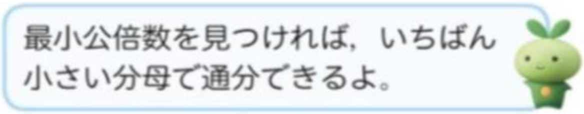

Fig. 3: Figure from the textbook

As a consequence of such emphases in textbooks and in the curriculum, τ_2 is hugely prevalent among Japanese teachers and pupils, leading them to favor τ_2 over τ_1. As previously mentioned, within the Japanese context, the process of "converting the fractions to equivalent ones with common denominators" is embedded in the discourse of at least two techniques (τ_1 and τ_2). The specific mathematical term "*tsubun*", attributed to this operation, facilitates instant recognition for both teachers and pupils. Consequently, "tsubun" assumes a central role in the technology. Specifically, "*tsubun*" is synonymous with "converting the fractions to equivalent ones with common denominators", and pupils recognise that it involves at least two techniques (τ_1 and τ_2). Leveraging this understanding, textbooks and teachers support pupils in forming the concept of *tsubun* = τ_2 in their minds instantly. In Swedish schools, there is no equivalent term. Consequently, this process is not reified as a mathematical object in the sense of Sfard (1991). Additionally, τ_2 is not commonly demanded as a first priority of the two techniques in the Swedish context; in fact, the term "the least common multiple" is not associated with the procedure of converting fractions to obtain common denominators.

The following episode is part of the introduction of the third lesson. The teacher began by reviewing what the pupils have learned the previous week. Initially, the teacher confirmed that $\frac{3}{5}$ is read as "*go* (5) *bunno* (parts) *san* (3)", and the lower and upper parts of the fraction ($\frac{3}{5}$) are referred to as the "*bunbo* (the denominator)" and "*bunshi* (the numerator)" respectively. The teacher highlighted the difference in the way fractions are read in Japanese compared to the Swedish way. Following this, she asked how to calculate $\frac{2}{3}+\frac{1}{6}$, a task of a type which pupils had learned in the previous week.

1 Teacher: How to do it [$\frac{2}{3}+\frac{1}{6}$], Sam?
2 Sam: To make them have common denominators, I find the *saishokobaisu* (least common multiple), and then I do the addition using it.
3 Teacher: Yes. Converting the denominators to determine the common denominators. Emil, please.

Emil came to the front of the whiteboard to write down his answer for the task $\frac{2}{3}+\frac{1}{6}$. The dialogue between the teacher and pupils continued as follows.

4 Teacher: Converting the denominators. What do we call it?
5 Pupils: *Tsubun* (converting the denominators to determine the common denominators).
6 Teacher: *Tsubun.* Yes. Do you all remember? *Tsubun.* One, two, three…
7 Pupils: *Tsubun.*
8 Teacher: *Tsubun.*

Emil first wrote $\frac{2}{3}+\frac{1}{6}=\frac{2\times 6}{3\times 6}$ ... (τ_1) at the whiteboard. However, some of the pupils intervened in Swedish from their seats, and Emil changed his writing to $\frac{2}{3}+\frac{1}{6}=\frac{2\times 2}{3\times 2}+\frac{1}{6}=\frac{4}{6}+\frac{1}{6}=\frac{5}{6}$ (τ_2).

9 Teacher: What do you think? Okay? Do you remember? Do *tsubun*. Converting the denominators. As Sam said before, this is *saishokobaisu*. This is *saishokobaisu* of 3 and 6.

However, Alice asked, "Why is 3 multiplied by 2?". Immediately, Emil answered, "This is because of *saishokobaisu*". The teacher, however, asked "Why did Emil change his answer from this (τ_1) to that (τ_2)". Theo responded, "The smaller denominator is easy". Drawing from the concept of equivalent fractions, the teacher and pupils concluded that the resulting fraction with the smaller denominator might be more convenient. Nevertheless, Alice continued to ask the same question to the teacher.

10 Alice:	Eh...But why 2? Why not like 3 times 4?
11 Teacher:	Why here [pointing at the 2 in the denominator of $\frac{2\times 2}{3\times 2}$], Why did we multiply by 2 here [pointing at the 2 in the denominator of $\frac{2\times 2}{3\times 2}$]?
12 Emil:	Because it is smallest number, and it is easy.
13 Teacher:	Why did we multiply by 2 here [2 in the denominator of $\frac{2\times 2}{3\times 2}$]. Because… why?
14 Alex:	Well… *bunbo* (the denominator)? will be the same... well…because it is the same if we multiple the top and the bottom… if you convert the denominators, it will be much easier.
15 Teacher:	Yes. Then…When converting the denominators… As Emil said, before… [pointing at the 3 and 6 of $\frac{2}{3}+\frac{1}{6}$] we want to convert them to *saishokobaisu* (the least common multiple), which is 6.

In the above episode, we can observe two praxeological anomalies. The first pertains to the level of logos in converting the fractions to achieve the common denominators. In turns from 4 to 8, the teacher asked what that operation is called, and the pupils confirmed that it is called "*tsubun*" in Japanese. Subsequently, she had the pupils pronounce "*tsubun*" aloud. It is obvious that the teacher emphasised the term "*tsubun*" and wanted the pupils to thoroughly grasp its meaning. This emphasis is confirmed by the fact that the term "*tsubun*" appears multiple times in the exercises of the textbook. For instance, we find tasks such as "compare the value of the following fractions by doing *tsubun*, and then write the appropriate inequality sign in each blank" (Fujii, T., & Majima., 2021a, p.9), and "do *tsubun* fractions in the blank" (ibid, p.9). Consequently, the lack of knowledge about the term "*tsubun*" and its meaning poses an obstacle in the Japanese context. The teacher in this episode likely understands experientially that this obstacle will occur in $\mathrm{JSS_{SE}}$ where the aim is to learn the Japanese praxeology. As a result, this episode highlights an anomaly pertaining to the level of logos, particularly the level of technology, which arises in the situation considered as a result of the pupils' mathematical experiences in two school institutions.

The second praxeological anomaly also pertains to level of logos (and the terminology on which technology is based) influences the praxis (use of the techniques), specifically in relation to the term *tsubun* and the technique (τ_2): determining the least common multiple of the denominators.

In the above episode, the teacher asked Sam how to calculate $\frac{2}{3}+\frac{1}{6}$. He immediately answered, "To make them have common denominators, I find the *saishokobaisu* (least common multiple), and then I do the addition using it". His explanation in Japanese is somewhat ungrammatical, but we can still understand from the phrases that he is explaining the process of converting the fractions to have common denominators based on τ_2, essentially highlighting the preference for τ_2 to solve tasks of type T. It is evident that he unconsciously and naturally follows Japanese praxeology − the *normal* praxeology. The teacher also confirmed that the denominator that should result after *tsubun* is 6, which corresponds to the least common multiple of 3 and 6, as opposed to Emil's initial production. Afterward, Alice raised the question, "Why is 3 multiplied by 2?" and Emil promptly responded, "That is because this is the *saishokobaisu".* Emil's statement can be paraphrased as follows: "The reason is that the least common multiple of the denominators 3 and 6 is 6, so I multiplied 3 by 2 to align the denominators at 6." Similar analyses can be derived from Emil's statement as discussed with Sam. However, in turn 10, Alice asked "Eh...But why 2? Why not like 3 times 4?". From this statement, it can be inferred that although Alice comprehends the need to convert fractions to obtain common denominators when calculating $\frac{2}{3}+\frac{1}{6}$, she does not grasp why it is crucial to convert them with the least common multiple of 6. Indeed, the question of why it is effective to align denominators using the least common multiple is likely to arise even from pupils attending local schools in Japan. However, as mentioned earlier, the specific mathematical related terms *saishokobaisu* and *tsubun* have no equivalent in the Swedish context. Therefore, this question may not be raised in Swedish schools. In other words, the praxeological difference between the two institutions illustrates that the differences between the logos (in particular, the terminology on which technology is based) affects the praxis (use of the techniques), and gives rise (in the Japanese context) to at least passing praxeological anomalies.

7 Discussion and conclusion

In the first episode, we identified three praxeological anomalies. The first anomaly pertains to the pronunciation of fractions. When Alex reads out fractions in Japanese, he utilised the syntactical structure of Swedish. Specifically, he articulated them in the order of the numerator to the denominator, whereas Japanese syntax typically follows the order from the denominator to the numerator. The second anomaly involves the amalgamation and interchange of specific mathematical terminologies from Japanese and Swedish. The third anomaly concerns the way of representing the multiplication symbol. Alex utilised the notation "·" commonly employed in Swedish schools, instead of "×", which is used in Japanese primary schools. As a result, the primary factor behind the observed anomalies is that Alex utilises the Swedish praxeology instead of employing the Japanese praxeology expected in JSS_{SE}. The fourth anomaly was elucidated through Episode 2. In this episode, the task was to determine how to read the fraction "$\frac{3}{4}$" in Japanese, and Jonas provided an unexpected response. Rather that the expected answer of *yon bunno san* (3 over 4), Jonas responded with *San waru yon* (3 divided by 4). This, too, stemmed from Jonas applying the Swedish praxeology. This anomaly arising from differences in the way fractions are taught in Japanese and Swedish schools. In the third episode, a specific mathematical term related to learning fractions in the Japanese context, *tsubun*, was introduced. The term *tsubun* appears in a technology comprising two techniques (τ_1 and τ_2). This term is absent in the Swedish context. Furthermore, the concept observed in textbooks and the episode associating *tsubun* to τ_2 in the Japanese context led to the conclusion that anomalies at the level of logos may substantially influence the praxis (use of the techniques) in the context of adding and subtracting of fractions with different denominators. If the first four anomalies out of the six were to occur in Swedish local schools, they would not be praxeological anomalies, but simply be part of the normal praxeology. Also, if the purpose is to learn and utilise two praxeologies in JSS_{SE}, and this is expected and intended, then they would not be anomalies. Therefore, the decisive reason they are referred to as anomalies here is that they depend on the constraints of the JSS, where learning and using Japanese praxeologies is expected and recommended. The fifth and sixth anomalies also revolve around the concept of institution, stemming from the cultural differences between institutions, ultimately arising from the cultural realms to which the institutions belong.

In this study, based on the main assumption of ATD that praxeologies depend on (and often differ between) institutions, we analysed the fraction lessons of 5th-grade pupils at JSS_{SE}, who are studied in their capacity of subjects of this institution, while they are concurrently learning the praxeologies of Swedish school. The study incorporated the concept of praxeological anomaly as a counterpoint to normal praxeology. It is crucial to emphasise that we do not claim any detrimental effects or resource constraints in the bi-multilingual settings, while emphasising the objective experience of anomaly experienced by pupils in each instiution. The use of praxeological analysis from ATD is seldomly employed in multilingual contexts, and enabled us to transcend the micro dimension of language, encompassing not only the linguistic aspect but also the macro dimension of curriculum, with a particular focus on institutional differences. The emergence of (relative) anomalies arises naturally from the specific circumstance of simultaneously learning fraction arithmetic in two school systems, while being encouraged to apply the praxeology of one specific context. The pupils occupy *positions* of pupils of JSS_{SE} to develop certain relations to praxeologies, which Chevallard (2019) calls the *public part* of these relations. That means, the pupils can realise what is the prioritized technique to calculate a sum of two fractions within the praxeology of JSS_{SE}, as they place themselves in the position of the pupils of JSS_{SE}. In other words, this experience of different expected praxeologies extends beyond learning to speak mathematically in different languages; it can be argued that they are, more broadly, experiencing explicit cultural differences. While our study concentrated on fraction teaching at JSS_{SE}, similar studies of the pupils' experience at Swedish school could be considered essential further to elucidate how the interplay between praxeologies from the two institutions impacts learning of mathematics for pupils. The theoretical approach exemplified by the analysis of our episodes represents a promising direction for such research. It is also an interesting hypothesis – to be further investigated – that pupils experiencing two praxeological norms by attending two schools in parallel, may at least to some extent become aware of the institutional relativity of such norms, even in mathematics, unlike pupils who do not have such an experience.

Acknowledgements We acknowledge all who enabled this study, including pupils, teachers and the school principal.

Declarations On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

Aoki, M., & Winsløw, C. (2022). How to map large parts of the mathematics curriculum? The case of primary school arithmetic in Japan. In Hodgen, J., Geraniou., E., Bolondi G., & Ferretti, F. (Eds.), (2022). *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)*. (pp.328-335). ERME / Free University of Bozen-Bolzano.

Austin, J. L., & Howson, A. G. (1979). Language and mathematical education. *Educational Studies in Mathematics*, *10*, 161-197. https://doi.org/10.1007/BF00230986.

Bartolini Bussi, M. G., Baccaglini-Frank, A., & Ramploud, A. (2014). Intercultural dialogue and the geography and history of thought. *For the Learning of Mathematics*, *34*, 31-33.

Barwell, R., Clarkson, P. C., Halai, A., Kazima, M., Moschkovich, J., Planas, N., Setati-Phakeng, M., Valero, P. & Villavicencio Ubillús, M. (2016). *Mathematics education and language diversity: The 21st ICMI Study*. Springer Nature. https://doi.org/10.1007/978-3-319-14511-2.

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin*, *58*(58), 51-65.

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: an attempt at a principled approach. *Hiroshima Journal of Mathematics Education, 12*, 71–114. https://doi.org/10.24529/hjme.1205.

Clarkson, P. C. (1992). Language and mathematics: A comparison of bilingual and monolingual students of mathematics. *Educational Studies in Mathematics*, *23*(4), 417-429. https://doi.org/10.1007/BF00302443.

Clarkson, P. C. (2007). Australian Vietnamese students learning mathematics. *Educational Studies in Mathematics, 64*(2), 191–215. https://doi.org/10.1007/s10649-006-4696-5.

Cuevas, G. J. (1984). Mathematics learning in English as a second language. *Journal for Research in Mathematics education*, *15*(2), 134-144. https://doi.org/10.2307/748889.

Farrugia, M. T. (2022). Engaging in the discourse of fractions in a bilingual Maltese classroom. *Teachers College Record*, *124*(5), 89-107. https://doi.org/10.1177/01614681221103977.

Fujii, T., & Majima, H. (2021a). *Atarashi sansu 5* [*New mathematics for elementary school for grade 5*]. Tokyo Syoseki Shuppan.

Fujii, T., & Majima, H. (2021b). *Atarashi sansu 5: Kyoushi you shidousyo* [*New mathematics for elementary school for grade 5: teachers' guide*]. Tokyo Syoseki Shuppan.

Kiselman, C., & Mouwitz, L. (2008). *Matematiktermer för skolan*. [*Glossary of school mathematics terms*]. NCM, University of Gothenburg.

Moschkovich, J. (2007). Using two languages when learning mathematics. *Educational Studies in Mathematics, 64*(2), 121–144. https://doi.org/10.1007/s10649-005-9005-1.

Ministry of Education, Culture, Sports, Science and Technology. (2022). *Zaigai kyoiku shisetsu no gaiyo* [*Overview of overseas educational institutions*]. Ministry of Education, Culture, Sports, Science and Technology. https://www.mext.go.jp/a_menu/shotou/clarinet/002/002.htm.

Morgan, C., Craig, T., Schütte, M., & Wagner, D. (2014). Language and communication in mathematics education: An overview of research in the field. *ZDM*, *46*, 843-853. https://doi.org/10.1007/s11858-014-0624-9.

Petersson, J, & Norén, E. (2017). To halve a fraction: An issue for second language learners. *Education Inquiry, 8*(3), 173-191, https://doi.org/ 10.1080/20004508.2016.1275187.

Pimm, D. (1987). *Speaking mathematically. Communication in mathematics classrooms*. Routledge.

Planas, N. One speaker, two languages: Learning opportunities in the mathematics classroom. *Educational Studies in Mathematics 87*, 51–66 (2014). https://doi.org/ 10.1007/s10649-014-9553-3.

Planas, N., Morgan, C., & Schütte, M. (2018). Mathematics education and language: Lessons and directions from two decades of research. *Developing Research in Mathematics Education. Twenty years of communication, cooperation and collaboration in Europe*, 196-210. https://doi.org/10.4324/9781315113562.

Planas, N., & Pimm, D. (2023). Mathematics education research on language and on communication including some distinctions: Where are we now?. *ZDM–Mathematics Education*. https://doi.org/10.1007/s11858-023-01497-0.

Prediger, S., & Wessel, L. (2011). Relating registers for fractions–multilingual students on their way to conceptual understanding. In Setati, M.; Nkambule, T. & Goosen, L. (Eds). *Proceedings of the ICMI Study 21 Mathematics and language diversity* (pp. 325-333). Sào Paulo, Brazil.

Prediger, S., Renk, N., Büchter, A., Gürsoy, E., & Benholz, C. (2013). Family background or language disadvantages? Factors for underachievement in high stakes tests. In *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 49-56). PME.

Prediger, S., Kuzu, T., Schüler-Meyer, A., & Wagner, J. (2019). One mind, two languages–separate conceptualisations? A case study of students' bilingual modes for dealing with language-related conceptualisations of fractions. *Research in Mathematics Education*, *21*(2), 188-207. https://doi.org/10.1080/14794802.2019.1602561.

Setati, M., Adler, J. Between languages and discourses: Language practices in primary multilingual mathematics classrooms in South Africa. *Educational Studies in Mathematics* 43, 243–269. https://doi.org/10.1023/A:1011996002062.

Sfard, A. (1991). On the dual nature mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics* 22, 1-36. https://doi.org/10.1007/BF00302715.

Whorf, B.L. (1956). Language, Thought and Reality. In J.B.Carroll (Ed.), *Selected Writings of Benjamin Lee Whorf* (134-159). MIT Press.

PaperⅢ

LEARNING FRACTIONS IN TWO SCHOOLS

Mayu Aoki

Department of Science Education, University of Copenhagen, Copenhagen, Denmark

mayu13ak@gmail.com

The complexity of learning fractions is widely acknowledged globally. Learning fractions in a single language poses significant challenges for pupils. Then what happens when pupils are required to learn fractions in two different languages and under two different curricula, in parallel ? This study focuses on bilingual students (Japanese and Danish or Swedish) who attend two separate school systems simultaneously. It investigates how differences in language and curricula influence students' knowledge on how to solve specific fraction-related tasks utilizing the framework of the Anthropological Theory of the Didactic. Through semi-structured interviews, we discovered that some students have parallel praxeologies, to some extent, particularly concerning: (1) the process of writing fractions, (2) techniques for converting two fractions into equivalent forms with the same denominator, (3) the discourse related to explaining fraction calculations. Furthermore, our findings reveal that students recognise and navigate differences in praxeological norms between the two different institutions.

Keywords: Fractions, Language, bilingual, curriculum, The Anthropological Theory of Didactic

1 Introduction

Comparative studies of students' mathematical knowledge tend to claim that the differences observed can be explained from characteristics of the schools (used here in the institutional sense, i.e. school systems) involved. That can then be challenged by the fact that students in the compared schools live in different societies with possibly very different general conditions. But what if the same student attended two schools (and school systems) simultaneously? Such a situation would make an ideal case to investigate how mathematical knowledge learned in different schools may differ, due to the teaching at the school rather than to other factors. In this paper, we consider such a case, namely students who attend simultaneously a regular school in Denmark or Sweden, and a so-called "Japanese supplementary school" (more details on this in Sec. 3). These students are, in particular, taught mathematics according to two different curricula, and in different languages. Significant differences exist between the Japanese and Danish (Swedish) curricula, both concerning their structure and content. While the Japanese mathematics curriculum meticulously outlines specific teaching contents for each grade (Aoki, 2023), the Danish (Swedish) curricula adopt a more open structure and describes contents in broader terms, affording teachers substantial autonomy (Pedersen, 2021, p.69). Additionally, Aoki et al., (under review) have investigated variations in language usage pertaining to fractions teaching. In the light of these significant differences concerning curricula and language conditions, this study aims to investigate their impact on students who attend Japanese and Danish or Swedish schools, focusing on the case of learning of fractions arithmetic. The institutional contingency of knowledge is, in general, a main tenet of the anthropological theory of the didactic, the theoretical framework for the present study (cf. section 4); our focus therefore lies in how the two schools contribute to (or influence) students' practice and knowledge in this mathematical context.

2 Learning fractions at bi-multilingual context

Mathematics is popularly considered a more or less language-independent subject, focused on computation; however, research indicates that language significantly influences cognition and learning in mathematics, as evidenced by studies spanning over several decades (e.g., Austin & Howson, 1979; Pimm, 1987; Andle, 2001; Morgan et al., 2014; Halai & Clarkson, 2006; Planas & Pimm, 2023). Such research has explored the use of various languages in the learning of specific mathematical topics (e.g., Clakson, 2006; Planas &Setati, 2009; Planas, 2014; Setati & Adler, 2000), and their interaction with mathematical conceptualization in the increasingly common situation of students with a migrant background (e.g., Prediger and Wessel, 2011; 2013; Prediger et al., 2019).

This paper focuses on the learning of fractions in bilingual settings, which has been investigated by several researchers (e.g., Prediger and Wessel, 2011; 2013; Petersson & Norén, 2017; Prediger et al., 2019; Farrugia, 2022). Prediger et al. (2019) examined four different nuances of conceptualization related to the part-whole concept of fractions among Turkish-German speaking students. Their study revealed that multilingual students navigate between different conceptualizations of fractions that depend on both languages. Additionally, they highlighted the potential of two bilingual modes, namely the *bilingual complementarity mode* (e.g. Moschkovich, 2007) and the *bilingual connection mode* (Kuza & Prediger, 2017), in fostering a deeper conceptual understanding of fractions. Farsani (2014; 2016) pointed out that research on learning mathematics, encompassing not only different languages but also curricula, is still in progress. His focus was on British-Iranian bilingual students attending two different institutions in the UK: a complementary school operating on weekends and a mainstream school running from Monday to Friday. In the complementary schools, where multiple languages are used by students and teachers, while the languages of instruction is English in regular schools of the UK. Particularly noteworthy is Farsani's (2016) identification of differing techniques for solving complex fractions between the two institutions, with a student integrating these techniques across institutions when working on similar tasks in the regular school. Farsani (2016) concluded that learning complex fractions using two different languages in the complimentary school contributes to integrate not only their languages but also to transfer knowledge across different tasks and settings. As the students appearing in this paper are learning fractions in two monolingual settings in parallel, and we believe that our study contributes a different perspective on the subject , by identifying both potentials and obstacles related to students' learning about fraction while being exposed to two different curricula and two different languages of instruction, in terms of how their learning is affected by attending these different institutions.

3 The Japanese supplementary school in Denmark

The context of our study lines within the Japanese Supplementary Schools (JSS), specifically those in Denmark, (JSS_{DK}), which were established by the Japanese Ministry of Education, Culture, Sports, Science and Technology (MEXT). While the main purpose of JSS is to facilitate the learning and practice of Japanese language and also experiencing (maintaining) Japanese culture, children enrolled in JSS_{DK} also have the opportunity to learn mathematics in Japanese. At JSS_{DK}, children learn Japanese language and mathematics on Saturday mornings, following the Japanese national curricula. Despite covering the entire mathematical content taught in Japanese schools, JSS_{DK} dedicates only approximately 56 hours per year to mathematics education, a significant contrast to the 136-175 hours per year in local schools in Japan. The previous study of the teaching in these schools (Aoki, 2023) suggest that the faster pace is made possible by more teacher centered lessons, with less focus on students' autonomous problems solving than is common in regular schools in Japan. JSS_{DK} serves as a supplementary education institution; during weekdays, enrolled children attend regular schools in Denmark or South Sweden where only Danish or Swedish are used as the medium of instruction. Notably, children in JSS_{DK} learn mathematics parallelly in two monolingual settings, utilizing two different languages and following two different curricula – that of the regular school and the Japanese national curriculum. Therefore, proficiency in the Japanese language varies among children, with the majority having Danish or Swedish as their first language.

4 Theoretical framework, Research Question

This study is grounded in the Anthropological Theory of Didactic (ATD) proposed by Yves Chevallard since the early 1980s (Bosch & Gascòn, 2006), and widely disseminated by researchers globally. In this study, we rely in particular on the notion of praxeology, created for elucidating knowledge and practice in didactical research. Praxeologies are comprised of a four-tuples: types of task (T), techniques (τ), technology (θ) and theory (Θ). The type of task (T) encompasses the set of all tasks (t) solvable by a specific technique (τ). A technology (θ) elucidates and justifies one or more techniques(τ), while a theory (Θ) furnishes universal explanations and justifications for the technology (θ). The former two components are denoted together as praxis, while the latter two are recognised as logos.

The basis of this study relies on the findings of Aoki, Johansson and Winsløw (under review), and their positioning is elucidated using the didactic transposition theory as follows.

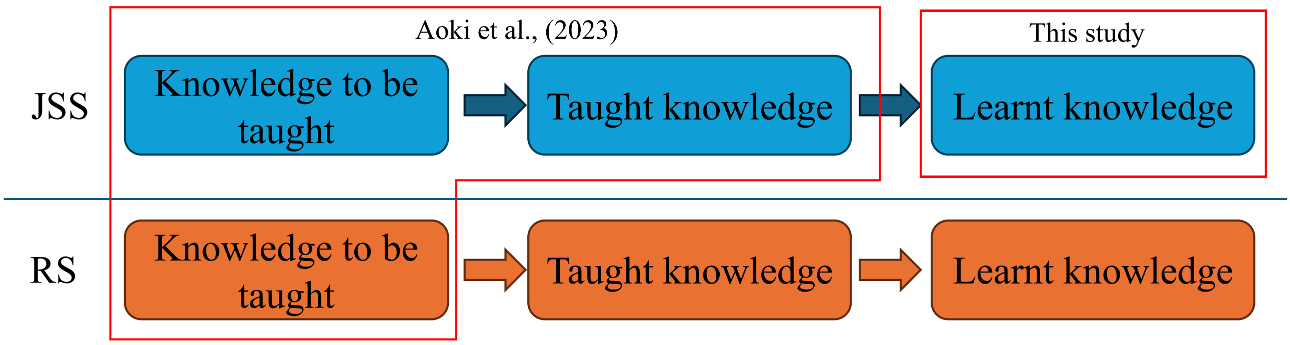

Figure 1: The relation between two studies

The theory of didactic transpositions, introduced in mathematics education by Yves Chevallard (Chevallard & Johsua, 1985), pertains to the transformations undergone by a body of knowledge to be disseminated in institutions, most commonly school institutions. Learning outcomes detected in pupils (learnt knowledge) often originates in other institutions, encompassing scholarly knowledge (e.g., scholarly mathematics), knowledge to be taught (e.g., curriculum and textbooks), taught knowledge (as observed in classroom teaching). This perspective acknowledges that construction of knowledge occurs through transpositive work (Chevallard & Bosh, 2020) that originates outside of school institutions themselves. Aoki et al., (under review) examined pupils' incorrect utterances observed during lessons on addition and subtraction of fractions with different denominators at JSS in Sweden (JSS_{SE}). They retributed these incorrect utterances to institutional differences and natural languages. The findings are elaborated below, utilizing the following notational conventions: Japanese and Swedish text presented in *italics* and **bold**, respectively**.** Pupils are all referred to by pseudonyms. The first and second differences pertain to the reading order and the process of writing of fractions. In Japanese, fractions are read with the denominator first, and then the numerator. By contract, in Swedish, this order is reversed. As a result, the process of writing of fractions also differs.

The second difference involves the amalgamation and inconsistent use of specific mathematical terminologies, such as denominators and numerators. These terms are elements of technology coming from both Japanese and Swedish.

The third one concerns the use of different multiplication symbols, and it is related to the technology. Alex utilized the notation "·" commonly employed in RS_{SE}, instead of "×", which is preferred in Japanese regular schools at the primary level.

The fourth one pertains to technological and theoretical differences, particularly in the way fractions are taught in Japanese and Swedish schools. The teacher queried Jonas about the correct way to read out the fraction $\frac{3}{4}$. In Japanese regular schools, *yon bunno san* (3 over 4 in Japanese) is the expected or normal response. However, during the actual interaction, Jonas initially responded "*San waru yon* (3 divided by 4)" . In the context of Japanese regular schools, such an expression is highly unlikely, as "3 divided by 4" uniquely corresponds to $3 \div 4$ and not to $\frac{3}{4}$, which denotes a number represented by a fraction, not to be confused with a division. Conversely, in the context of Swedish regular schools, reading out fractions in either way, "**tre över fyra**" (literally "three over four"), and "**tre delat med fyra**" (literally, three divided by four). Consequently, this phenomena arising from differences in the pedagogical approaches to teaching fractions in Japanese and Swedish regular schools.

The fifth and sixth one considers at the level of logos, where the differences in elements of technology and theory substantially influence the praxis (use of the techniques). The teacher and pupils engaged in a task: how to convert two fractions to forms with the same denominator (T), motivated by having to solve the specific task of calculating. During their interactions, we observed the teacher placing emphasis on a specific term, "*tsubun*" (a noun representing the procedure of converting fractions to obtain common denominators). This term is encountered by pupils in Japanese regular schools, and is exclusively used within the school context; a similar term does not occur in Swedish context. Moreover, there are two techniques employed when solving type of task T. The first technique accomplishes calculating by utilising the product of the original denominators as a common denominator (τ_1), while the second one involves determining the least common multiple of the denominators (τ_2). In τ_2, a term "*saishokobaisu*" (the least common multiple) is involved. We identified that, in the Japanese context, the term *tsubun* and the technique τ_2 are closely intertwined, while similar connections do not exist in the Swedish context. Consequently, the differences between the logos (especially the terminology upon which technology is based) have an impact on the praxis (the use of techniques).

Hence, Aoki et al., (under review) elucidated the taught knowledge at JSS_{SE} in relation to the knowledge to be taught at JSS and Swedish regular schools (RS), as well as natural languages. Consequently, this study addresses the following research questions concerning learnt knowledge:

RQ1: How do the praxeological differences previously identified, regarding fractions, influence when solving a specific task?

RQ2: To what extent do students have parallel mathematical praxeologies, used by them according to the school institution they are in as they solve a specific task?

5 Data and Methodology

To address the research questions, the author conducted semi-structured interviews in Japanese with three seventh-grade students: Luna, Andreas, and Sofie at JSS_{DK}. Pseudonyms were used to refer to all the students. Luna attends a regular school in Denmark, while the other two does so in nearby Sweden. During the interview, a worksheet containing the following questions was distributed to the students. Regarding

Tasks 1 and 3 were conducted twice in total, with the second interview taking place five months after the first one.

1. Solve $\frac{6}{5}+\frac{7}{5}$ and write the answer in a worksheet.
2. Explain how to calculate $\frac{6}{5}+\frac{7}{5}$ to the interviewer.
3. Solve $\frac{9}{8}-\frac{5}{6}$ and write the answer in a worksheet.
4. Explain how to calculate $\frac{9}{8}-\frac{5}{6}$ to the interviewer.

Two tasks where anomalies could occur were chosen from the textbook written by Fujii, T., & Majima (2021). This textbook was used when the students were in fifth grade. The interview was conducted in the order of questions 1 through 4. The interviewer instructed the students to write their answers on their worksheet in tasks 1 and 3, while the students were instructed to orally explain their written solutions to the interviewer in tasks 2 and 4. After these questions, the interviewer also asked the students two questions orally: “Do you have any thoughts on learning mathematics at the Japanese supplementary school and at a regular school?” and “How do you manage learning mathematics in Japanese language at the supplementary school and in Swedish language (or Danish for the student attending regular school in Denmark) at a regular school?”. All interviews were recorded from the bigging to the end, and interviewer took notes during the interview. The voice recordings were analysed in two steps. Firstly, all voice recordings were transcribed in Japanese (the original language). Secondly, using the Japanese transcriptions and students’ worksheets, episodes related to the six praxeological differences mentioned in Sec. 4 were identified and translated into English.

6 Findings

We present four episodes that illustrate praxeological differences associated with language and institutional norms. While all students are present throughout each episode, we focus on the transcript of a single student in Episode 3 and 4, followed by a summary of outcomes for all students. It is assumed that conducting interviews with more students and including a wider array of tasks is necessary to estimate the actual *extent* of the following phenomenon.

6.1 The process of writing fractions

The first episode illustrates fundamental differences related to the process of writing fractions. Table 1 shows the manner in which students wrote fractions while solving the tasks: $\frac{6}{5}+\frac{7}{5}$ (t_1) and $\frac{9}{8}-\frac{5}{6}$ (t_2). During the interviews, the interviewer prompted them to write both problems and answers. To avoid language bias, participants were instructed to write the given problems and their corresponding answers on the worksheet while pointing to the problems.

	Andreas	Luna	Sofie
t_1: $\frac{6}{5}+\frac{7}{5}$	Swedish way	Japanese way	Mix of Japanese and Swedish way
t_2: $\frac{9}{8}-\frac{5}{6}$	Swedish way	Japanese way	Mix of Japanese and Swedish way

Table 1: The outcome of students' process of writing fractions

Andreas consistently wrote all fractions in the Swedish way (i.e., writing fractions from the top down), while Luna consistently wrote all fractions in the Japanese way (i.e., writing fractions from the bottom up) in both tasks. Sofie wrote fractions in a mixed way, incorporating elements of both the Japanese and Swedish ways. Specifically, she wrote the first fraction in the equation ($\frac{6}{5}$ and $\frac{9}{8}$) wrote in the Swedish way, while revising the second fraction ($\frac{7}{5}$ and $\frac{5}{6}$) in the Japanese way. Furthermore, she expressed the answer in the Swedish way in both tasks. When calculating of t_2, Sofie depicted the calculation process as $\frac{27}{24} - \frac{20}{24}$, writing both fractions ($\frac{27}{24}$ and $\frac{20}{24}$) in the Japanese way. Notice that the technique Sofie employed to convert the denominators by finding the least common multiple (τ_2) may naturally involve writing from the numerator (following the Japanese way). In fact, Luna also employed τ_2 and followed the Japanese way, while Andrea employed τ_1 and follow the Swedish way. Therefore, techniques may dictate the process of writing of fractions. Consequently, Sofie appears to demonstrate parallel praxeologies in the process of writing fractions.

6.2 Different techniques to convert two fractions into forms with the same denominator (technique)

The second episode exposes how students convert two fractions to forms with the same denominator when solving $\frac{9}{8} - \frac{5}{6}$. In section 4, we mentioned two common techniques to solve this type of task, τ_1 and τ_2. In the Japanese context, pupils are ultimately trained to use τ_2 as they gradually grasp the advantages of τ_2 over τ_1, and also as a result of exercising this technique to become habitual and almost automatic. By contrast, in the Swedish and Danish context, τ_1 is commonly utilized in schools, while τ_2 is not commonly prioritized and may not even be taught, judging from text books like (e.g., Gregersen et al., 2020, pp.60-73; Karppinen et al., 2019, pp.22-23). Table 1 shows the students' calculations of $\frac{9}{8} - \frac{5}{6}$ during the two interviews.

	Sofie	Andreas	Luna

1st time	$\frac{27}{24}-\frac{20}{24}=\frac{17}{24}$	$\frac{27}{24}-\frac{20}{24}=\frac{17}{24}$	$\frac{54}{48}-\frac{40}{48}=\frac{14}{48}=\frac{7}{24}$
2nd time	$\frac{27}{24}-\frac{20}{24}=\frac{7}{24}$	$\frac{54}{48}-\frac{40}{48}=\frac{14}{48}=\frac{7}{24}$	$\frac{27}{24}-\frac{20}{24}=\frac{7}{24}$

Table 2: Students’ calculations of $\frac{9}{8}-\frac{5}{6}$

Sofie consistently employs τ_2 in both interviews. Andreas first employed τ_2 but shifted to τ_1 at the second interview, and Luna initially employed τ_1 and later τ_2. These technical choices were done by students during interviews held 5 months apart, and with no previous questions to influence them.

While Sofie seems to have adopted the praxeological norm of the JSS, the two other students display more inconsistency or lack of adherence to the norms of the two schools they attend – so that in terms of the second research question, we have here a simple example showing the existence, for some students, of “parallel” praxeological norms.

6.3 Order of explaining calculation of numerator and denominator (technology)

In section 4, we mentioned reading order of fractions is different in Japanese and Swedish (Danish). In this episode, particular attention is given to whether students begin explaining their calculations by referring to the denominator or the numerator, particularly in the cases of $\frac{6}{5}+\frac{7}{5}\,(t_1)$ and $\frac{9}{8}-\frac{5}{6}\,(t_2)$ during first interview. First, we provide a detailed explanation of Andreas’s case, followed by a summary of all students. Initially, Andreas wrote the answer $\frac{13}{5}$ for t_1 on his worksheet and explained as follows:

1 Andreas: And 13. I just added 6 and 7. And 5 remained as it is.

Next, Andreas wrote the answer for t_2 to the left in Fig. 2 on his worksheet and explained this method to the interviewer as follows:

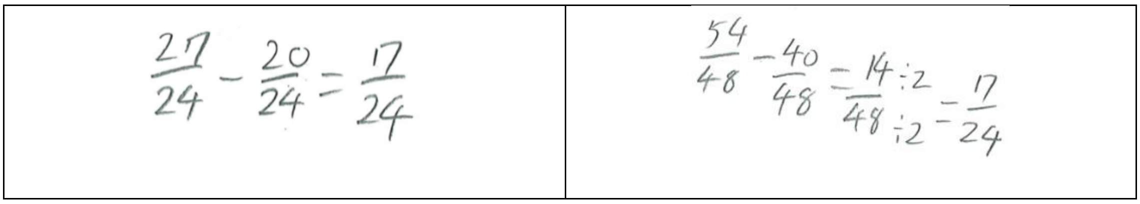

Figure 2: Andreas's handwriting to $\frac{9}{8} - \frac{5}{6}$

2 Andreas: I made *bunbo* (the denominators) the same number, and it is…*kobaisu* (the common multiple) of 8 and 6, maybe? *Saisyokobaisu* (the least common multiple) is 24. Well… 8 times 3 is 24, and then I multiplied 9 by 3 to make it 27. And 6 times 4 is 24, and I multiplied 5 by 4 to make it 20.

After the above explanation, when the interviewer asked if he knew any other calculation methods, Andreas said, "I can also do it with 48" and wrote the answer to the right in Fig. 2, but only up to the result $\frac{14}{48}$. Then he explains (parts of) the calculation as follows:

3 Andreas: 54 minus 40 is 14. And then I keep *bunbo* (the denominator) of 48 as it is.

4 Interviewer: Okay. So, there are two methods, right? But why did you use first method?

5 Andreas: This is because this [pointing to $\frac{14}{48}$] requires *yakubun* (reduction of fractions). If we divide both 14 and 48 by 2, the number remains the same, but it would be nice to have cleaner numbers, so to speak.

6 Interviewer: What does *yakubun* mean?

7 Andreas: When we divide both the numerator and the denominator by the same number, that is called *yakubun* (the last part of the right hand side of Fig. 2). Dividing this by 2 and then dividing this by 2 we get *nana bunno nijuyon* (24 over 7). No! It becomes *nijuyon bunno nana* (7 over 24).

In Turn 1, he first explains the process for obtaining the numerator, and then proceeds to explain the process for obtaining the denominator (Swedish syntax, while explaining). In Turn 2, he first explains how to obtain the denominator 24, then how to get the numerators (Japanese syntax, while explaining). However, in Turns 3 and 5, when explaining the right side calculation in Fig. 2, he says nothing about where the two first fractions come from, but explains how to subtract them, starting from the numerators and how to reduce $\frac{14}{48}$, beginning from the numerators (Swedish syntax, while explaining). In Turn 7, he explains definition of reducing fractions, again in the order from numerator to denominator (Swedish syntax). Subsequently, he reads out $\frac{7}{24}$ as "*nana bunno nijuyon*", but immediately corrects himself, exclaiming "No!" – then provides the correct Japanese read-out "*nijuyon bunno nana*". So, he initially reads out the resulting fraction using Japanese words, but following the Swedish syntax for reading out fractions, with the numerator stated first. Therefore, while Andreas adheres to Japanese syntax in Turn 2, it seems that he confuses with Swedish syntax some of the time.

When asked about his learning experiences in both a Japanese supplementary school and a Swedish regular school, Andreas stated " I am in sixth grade in Sweden now, and in Sweden, I am learning the maths I learned in Japan (referred to here as JSS)." Furthermore, when asked how he manages encountering math in Swedish regular school that is already being taught in the supplementary school?" Andreas stated, "Japan. Well… I use the methods taught in the supplementary school, but I am thinking in Swedish language." This assertion implies that despite Andreas using Japanese language during the interview and mastering τ_2, a dominant technique within the Japanese context, he may, to some extent, be translating from Swedish language, particularly during cognitive processes.

The analysis was extended to the other two students. Sofie demonstrated no mistake in the reading order of fractions, and she adhered to Swedish syntax while explaining the calculation of t_1. Furthermore, she utilized τ_2 for the calculation of t_2 and explained a method fowling Japanese syntax. By contact, Luna exhibited no mistakes in the reading order of fractions and consistently employed Japanese syntax while explaining in both tasks.

Hence, the different reading order of fractions may also influence the order in which the calculation of the numerator and denominator is explained. Here, we presented simple examples from Andreas and Sofie, which demonstrated the existence, for certain students, of parallel praxeological norms. Notice that the technique: τ_2 may naturally involve explaining from the numerator (following the Japanese way), as observed in the writing process as well (section 6.1).

6.4 Is a fraction the same as a division?

This final episode addresses students' confusion regarding certain differences between two institutions. During the interview, all students mentioned specific variations at the level of techniques. For instance, Sofie and Luna mentioned the form of long multiplication and long division, respectively. Luna also brought up the distinct reading order of fractions. In the following episode, Andreas not only mentioned differences in long division (at the level of techniques) but also exhibited confusion at the level of logos, particularly, concerning how division and fractions are taught in the two institutions, as evidenced, for instance, in textbooks. This particularly concerns whether a fraction represents just a number or could also indicate the division of two numbers. When asked about differences between the Swedish and Japanese contexts for learning mathematics, Andreas spontaneously mentioned "division". In the following transcript of the next turns in the interview, the left side of Fig. 4 is identified by Andreas as the

Japanese form of writing a division (here, 300 divided by 3), and the right side as the Swedish form.

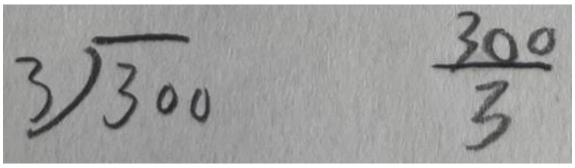

Figure 3: Handwriting by Andreas during the interview

1	Andreas:	Well…in Japanese, [writing the Japanese form] we use this, right? But, in Sweden, [writing Swedish form] we just write like this, *sanbyaku waru san* (300 divided by 3).
2	Interviewer:	I see. [Pointing to the Swedish form] this is a division.
3	Andreas:	Yes.
4	Interviewer:	But this one ($\frac{300}{3}$), is this not a fraction?
5	Andreas:	No. That also feels a bit odd.
6	Interviewer:	Does it feel odd? But you are not confused? For instance, [pointing at $\frac{300}{3}$ on his worksheet] this one is a fraction in Japan.
7	Andreas:	Yes.
8	Interviewer:	But, in Sweden, this is a division, right?
9	Andreas:	This ($\frac{300}{3}$) is also a fraction! In Sweden! That is why it is a bit difficult.

In Turn 1, while Andreas did not explicitly explain the algorithms, his mention of "using" the form in the left side of Fig. 4 implies an explanation of the typical long division used in Japanese primary schools. Subsequently, he proceeded to write the right side of Fig.4. Although he did not provide the algorithm for this either, his prior explanation of long division in Japanese context suggests that he explained "**Kort division**", the most commonly used division method in Swedish primary schools. Therefore, it can be assumed that Andreas want to highlight the disparity between the two institutions, in how division operations are performed and written. Thus, he mentioned the difference at the level of technique.

Between Turns 4 and 9, when asked whether $\frac{300}{3}$ represents a fraction as a number in Japanese schools and a division in Swedish schools, Andreas mentions $\frac{300}{3}$ is *also* considered a fraction (representing a number) in Swedish in Turn 9. In Turn 5, Andreas stats his confusion about this double meaning of a representation like $\frac{300}{3}$ (in Sweden), and in Turn 9, he even mentions this circumstance presents a challenge. Andreas's remarks in Turn 5 and 9 indicate that he is at least somewhat confused about the aforementioned differences. In Japanese school mathematics, when pupils learn division at primary schools, the specific notation "÷" is used to designate the operation, like $300 \div 3$. Although fifth-grade pupils learn (as mathematical results) about identities such as $300 \div 3 = \frac{300}{3}$, the distinction between the number (represented by $\frac{300}{3}$) and the operation (represented by $300 \div 3$) remains intact. Since this interview did not directly involve mathematical tasks directly related to this difference in praxeological norms, it is remarkable that Andreas spontaneously mentions of the difference not only at the level of technic but also at the logos level.

7 Discussion and conclusion

This study was based on semi-structured interviews with students who are learning fractions in two different languages and curricula simultaneously. It revealed that some students exhibit parallel praxeologies to varying degrees, both in terms of logos and techniques applied when solving specific fraction-related tasks. These parallel praxeologies were notably evident, firstly, in the process of writing fractions and, secondly, when converting fractions into equivalent forms with the same denominator. Additionally, we observed empirical evidence of differences in the process of writing and reading fractions, which aligns with findings from previous studies such as Bartolini et al. (2014) and Aoki et al. (under review). Furthermore, we found indications that these differences also influence whether students begin explaining their calculations by referring to the denominator or the numerator. Some students also demonstrated an awareness of differences in praxeological norms between the two institutions, both at the level of technique and logos. Therefore, students experiencing two systems of praxeological norms, while attending two schools in parallel, at least to some extent become aware of the institutional relativity of such norms for mathematical praxeologies, unlike students who do not have such an experience.

Learning about fractions in two monolingual environments simultaneously, may at first sight appear as independent processes, since students are expected to learn the specific mathematical praxeologies and languages associated with their respective institutions. Despite the expectation for students to apply the mathematical praxeology and language of the institution they are in at a given moment, the interaction or mixing between these two are observable for students in this study. Our study underscores the potential of utilizing praxeological analysis from ATD in multilingual contexts, which is also a main methodological novelty of the study. This analytical approach enabled us to transcend the micro-level language dimension and delve into broader curricular macro-dimensions, particularly focusing on institutional disparities. Additionally, it is noteworthy that this study was carried out based on the findings and theoretical constructions of Aoki (2023) and Aoki et al., (under review); similar studies would benefit from similar preliminary analyses of curricula and of classroom observations.

To further expand upon the findings of this paper, additional research is needed, including: (1) incorporating a larger sample of students and a wider range of mathematical tasks, (2) considering students with different language backgrounds and contexts and (3) investigating in multilingual setting where students can used both Danish or Swedish along with the Japanese language.

Acknowledgements

We acknowledge all who supported this study, including pupils, teachers and the school principal.

Declarations

On behalf of all authors, the corresponding author states that there is no conflict of interest.

ORCID

Mayu Aoki [*https://orcid.org/0000-0001-6885-5855*](https://orcid.org/0000-0001-6885-5855)

References

Adler, J. B. (2001). *Teaching mathematics in multilingual classrooms*, Doedrecht, The Netherlanfs: Kluwer Academic Press.

Aoki, M. (2023). Mathematics in a Japanese overseas school: Connecting classroom study and curriculum. *Asian Journal for Mathematics Education*, *2*(2), 204-219. https://doi.org/10.1177/27527263231187804

Aoki, M., Asami-Johansson, Y., & Winsløw, C. (2023). Learning to speak mathematically at the Japanese supplemenrary school in Sweden critical cases of praxeological anomaly. [Manuscript submitted for publication]. Educational Studies in Mathematics.

Austin, J. L., & Howson, A. G. (1979). Language and mathematical education. *Educational Studies in Mathematics*, *10*, 161-197. https://doi.org/10.1007/BF00230986.

Bartolini Bussi, M. G., Baccaglini-Frank, A., & Ramploud, A. (2014). Intercultural dialogue and the geography and history of thought. *For the Learning of Mathematics*, *34*, 31-33.

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin*, *58*(58), 51-65.

Chevallard, Y., & Johsua, M. A. (1985). *La transposition didactique: du savoir savant au savoir enseigné*. La Pensée Sauvage,.

Chevallard, Y., & Bosch, M. (2020). Didactic transposition in mathematics education. *Encyclopedia of mathematics education*, 214-218. https://doi.org/10.1007/978-3-030-15789-0_48

Clarkson, P. C. (2007). Australian Vietnamese students learning mathematics: High ability bilinguals and their use of their languages. *Educational studies in mathematics*, *64*(2), 191-215. https://doi.org/10.1007/s10649-006-4696-5

Farrugia, M. T. (2022). Engaging in the discourse of fractions in a bilingual Maltese classroom. *Teachers College Record*, *124*(5), 89-107. https://doi.org/10.1177/01614681221103977.

Farsani, D. (2014). *Making multi-modal mathematical meaning in multilingual classrooms* (Doctoral dissertation). University of Birmingham, England.

Farsani, D. (2016). Complementary functions of learning mathematics in complementary schools. In Anjum, H & Clarkson, P (Eds.), *Teaching and learning mathematics in multilingual classrooms*, 227-247. Sense. 10.1007/978-94-6300-229-5_15

Fujii, T., & Majima, H. (2021). *Atarashi sansu 5* [*New mathematics for elementary school for grade 5*]. Tokyo Syoseki Shuppan.

Halai, A., & Clarkson, P. (2016). Teaching and learning mathematics in multilingual classrooms: An overview. In Anjum, H & Clarkson, P (Eds.), *Teaching and learning mathematics in multilingual classrooms*. 1-10. Sense.

Karppinen J., Ronkainen-Salminen, J., kiviluoma P., & Urpiola T. (2019). *Bas Faorit matematik 6A* [*Base favorite mathematics 6A*]. Studentlitteratur

Morgan, C., Craig, T., Schütte, M., & Wagner, D. (2014). Language and communication in mathematics education: An overview of research in the field. *ZDM*, *46*, 843-853. https://doi.org/10.1007/s11858-014-0624-9.

Moschkovich, J. (2007). Using two languages when learning mathematics. *Educational Studies in Mathematics, 64*(2), 121–144. https://doi.org/10.1007/s10649-005-9005-1.

Pedersen, P. L. (2021). Learning and understanding the complexity of fractions (Doctoral dissertation). Aalborg Universitetsforlag, Denmark. https://doi.org/10.54337/aau441584586

Gregersen, P., Højgaard-Jensen, T., Petersen, K., Thorbjørnsen, H., Kathrine- Petersen, L., & Højgaard Jensen, T. (2017). *Matematrix 6* [*Math matrix for grade 6]*. Alinea.

Petersson, J, & Norén, E. (2017). To halve a fraction: An issue for second language learners. *Education Inquiry, 8*(3), 173-191. https://doi.org/10.1080/20004508.2016.1275187

Pimm, D. (1987). *Speaking mathematically. Communication in mathematics classrooms*. Routledge.

Planas, N., & Setati, M. (2009). Bilingual students using their languages in the learning of mathematics. *Mathematics Education Research Journal*, *21*(3), 36-59. https://doi.org/10.1007/BF03217552

Planas, N. (2014). One speaker, two languages: Learning opportunities in the mathematics classroom. *Educational Studies in Mathematics*, *87*(1), 51-66. https://doi.org/10.1007/s10649-014-9553-3

Planas, N., & Pimm, D. (2023). Mathematics education research on language and on communication including some distinctions: Where are we now? *ZDM–Mathematics Education*, 1-13. https://doi.org/10.1007/s11858-023-01497-0

Prediger, S., & Wessel, L. (2011). Relating registers for fractions–multilingual students on their way to conceptual understanding. In *Proceedings of the 21 ICMI study conference*, 324-333.

Prediger, S., & Wessel, L. (2013). Fostering German-language learners' constructions of meanings for fractions—design and effects of a language-and mathematics-integrated intervention. *Mathematics Education Research Journal*, *25*(3), 435-456. https://doi.org/10.1007/s13394-013-0079-2

Prediger, S., Kuzu, T., Schüler-Meyer, A., & Wagner, J. (2019). One mind, two languages–separate conceptualisations? A case study of students' bilingual modes for dealing with language-related conceptualisations of fractions. *Research in Mathematics Education*, *21*(2), 188-207. https://doi.org/10.1080/14794802.2019.1602561.

Setati, M., & Adler, J. (2000). Between languages and discourses: Language practices in primary multilingual mathematics classrooms in South Africa. *Educational studies in mathematics*, *43*(3), 243-269. https://doi.org/10.1023/A:1011996002062