UNIVERSITY OF COPENHAGEN
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# PhD Thesis

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# A Praxeological Study of Proportionality in Mathematics Lower Secondary Textbooks

This thesis is submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

Supervisor: Carl Winsløw

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# ABSTRACT

Research on the uses and contents of mathematics textbooks has expanded over the past decades, due to the central role textbooks occupy in mathematics teaching worldwide. However, the methodology of analysing the texts themselves often appears underdeveloped or even naïve, especially when it comes to specific mathematical content.

The central idea of this thesis is to deploy the anthropological theory of the didactic, and especially the notion of praxeology, to analyse how textbooks treat three specific and related areas (or more precisely, sectors) of mathematical contents for lower secondary school, namely "proportion and ratio" (in Arithmetic), "similar plane figures" (in Geometry), and "linear functions" (in Algebra). This leads to a new and very precise methodological tool for analysing the practices (types of tasks, techniques) supported by the textbooks through examples, explanations and exercises; it also allows us to analyse how these practices are organised or unified through the theoretical contents, and how the theoretical level may enable the texts to establish explicit links between different sectors.

By constructing praxeological reference models, we obtain a powerful and precise tool to compare the priorities or profiles of different textbooks, their alignment with national examination practices, curricula, etc. We have devoted separate attention to study why certain mathematical links between the three sectors seem to be absent or underdeveloped in textbooks, and explanations are found both in terms of the mathematical notions used (for instance, how "linearity" is defined) and in terms of the historical development of the didactic transposition, as evidenced by textbooks from different periods of time.

# ABSTRACT IN DANISH

I de seneste årtier er der blevet forsket stadig mere i matematiklærebøgers brug og indhold, i erkendelse af den centrale rolle som lærebøger spiller i matematikundervisning verden over. Ikke desto mindre fremstår disse studiers metodologi ofte underudviklet og naiv, specielt hvad angår analysen af specifikt matematisk indhold.

Denne afhandlings hovedidé er at anvende den antropologiske teori om det didaktiske i spil, og specielt begrebet prakseologi, til at analysere hvordan lærebøger behandler tre specifikke og forbundne områder (mere præcist sektorer) af matematisk stof for grundskolens ældste klasser, nemlig ”proportion og forholdsregning” (i Aritmetik), ”ligedannede plane figurer” (i Geometri), og lineære funktioner (i Algebra). Dette giver et nyt og meget skarpt værktøj til at analysere de praksisblokke (opgavetyper, teknikker) som understøttes af lærebøgerne gennem eksempler, forklaringer og opgaver; det tillader os også at analysere hvordan disse praksisblokke organiseres eller forenes af teoretisk indhold, og hvordan det teoretiske niveau kan sætte teksterne i stand til at knytte eksplicitte forbindelser mellem forskellige sektorer.

Ved at konstruere prakseologiske referencemodeller opnår vi således et stærkt og præcist redskab til at sammenligne forskellige lærebøgers profiler eller præferencer, deres overensstemmelse med nationale eksamensopgaver, læreplaner etc. Vi har viet særlig opmærksomhed til at undersøge hvorfor bestemte matematiske forbindelser mellem de tre sektorer synes at være fraværende eller underudviklede i lærebøgerne, og finder forklaringer både i den måde matematiske begreber bruges på (fx hvordan ”linearitet” defineres”), og i den historiske udvikling af den didaktiske transposition, som kan iagttages ved at se på lærebøger fra forskellige tidsperioder.

# LIST OF PAPERS

## I. Papers included in this thesis

The PhD thesis is constituted by the following papers; an outline of how these papers each contribute to answer the overall research questions of the thesis, is given in section 1.4. Here we will just state the main idea and status of the papers.

A: Mathematical practice in textbooks analysis: praxeological reference models, the case of proportion

- Submitted to *Research in Didactics of Mathematics* (REDIMAT)
- Construction and applications of a praxeological reference model for the proportion sector in the Domain of Arithmetic

B: A reference model for analysing textbook treatment of similarity in plane geometry

- Submitted to *Annales de Didactique et des Sciences Cognitives*
- Construction and applications of a praxeological reference model for the similarity sector in the Domain of Geometry

C: Two notion of ‘linear function’ in lower secondary school and missed opportunity for students’ first meeting with functions

- Submitted to *The Mathematics Enthusiast*
- The theoretical and practical analysis of how proportion functions and linear functions are presented and worked with in Indonesian textbooks

D: Linking proportionality of arithmetic, algebra and geometry domain in Indonesian lower secondary textbooks

- Accepted in *Educaçao Matemática Pesquissa*
- An analysis of links, established in Indonesian textbooks, between the sectors of proportion in Arithmetic, similarity in Geometry, and linearity in Algebra.

E: Didactic transposition phenomena through textbooks: the case of proportionality

- Manuscript to be submitted
- A historic-epistemological analysis of the didactic transposition of “the mathematics of proportions”, with examples from Indonesian, Spanish and French textbooks

## II. Other papers

Here is a list of other papers published during my time as a PhD student; all are strongly related to the included papers, usually initial or partial versions of these.

F. Wijayanti, D. (2015). Relating arithmetical techniques of proportion to geometry: The case of Indonesian textbook. In K. Krainer, & N. Vondrova (Eds.), Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (pp. 3157-3163). European Society for Research in Mathematics Education .

G. Wijayanti, D. (2015). Introducing proportion in arithmetics domain: The case of current and historical textbooks: Papers from a doctoral course at the University of Copenhagen. In M. Achiam, & C. Winsløw (Eds.), Mathematics and Science: and education: The relationships and disconnections between research (Chapter 12, pp. 131-140). Institut for Naturfagenes Didaktik, Københavns Universitet. (IND's Skriftserie; No. 39)

H. Wijayanti, D. (2016). A collective aspect of mathematics textbook production based on the anthropological theory of the didactic. In Educational design in math and science: The collective aspect: Peer-reviewed papers from a doctoral course at the University of Copenhagen (pp. 71-76). Institut for Naturfagenes Didaktik, Københavns Universitet (IND's Skriftserie, Vol. 46).

I. Wijayanti, D & Purnomo, Y.P. (2016). A historical study of how proportion in arithmetic appears in Indonesian lower secondary textbooks. Paper presented at 13th International Congress on Mathematical Education, Hamburg, 24-31 July 2016.

# 1 INTRODUCTION

This PhD thesis presents five papers which reflect my main academic work during 3,5 years. All of the papers are written within this period, and their main focus is on lower secondary textbooks, proportion, and ATD. The first four papers analyses relevant parts of Indonesian textbooks, while the last paper is a comparative study of textbooks from Indonesia and Spain (*and more*). All of the papers are in review process for publication in international journals. Exceptionally, the fourth paper appears in a special issue of *Educaçao Matemática Pesquissa*, which will publish the proceedings of 5th International Conference on the Anthropological Theory of the Didactic (ICATD) in 2016. Additionally, during my Ph.D. years, two conference papers and two course papers as preliminary versions of the five main papers have been published (see page 8). The papers presented in this thesis are essentially in the same form as the submitted/published papers. However, minor modifications have been made to adjust to the format of the thesis.

In the next session, we will discuss my motivation for the overall study and we will continue by present some background on textbook analysis in general. Afterwards, a discussion about ATD will be presented. Then, we will resume the original motivations, to present the general purpose and theoretical framework along with my research questions. The answers obtained will be discussed in a section on results followed by conclusive remarks. In the last chapter, we will present an overview of the papers constituting the thesis. Finally, we attach the five main papers which constitute the core of this thesis.

## 1.1 Overall motivation

It is found that the number of teachers in Indonesian lower secondary school who have the officially required education level is 598.989 in 2016, which corresponds to 92% of the total numbers of teachers (National Indicators for Education Planning, 2016a). This percentage has increased significantly from the percentage (37%) in the year 2011 (The World Bank, 2011). However, does this increasing percentage really imply a growth in teachers' mathematical knowledge? In fact, some researchers have shown that there is insufficient development of Indonesian teachers' mathematical knowledge in pre-service education and at school. Widjaja, Stacey, and Steinle (2011) argued that there are considerable numbers of students in pre-service education who are facing serious challenges with the placement of negative decimals on a number line. The result of this study indicates students' misconceptions regarding the ordering of negative decimals either by creating separate negative numbers line which appear in the same numerical order (-0, -1, -2, …, 0, 1, 2, …) as positive numbers, or by interpreting negative numbers (e.g. -1.2 =-1-0.2) with same, but wrong characteristics of positive numbers (-1.2 =-1+0.2). In the domain of geometry, Ng (2012) also found depressing results while examining Indonesian elementary teachers' mathematical knowledge of basic measurement. These shortcomings of Indonesian mathematical teachers' knowledge will potentially lead to larger dependency on textbooks.

From the point of view of international research, studies on textbooks role by teachers have got increasing attention in recent years. For example, Pepin, Gueudet, and Trouche (2013) argued that textbooks function as a crucial resource in teaching practice in France and Norway. A similar situation was also found in Singapore (Yan & Lianghuo, 2002), China (Lianghuo, Ngai-Ying, & Jinfa, 2004), and Croatia (Gracin & Matić, 2016). Interestingly, the exercise part of textbooks was intensively studied by Swedish teachers, as was their use in the classroom (Johansson, 2006).

The National Indicators for Education Planning (2016b), a government institution under the Indonesian Ministry of Education and culture, mentioned that the numbers of lower secondary student from grade 7-9 is 10.133.648 in the school year 2016/2017. This daunting number makes school textbooks turn into a big enterprise for publishing companies. On the one hand, teachers have many textbooks to choose as teaching resources and student companion. On the other hand, teachers need a precise tool to select among a lot of textbooks that match with teaching situation or students need.

The fact that there is potentially a situation in which teachers depend heavily on textbooks and tools for appropriately selecting textbooks motivated me to write a master thesis about textbooks analysis. The study purpose was trying to have more precise procedure (mainly concerning the exercise part) to choose a textbook. I analysed elementary school textbooks with respect to "problem solving tasks". After I considered several definitions of problem solving task, I defined problem solving as involving non-routine and open questions for students to solve. I realized that the distinction among routine and non-routine tasks is not easy to define precisely. What constitutes routine and non-routine tasks depends on assumptions on students' knowledge and experience. In this case, the target mathematical knowledge is hard to analyse because it is relative to mathematical knowledge of subject (including cognitive process of subject). In other words, there is no objective measure of the distance between knowledge to be taught, and students' knowledge. As a result, mathematics will be seen as given and unquestionable. Therefore, an epistemological analysis is needed to question mathematical knowledge that is located in an institution (c.f Gascón, 2003; Winsløw, 2007).

Concretely, we are going to use epistemological models to analyse textbooks in my PhD thesis. Later, we will replace epistemological analysis by praxeological analysis concerning certain themes related to proportionality. Even though specific mathematical knowledge is important for this study, the aim of this study is to apply a precise methodology to analyse textbooks. The original idea of this thesis was comparing different approaches to proportionality in textbooks and using a unique model for a comparative study of how proportion is treated in textbooks in Indonesia and Denmark. After finishing the model for the arithmetical proportion sector in Indonesian textbooks, we developed the idea to analyse connections between this sector and related sectors in Geometry and Algebra. The issue of connectivity is not a new idea in mathematics and we found that research about connectivity is closer to epistemological analysis of textbooks than comparing textbooks for a single sector, in several countries.

## 1.2 Textbook analysis in general

A large literature review study on textbook research has been done by Fan, Zhu, and Miao (2013). They mentioned that most textbooks analysis and comparison tend to focus on specific mathematics content and topics and how they have been treated in a variety of textbooks. The idea of textbooks analysis and comparison by Fan et al. (2013) is well aligned with the idea of epistemological analysis that we mentioned in the above discussion. They further analysed examples of research in textbooks analysis. However, they did not discuss specifically the differences of methodologies among these studies. In the following paragraph, we will describe what types of methodological tools these studies use, and we will compare them according to their methodology.

The modes of reasoning in explanations provided in Australian eighth grade mathematics textbooks were analysed by Stacey and Vincent (2009). Nine best seller textbooks were analysed and seven topics were chosen which require some kind of proof, e.g. the angle sum of a triangle, the formula for the area of a circle, the distributive law, etc. They investigated if “proofs” were based on seven modes classification that can be refined into deductive reasoning, empirical reasoning, and external conviction. The results show that for two thirds of topics in every textbook, empirical reasoning dominates most topics. Teachers can use this result to choose a textbook based on the treatments which is offered for reasoning and proving. However, teachers who need specific information regarding one particular topic (content) cannot use their results, especially for topics that do not come with proving and reasoning problems.

Pickle (2012) did a study on the treatment of statistical content in middle grades mathematics textbooks. They both considered non-commercial textbooks (edited and funded by a government institution) and textbooks published commercially. To select relevant parts of the text to include in the study, they looked for chapters or sections that contained statistical titles or terms (e.g. mean, mode, median, midrange, central tendency, range, variance, standard deviation, dispersion, best fitting line, and frequency distribution table). Then, the this data was analysed based on lesson components for lesson narrative, e.g. description of statistical situations, definitions of vocabulary, questions, activities, worked examples, etc. The results show that the commercial textbooks had lesson narratives composed mainly of worked examples, while the non-commercial textbooks aimed for a more conceptual or theoretical understanding. This study is useful for curriculum developer and publishers to improve the textbooks. Teachers can also use this result to choose a textbook, based on the dominance of worked examples of statistic content. However, this study does not give a detailed characteristic of the statistic content as such.

Closer to our topic in terms of the content, a study of the capacity of two Australian eight grade textbooks for promoting proportional reasoning was presented by Dole and Shield (2008). The material analysed is chapter titles, subheadings, page count, presentation of material, and additional features that relate proportional reasoning (e.g. ratio, ratio and proportion, similarity, modelling). One of their analysis tools relates to common description of multiplicative structure and inverse multiplicative structure (division) in proportional situation. For examples, students are asked to decide the value of one missing number of quantity in forth value numbers (a/b=c/?). The structure of the task can be solved using division to find unit of quantity (a/b) and the missing number. However, students can also use different solution, e.g. cross product solution. As a result, in one textbook (e.g. in scale, ratio, rate) it is found that eight worked examples are structurally similar, but all present different solutions and they do not make a reference to previously covered procedures. This study focuses on specific content and uses its characteristics to analyse the textbook. Here, they focus on chapters that relate to proportional reasoning (e.g. ratio and proportion) and multiplicative structure of proportional reasoning task. However, they mostly concentrate on the general result, while the detail nature of worked example is left untouched, e.g. types of solution that the work examples had.

A textbooks analysis focusing on the development of computational skills in the Japanese primary grade has been done by Reys, Reys, and Koyama (1996). They analysed tasks and techniques that textbooks provide. As results, they identified some common types of task in the first grade, e.g. addition and subtraction of two digit multiples of 10 and numbers requiring no grouping (30+60, 70-40, 34+12), of one and two digits numbers (32+5, 47-3), and multiplies of 10 to two digit numbers (97-20, 43+50). However, they generalize the technique to solve them;

‘when basic facts for addition are being developed, they are illustrated in the textbook by one method of thinking’. Moreover, they classified task in the second and the third grade in a general definition. For example addition and subtraction algorithms in the third grade are extended to computations with four or more digit. The purpose of this analysis is to characterize the development of computation in the Japanese primary grades, but we can also use this analysis to see what kind of computation can be offered in each grade. For example, teachers can consider types of task covered in the first grade to assess textbooks. However, the analysis does not really capture precise types of task, or the tasks analysed can be interpreted as belonging to more than one classification. For example, addition and subtraction of one and two digits numbers (32+5, 47-3) can be interpreted into; 1. addition and subtraction of one and two digit multiples of 10, 2. addition and subtraction of numbers requiring no grouping. Additionally, the theory of this study is at a fairly informal stage.

Based on the studies above, a content analysis is needed to describe not only general characteristics of textbooks (extension of conceptual discussion part, numbers of worked examples) but also the specific characteristics in theory and practice of the content itself, which require a much more detailed theoretical framework. In the following discussion, we will propose ATD as our theoretical framework to achieve this aim.

## 1.3 Anthropological theory of the didactic

This study was based on ATD initiated by Chevallard (1999), and further described e.g. by Chevallard and Sensevy (2014), Barbé, Bosch, Espinoza, and Gascón (2005), and Winsløw (2011). “The didactic” here refers to the sharing of objects of knowledge and practice in institutions, such as when a teacher is given the responsibility to teach mathematics to a group of students within a school. The didactic can be located anywhere, not only in schools. Objects of knowledge and practice shared in the school context can be a mathematics subject, theory or method. In fact, mathematics is often accepted as “given knowledge” at schools because of the dominance of the paradigm of visiting monuments (Chevallard, 2012) in which students are encouraged to accept knowledge and practice as it is (monumentalization) without knowing the meaning or uses it has, and often without realizing that mathematical knowledge can be connected internally to other pieces of knowledge. ATD in this case is trying to questioning what mathematics is about, particularly the alerting researchers to the differences between mathematics in the scholarly world, and mathematics at school, and to the distance between how mathematics is taught by teachers, and how mathematics is learnt by students. These different forms of mathematics are at the foundation of a sub theory of ATD, the theory of didactic transposition (Chevallard, 1985, 1988; Chevallard & Bosch, 2014). The idea of didactic transposition is to consider knowledge, e.g. mathematical knowledge, as contingent upon institutions which functions as ecologies for that knowledge. In particular, we can distinguish three categories; scholarly knowledge (found in scholarly institutions such as universities), knowledge to be taught (specified in curricula), and knowledge actually taught (in schools). Through the external didactic transposition, mathematics in scholarly knowledge is transposed into a teachable knowledge for students and it is normally documented by ministries of education and publishers through texts such as curricula and textbooks. Then, teachers follow these documents and try to share the knowledge to be taught with students while considering students' conditions (internal didactic transposition). The origin of the transposition is not always scholarly knowledge, of course, but may also be mathematics as found in other societal institutions.

Didactic processes cannot achieve any specified transposition among institutions, but it is constrained by particular conditions, e.g. school regulation, government policy, teachers' knowledge, available resources, culture, etc. To study these conditions, ATD proposes the so-called levels of didactic co-determination**,** considering that didactic systems are impacted from higher levels, including external institutions. The scale of didactic co-determination can be summarized as follows: civilization, society, school, pedagogy, discipline, domain, sector, theme, and subject. Additionally, we need to consider that the conditions at any given level may influence the other levels. Mathematics is an example of a discipline and it consists of some domains, such as algebra, arithmetic, geometry. Then, similarity in the geometry domain can be seen as a sector that can be declined in themes such as polygon similarity. Lastly, some subjects can be constructed based on the subject (e.g. how to show that two polygons with given measures are similar).

The three basis levels of the didactic level co-determination correspond to praxeological elements that consist of type of task and technique (subject), technology (theme), theory (sector). A type of task $(T)$ is solved by a particular technique $(\tau)$ that is explained by a technology $(\theta)$. Additionally, a theory $(\Theta)$ is used to justify a technology. Types of task and technique are together called a *practice block*, while technology and theory constitute a *theory block*. This four-tuple $(T,\tau,\theta,\Theta)$ and their categorization of theory and practice block in ATD is called praxeological organization (Chevallard, 2002). Here, the fact that knowledge is transposed from one institution to another institution requires researchers to construct their own reference epistemological models (REM). Such reference epistemological models furnish a tool to describe and compare the praxeologies occurring in different institutions.

### A glance of proportionality

As our focus is textbooks analysis, but textbooks as a whole are very complex and large objects, we decided to focus on some part of the mathematical content. As a first such area, we chose the sector of "proportionality" as it appears largely in all grades in middle school. Proportionality is informally defined as a particular relationship between quantities: One quantity A is said to be proportional to another quantity B if, when one reduced is multiplied by some number, B is also multiplied by the same number. Thus, if the price and the amount of sugar are proportional, you can get the double amount of sugar with the double amount of money. Proportionality can also be found in geometry where the lengths of corresponding sides of similar figures are in a proportional relation to each other. The proportionality relationship can also be expressed as an identity of "ratios" of quantities, $\frac{a}{b} = \frac{c}{d}$ , in which a,b are two values of one quantity and c,d are the corresponding values of another quantity. The use of algebraic variables permits other ways to explain what proportionality is. Then, the general proportionality of two quantities, viewed as a free variable $x$ and a dependent variable $y$, can be represented as '$f(x)$=$r.x.$' where r is some fixed number. Thus, in the algebraic domain, the idea of proportionality is more or less identical to the notion of linear function. Thus we see that the vague and general idea of proportionality stretches across several domains and it can therefore be discussed in different and distant chapters in textbooks, with headings such as ratio, similar triangles, and linear functions.

### 1.4 Textbook research based on ATD

In the following discussion, we will present the current research of textbooks analysis in ATD that used praxeology organization. The way to define a sector is flexible. For example, Miyakawa (2017) defined "proof" as a sector in the domain of geometry, instead of choosing topic-driven sectors such as similarity or volume. By contrast, González-Martín, Giraldo, and Souto (2013) chose to study the sector of real numbers constituted by six "types of task". However, at a close look, they are not types of tasks in the sense of ATD, but rather large classes of tasks covering several types. Thus, to make full use of the ATD framework, a clearer categorization is needed to provide a closer picture of the student activities proposed by the textbook. In line with that aspiration, Hersant (2005) did a research on different techniques to solve so-called fourth missing value proportional problems. Here, she found six different techniques. Thus, is it a possible to analyse more than one specific type of task e.g. in proportion, similarity and linear function? Furthermore, is it possible to analyse not only in practice level but also in theory level?

The general fact that proportion, similarity and linear function are related one to another has been discussed above. This situation leads us to the question how this three sectors are linked in the recent research. García (2005) studied about link between proportion and linear function. The results show that the two sectors have different relations Proportion is seen as relation between numbers, while in linear function is seen as a relation between quantities or variables. Then, it remains a question on how these relations are treated at the practice level and at the level of theory. Moreover, what are the explicit connections made in textbooks between proportion, similarity and linear function? The fact that proportion is disconnected from linear functions leads me to question the effects of the didactic transposition. This questioning also relates to the situation of proportion from a historical perspective: how was the topic treated in the past, for instance in textbooks? How is it treated in current textbooks? Based on the discussion above, the research question of this study are:

RQ1. How can the mathematical praxeologies of proportion, similarity, and linear function be described? And how can the praxis part be used to do quantitative analysis on textbooks' proposals for students' activities, primarily problems or exercises?

RQ2. What are the explicit connections, in the textbooks, among these three domains?

RQ3. What didactic transposition phenomena explain the current situation and the difficulties to change them?

These research questions are investigated in the papers listed on p.8 and included in this thesis. We analyse textbooks for the parts corresponding to the proportion sector (in Arithmetic) in the paper A. Then, in the paper B, we continue to analyse chapters corresponding to the similarity sector (in Geometry) and also consider how these chapters are related to the chapters about proportion. In the paper C, we analyse chapters corresponding to the linear function sector, and outline the relation between this sector and proportion. We summarise the links between the three sectors, and also add some further evidence, in the paper D. Lastly, in the paper E, we present a broader analysis of current and historical didactic transpositions, in order to explain the current situation and in particular the fact that the three sectors are presented in a relatively disconnected way. Notice that RQ1 is investigated in the paper A, B, and C, RQ2 is studied in the paper B, C, and D, andRQ3 is attacked in the paper E.

### 1.5 Outline Results

Schools in Indonesia are obliged to provide textbooks for students, based on limited government funding. Textbooks from private companies are expensive. To offer a more affordable solution, e-textbooks were launched by government from 2008. Students and teachers can download these textbooks from http://bse.kemdikbud.go.id/, for free. All the e-textbooks have been authorized by the government, but still, each of them has their own characteristics. Thus, schools and teachers continue to have a responsibility to choose which one is suitable for their students. In this study, for practical reasons, we mainly considered Indonesian textbooks corresponding to the 2006 curriculum to answer RQ1 and RQ2. However, we also use Indonesian textbooks based on the 2013 curriculum to answer RQ2 and RQ 3.

#### 1.5.1 Mathematical praxeology of proportion, similarity, and linear function and its quantitative analysis.

First of all, we will use didactic level of co-determination to locate proportion, similarity, and linear function in curricula and in textbooks. Based on the 2006 curriculum (Republic of Indonesia; Ministry of Education and Culture, 2006), proportion and linear functions are located within the algebra competence area, and similarity appears in the "geometry and measurement" competence area. Superficially, competence areas are similar to 'domains' in ATD. The curriculum is then to be interpreted in textbooks, and the government authorizes textbooks based on whether they adhere properly to the curriculum. As a result, Indonesian textbooks are similar in many ways, for instance in the selection of competence areas treated in the volume of the textbook for a specific grade. Thus, we can easily find the terms 'proportion, similarity, and linear function' in the titles of chapters, sub chapters, and in the main text. We analysed the chapters treating proportion in Arithmetic, in three textbooks for grade 7 (Nuharini & Wahyuni, 2008a; Wagiyo, Surati, & Supradiarini, 2008; Wintarti et al., 2008), while chapters treating similarity were found and analysed in textbooks for the ninth grade (Agus, 2007; Djumanta & Susanti, 2008; Dris & Tasari, 2011; Marsigit, Susanti, Mahmudi, & Dhoruri, 2011; Masduki & Budi Utomo, 2007; Wagiyo, Mulyono, & Susanto, 2008). Meanwhile, the sector on linear functions is located in Algebra for the eighth grade (Agus, 2008; Marsigit, Erliani, Dhoruri, & Sugiman, 2011; Marsigit, Susanti, et al., 2011; Nugroho & Meisaroh, 2009; Nuharini & Wahyuni, 2008b).

The first step in our method of analysis is to consider types of task and the corresponding techniques, as appearing in the "worked examples" of the main text. Here, techniques are explicit in the demonstrated solutions. We can then also identify types of task in the exercises appearing the chapter, assuming that techniques demonstrated in the worked examples will also be used to solve the tasks in the exercises, whenever it is possible. For instance, we may encounter the following worked example (translated from Indonesian):

The price of 2 m fabric is Rp. 45.000,00. How much does 10 m fabric cost?
Answer: The price of 2 m fabric is Rp. 45.000,00. So, the price of 1 m fabric is $=\frac{Rp.\ 45.000,00}{2}=$ $Rp. 22.500,00$. Thus, the price of 10 m fabric is: $10 \times Rp. 22.500,00 = Rp. 225.000,00$
(Wagiyo, Surati, et al., 2008, p. 120)

We introduce the relation $\sim$ on $\mathbb{R}^2$ defined by

$$(x_1, x_2) \sim (y_1, y_2) \Leftrightarrow \exists k \in \mathbb{R} : (x_1, x_2) = k(y_1, y_2).$$

Then we can categorize the above task as being of the following type:

T: Given $(x_1, x_2)$ and $y_1$ find $y_2$ so that $(x_1, x_2) \sim (y_1, y_2)$.

The corresponding technique is

$\tau$ : Calculate $y_2=\frac{x_2 . y_1}{x_1}$.

In this way, we identify the relevant types of task and techniques along with the data analysis. Whenever we find a new type of task, we extend the reference model with that types of task. It actually turns out that the same types of task appear frequently in all textbooks. We did find some types of task that appear only in one textbook and, usually, just once or twice. Some of these were not easy to categorize. For example:

Consider the statements below. Write B if the statement is always true, K if the statement is sometime true and S if the statement is always wrong.

Two parallelograms are similar
Two equilateral triangles are similar
Two rhombuses are similar
Two pentagons are similar

Sulaiman et al. (2008, p. 7)

In the task above, students are supposed to imagine a pair of polygons without specific angles and sides and are asked to decide if they are always similar, maybe similar, or never similar. Students need to use more theoretical knowledge on polygons to solve this task, which is therefore very different from common tasks in the textbook, calling on the use of more explicit and concrete techniques. Since such tasks appear very rarely and they are only in one textbook, we decided to omit it from the reference model.

In this first step, the reference model is developed based on the textbook itself. We do not discuss praxis that is not in textbook. Also, the collection of types of task and techniques does of course not take the theory blocks into account. Nevertheless the resulting model can be used for more than just to give a quantitative picture of the practice level of textbooks. We also used it to measure how close textbook's practice blocks are to the practices required at the national examination.

The second step of analysis relates to how textbooks discuss mathematics at the theory level. Here, we focus on the exposition in the main text. In lower secondary textbooks, theory is often quite 'informal'. Even so, we try to describe the technology that is discussed in the reference epistemological model section.

*Mathematical praxeology of proportion in Arithmetic*

At the practice level, we found seven types of tasks (and corresponding techniques) as our REM. These are divided into two clusters (corresponding to subsections in some textbooks): *ratio and proportion* and *direct and inverse proportion* (cf. table 2). At the theory level, we base the reference model mainly on technological explanations corresponding to the practice level. This is because the theory level for proportion does not contain many explicit, general definitions or results. Normally, textbooks only provide examples and informal discussions of properties.

To present the REM, we use algebraic symbolism. Positive real numbers and quantities are symbolized by letters ($x$, $y_j$ etc.). Our reference model includes an explicit theory in which proportionality that can be defined in terms of a relation between pairs of numbers or quantities. Specifically, two pairs $(x_1, x_2)$ and $(y_1, y_2)$ are said to be proportional if $x_1 \cdot y_2 = x_2 \cdot y_1$. In fact, this is equivalent to $(x_1, x_2) \sim (y_1, y_2)$ in the sense defined above. More generally, two n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are said to be proportional if $(x_i, x_j) \sim (y_i, y_j)$ for all $, j = 1, \ldots, n$ ; we then write $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$. The types of tasks, and corresponding techniques, which we identified in the textbooks analysed, can then be summarised as in Table 2.

| Ratio and proportion |
|---|
| $T_1$: Given $x_1$ and r, find $x_2$ so that $(x_1, x_2) \sim (1,r)$.<br>$\tau_1$: $x_2 = r.x_1$ (multiplying by the given ratio)<br>$T_2$: Given $x_2$ and r, find $x_1$ so that $(x_1, x_2) \sim (1,r)$.<br>$\tau_2$: $x_1 = x_2/r$ (dividing by the given ratio)<br>$T_3$: Given $x_1$ and $x_2$, find r so that $(x_1, x_2) \sim (1,r)$.<br>$\tau_3$: $r = x_2 / x_1$ (finding the ratio) |
| **Direct and inverse proportion** |
| $T_4$: Given numbers a, b and given that x > y are relations with a, b. Can it be true that $(x, y) \sim (a,b)$?<br>$\tau_4$: The answer is yes only if $a > b$.<br>$T_5$: Given $(x_1, x_2)$ and $(y_1, y_2)$, compare internal ratios.<br>$\tau_5$: Calculate $\frac{x_1}{x_2}$ and $\frac{y_2}{y_2}$, and compare.<br>$T_6$: Given $(x_1, x_2)$ and $y_1$ find $y_2$ so that $(x_1, x_2) \sim (y_1, y_2)$.<br>$\tau_6$: Calculate $y_2 = \frac{x_2 . y_1}{x_1}$.<br>$T_7$: Given $x_1$, $x_2$, $y_1$ find $y_2$ such that $(x_1, x_2)$ and $(y_1, y_2)$ are in inverse proportion.<br>$\tau_7$: Calculate $y_2 = \frac{x_1 . y_1}{x_2}$. |

Table 2. Type of tasks related to ratio and proportion in the Algebra domain

We use the REM to carry out a quantitative analysis of all tasks appearing the analysed textbooks, for instance to identify the dominant types of task in the different textbooks (i.e. many tasks of these types appear in all textbooks), and more generally to compare their "profiles" in terms of the mathematical practice blocks they support within the area of proportion. In relation to the model above, it is found that there are five dominant types of task, namely $T_1$, $T_2$, $T_3$, $T_6$, $T_7$.

*Mathematical praxeology of similarity*

There are eight types of tasks at the practice level; these are classified in two clusters, polygon similarity and triangle similarity (table 3). Proceeding as before, from constructing a REM to carrying out the quantitative analysis of all tasks, we find that $T_2$ and $T_3$ are the dominants types of tasks, and that these two are also dominant types of tasks in the written national examination on mathematics at the end of lower secondary school.

| Polygons similarity |
|---|
| $T_1$: Given two polygons P and Q, with given side lengths and given angles, determine if the two polygons are similar.<br>$T_2$: Given two similar polygons P and Q with given angles and given sides, identify what angles and sides correspond to each other.<br>$T_3$: Given two similar polygons P and Q as well as one side $p_1$ in P and two sides $q_1$, $q_2$ in Q with $p_1$ and $q_1$ being in correspondence, find the side $p_2$ in P that corresponds to $q_2$.<br>$T_4$: Given two similar polygons P and Q as well as sides $p_1, \ldots, p_n$ in P and $q_1, \ldots, q_n$ in Q, determine the scale factor of P and Q. |
| Triangles similarity |
| $T_5$: Given two triangles S and T with two or three angles known, determine if S and T are similar.<br>$T_6$: Given two triangles S and T as well as sides $s_1$, $s_2$, $s_3$ in S and $t_1$, $t_2$, $t_3$ in T, determine if they are similar.<br>$T_7$: Given two triangles S and T as well as sides $s_1$, $s_2$, and $\angle a_1$ that is located between $s_1$ and $s_2$ in S and sides $t_1$, $t_2$, and $\angle b_1$ that is located between $t_1$ and $t_2$ in T. Determine if they are similar.<br>$T_8$: Given a figure like Figure 1 with $\Delta ADE \sim \Delta ABC$, DE//BC, and given the length of three of four sides AE, AC, AD, AB. Find the remaining length. |

Table 3. Types of tasks about similarity in the Geometry domain

At the level of theory, the notion of similarity is actually quite complex and is difficult to state in a way which is both accessible to lower secondary students, and precise. In the lower secondary school textbooks, the definition of similarity is given in terms of a relation between polygons. Thus, two similar polygons are declared to be similar if only if the following two conditions are satisfied: 1. All pairs of corresponding angles are congruent, 2. All pairs of corresponding sides are proportional. We point out that the term 'corresponding' hides, in principle, a vicious circle. For example one cannot find out what angles "correspond" without knowing their angles; but according to the definition, the correspondence should be known *before* we check if the angles are congruent. A mathematically precise definition has to consider the different ways in which angles and sides of the two polygons can be ordered (assuming, for orderings, that consecutive sides are adjacent to the same angle, and that consecutive angles share a side). Specifically: two *n*-gons P and Q are called similar if there are *some* orderings:

$\angle a_1, \ldots, \angle a_n$ of the angles in P,

$\angle b_1, \ldots, \angle b_n$ of the angles in Q ,

$p_1, \ldots, p_n$ of the sides in P and

$q_1, \ldots, q_n$ of the sides in Q

such that $\angle a_1 = \angle b_1, \ldots, \angle a_n = \angle b_n$, and $(p_1, \ldots, p_n) \sim (q_1, \ldots, q_n)$.

The correspondence problem can also be solved more pragmatically. To check whether two polygons are similar, one can be found by arranging angles in increasing size, in both polygons, and observe if the same angles appear. Then, corresponding sides can be paired by applying those corresponding sides one located between corresponding angles. Finally, students can observe if the corresponding sides have equal ratios.

*Mathematical praxeology of linear functions*

"Linear functions" do not appear as the title a chapter nor as subchapter, but they are discussed in the main text of all textbooks analysed, more specifically in a more comprehensive chapter on functions. Thus, we investigate the tasks in this function chapter that relate to linear functions. Table 4 shows that five common types of tasks are found.

| | |
|---|---|
| $T_1$ | Given a function $f(x)$ on the set of integers, find the image of the function at specific integers. |
| $T_2$ | Given the closed form expression $f(x) = ax + b$ (where *a* and *b* are given), and given $f(x_0)$ for a given $x_0$, find $x_0$. |
| $T_3$ | Some values of a linear function are given at certain points, and students are asked to determine the correct function expression from a list. |
| $T_4$ | For a linear function $x \mapsto ax + b$, where $a$ or $b$ are given, along with one value of *f* at a point. Determine (the expression defining) *f*. |
| $T_5$ | Given the algebraic expression of a linear function $f(x)$ and another algebraic expression $E$ (depending on *x*, so $E = E(x)$). Compute $f(E(x))$. |

Table 4. Type of tasks about linear functions

At the theory level, we focus on how a textbook introduces or defines the term "linear function". Here is a typical textbook definition: A linear function is a function $f$ on the real numbers that is given by $f(\mathrm{x}) = \mathrm{ax} + \mathrm{b}$, where $\mathrm{a}, \mathrm{b}$ are real numbers and $\mathrm{a} \neq 0$ (Marsigit, Erliani, et al., 2011,

p. 51). However, what we could call *proportion functions* - functions given by formulae of the type $f(\mathrm{x}) = \mathrm{ax}$ could be an important class of functions to study, as it relates directly to the sectors of proportion and similarity discussed above. But it is never named in secondary level textbooks, even if proportion functions appear in examples.

The three studies of proportion, similarity, and linear function above can be considered as first effort to analyse the practice and theory level of textbooks based on praxeological reference models. This potentially contributes towards creating common measures for the analysis of content in textbooks within a country, or even between countries, and between textbooks and other official "generators of mathematical practice", such as the exercise collections from national examinations. Additionally, the analysis of the theory level can be used to capture less precise definition of key notions (above, we considered the examples of similarity and linear function) and thus locate potential didactical obstacles.

### 1.5.2 Connection between three domains

We have also studied the extent to which textbooks establish *connections* between the above sectors (in three different domains), that is, how proportion, similarity and linear functions are related in terms of explicit links between practice blocks, and between theory blocks. First, we discuss the question of connections at the practice level.  Then, we present the connections at the theory level. At the practice level, it is found that two techniques pertaining to proportion are potentially close to techniques related to similarity. For example, the missing value problems located in proportion and the missing sides located in similarity can be solved using very similar techniques represented by the formula $a = \frac{c}{d}.b$ (table 5).

Occasionally, we also find a type of task on linear functions that corresponds to a missing value numbers task on proportion.  In the following task, students are asked to identify the correct expression for an unknown function.

> The price of a pencil is Rp. 1.200,00, the price of two pencils is Rp. 2.400,00, and the price of 5 pencils is Rp. 6.000,00. Which of the following functions describe this?
>
> a. $f : x \to 1200x$
>
> b. $f : x \to 2400x$
>
> c. $f : x \to 1000x + 200$
>
> d. $f : x \to 1300 - 100$
>
> (Marsigit et al., 2011, p. 63)

Of course, similar "indirect" and implicit connections between sectors occur at the level practice, when, for instance, a technique related to proportion is adapted for use to solving tasks on similarity or linear function. However, no one of the textbooks we studied explicitly discussed these connections.

| Proportion | Similarity |
|---|---|
| T: Given a pair of numbers $(x_1, x_2)$ and a third number $y_1$, find $y_2$ so that $(x_1, x_2) \sim (y_1, y_2)$<br>$\tau$: Calculate $y_2 = \frac{x_2 . y_1}{x_1}$. | T: Given two similar n-gons P and Q as well as one side $p_1$ in P and two sides $q_1$, $q_2$ in Q with $p_1$ and $q_1$ being corresponding, find the side $p_2$ in P that corresponds to $q_2$.<br>$\tau$: Calculate the missing side using $p_2 = \frac{p_1}{q_1} . q_2$ |
| T: Given $x_1$ and $x_2$, find $r$ so that $(x_1, x_2) \sim (1, r)$.<br>$\tau$ : $r = x_2 / x_1$ (finding the ratio). | T: Given two similar polygons P and Q as well as sides $p_1$, …, $p_n$ in P and $q_1$, …, $q_n$ in Q, determine the scale factor of P and Q.<br>$\tau$ : $k$ is scale factor if $\frac{p_1}{q_1} = \frac{\cdots}{\cdots} = \frac{p_n}{q_n} = \frac{1}{k}$ (i.e.<br>$k = \frac{q_i}{p_i}$ for any corresponding scales $q_i$ and $p_i$). |

Table 5. Similar techniques are used in Proportion and Similarity

At theory level, proportion and linear function also implicitly share some similar elements (see table 6). As an example, to deal with a general notion of similarity, they may draw on prerequisite knowledge about quantities, and proportion. These relations are further discussed in paper D.

| Domain | | "Arithmetic" | "Algebra" | "Geometry" |
|---|---|---|---|---|
| Sector | Theory | "Ratio and proportion" | "Linear function" | "Similarity" |
| Theme | Technology | Quantities (e.g. mass)<br>Units (e.g. kg)<br>Ratio, scale variable<br>Direct proportion<br>Indirect proportion<br>Augmentation<br>Reduction<br>Table | Quantities (e.g. mass)<br>Units (e.g. kg)<br>Relation function<br>Linear function<br>Domain<br>Codomain variable<br>Range<br>Table<br>Graph<br>Cartesian diagram<br>Arrow diagram<br>Pair orders | Quantities (e.g. width, length)<br>Units (e.g. cm, km)<br>variable<br>Ratio, scale proportion<br>similar triangles<br>similar polygons<br>Polygon<br>Congruent<br>Augmentation and reduction<br>Parallel line<br>Pythagoras theorem<br>Thales theorem<br>Angles<br>Sides<br>Figures |

Table 6. Theoretical elements of proportion, linear function, and similarity

We suspect that many of these potential connections will not be well established for students until textbooks and teachers explicitly deal with the connections between 'proportion', similarity and linear function. For example, we have looked for the use of specialized terms like ‘scale’ and ‘proportion’. All of the textbooks use the term of ‘proportion’ in defining similarity. For example,

> ‘Two triangles are similar if the corresponding sides are proportional or the corresponding angles are equal’ Djumanta and Susanti (2008)

Meanwhile, the term of proportion is not appeared in linear function.

### 1.5.3 Didactic transposition phenomena of proportion

The weak connections found, at least in textbooks, between proportion similarity and linear functions, raise the question on what the “overarching” mathematical idea, proportionality, is all about. To answer this, we use (in Paper E) the notion of didactic transposition, from scholarly knowledge to knowledge actually taught. Additionally, we only consider three different periods of time in mathematics education e.g., classical mathematics, new mathematics, and the counter reform. For our analysis of scholar textbooks and lower secondary school textbooks, we only focused on chapters or subchapter entitled ‘proportion’. As a result of our investigations, in the classical mathematics, there were no functions, and relationships which we would now describe in terms of functions, were described in terms of proportion between quantities. The “New Math” reform, in many countries, replaced the classical mathematics by sets and functions between sets, here mainly number sets. During the counter reform and its variations, which last until today, there is a mixture: some parts of the classical organisation of ratios and proportions have come back, but without a thorough treatment of quantities, which are somehow superseded by notions originally related to the “New Math” organisation on functions (variables, sets, graphs, etc.). But the organisation of functions is not systematically used because it coexists with proportions; similar inconsistencies appear with fractions, decimal numbers, percentages and ratios. The result is a hybrid organisation, a mixture from different mathematical periods. We note that such “hybrid” organisations are not only found in mathematics; for example, such phenomena were described in relation to the teaching of gene function in upper secondary school by Gericke and Hagberg (2010), where inconsistent models from different historical periods also continue to be taught together.

## 1.6 Conclusions

The main point of this thesis is to show that praxeological analysis has many potentials for precise and critical textbook analysis, both when it comes to address the practice level (analysis of task types, including detailed profiling) and the theory level (links and transposition). Our praxeological reference models can be used both for qualitative and quantitative analysis at almost any level of detail. In a wider perspective, these reference models can potentially be used to analyse a wide range of didactical resources on proportionality, and thus also be used for international comparison of textbooks and for comparing textbooks with official documents such as curricula and national examinations. The praxeological reference model can also be used to discuss potential, but wholly or partially missed connections between sectors in different domains. In particular, we found that proportion and similarity have a similar pattern at practice level and share a similar element at the theory level, as do the sectors of proportion and linear functions. However, the textbooks we studied never mention explicitly these connections except in similarity sector. The fact that the textbook do not mention the connections explicitly contributes to our current interest in questioning how proportionality has been constructed by didactic transposition, and how it is presented not only in school textbooks but also in historic scholarly texts. In paper E we have presented some of our first takes on these questions. As a result, proportion is seen as connection between quantities in classical mathematics which, however, disappeared during the New Math reform. We found that there is a mixture mathematical organisation of proportion in today's curriculum and in particular, in textbooks. We suspect that this situation influence the disconnection between proportion and similarity and proportion in the textbooks. However, we need do further research to confirm this hypothesis.

The current weak disconnection of these three sectors also raises the question on whether and how links could help students profit more from each of them. The purpose of this link is not to say that any connection would improve the current curriculum. What we advocate is to build alternative choices on how to present potential connections between the particular mathematical pieces of knowledge which were considered in this thesis. Further research can also consider knowledge actually taught the final stage of didactic transposition. We can also broaden the scope of explanatory variables considered, in particular in terms of the levels of didactic codetermination *above* school level to consider, for example, how textbook companies affect the production of mathematics textbooks. But we need to keep the "roots" of our inquiry at the levels closest to the textbook users, with precise models at the level of themes and sectors, as demonstrated in this thesis.

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# 2 INCLUDED PAPERS

## A. Mathematical practice in textbooks analysis: praxeological reference models, the case of proportion

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**Abstract**

We present a new method in textbook analysis, based on so-called praxeological reference models focused on specific content at task level. This method implies that the mathematical contents of a textbook (or textbook part) is analysed in terms of the tasks and techniques which are exposed to or demanded from readers; this can then be interpreted and complemented by a discussion of the discursive and theoretical level of the text. The praxeological reference model is formed by the analyst to categorize various elements of the text, in particular the tasks and techniques which it explains or requires from readers. We demonstrate the methodological features of this approach by analyzing examples and exercises in three Indonesian textbooks, focusing on the chapters dealing with *arithmetic proportion* (defined theoretically by the model). We also illustrate how this rigorous analysis can be used to provide a quantitative "profile" of textbooks within a topic.

Keywords: textbooks, praxeology, proportion

### 1. Introduction

The importance of "tasks" (exercises, problems and so on) as a main component of students' mathematical activity is increasingly acknowledged by researchers (e.g. Watson & Ohtani, 2015). Indeed, it is a commonly held assumption of both mathematics teachers and researchers that "the detail and content of tasks have a significant effect on learning" (ibid., p. 3). While a school mathematics textbook may at first present itself as a treatise exposing various contents, one of its main functions is in fact be to be a repository of tasks - whether presented together with solutions (often in the "main text"), or proposed as work for students (often in a separate section or volume of "exercises"). Many teachers draw on textbooks as a main source of examples and exercises (Fan, Zhu, & Miao, 2013, p. 643). In choosing a textbook, teachers (or whoever make that decision) will therefore have a significant interest in the contents and quality of the tasks exposed or proposed in the book.

What can teachers (or others) do to examine textbooks from this angle? One can try to assess if the tasks are compatible with any official regulations of mathematics teaching (e.g., the national curriculum). However, such guidelines are not always precise to the point of specifying types of tasks which students should encounter or work on, and so they offer little guideline for analysing examples and exercises in a detailed way. One may also use any relevant national exams to see if the book aligns with types of tasks found there; but in many contexts, such a "measure" will be highly reductive or wholly irrelevant.

In practice, teachers will often depend on others' assessments and opinions about a textbook, such as reviews in magazines or websites for mathematics teachers. Some countries (e.g. Indonesia and Japan) even have a national agency that produces reviews of textbooks and authorizes their use in public schools. But whether such assessments are endorsed by authority or not, one can ask the question: what are they based on ? Against what common measure are books evaluated? Could this measure be based on explicit theoretical models, grounded in research? What kinds of theoretical models could enable a systematic and (ideally) reproducible means of analysing and synthesizing the qualities of textbooks, with a special emphasis on tasks?

Of course, analysing all tasks in a textbook could be quite time consuming. If indicators of the overall "quality" of a textbook are aimed for, it is natural to select a few topics which are usually considered problematic or challenging in teaching practice. These problematic topics will typically have attracted considerable attention of mathematics education research, so that the analysis of textbooks focusing on them will have a wide range of research literature to draw on. This could be helpful to set up a sharp theoretical model of the mathematical topic itself, understood as a practice and knowledge with which the text may engage the reader, through its explanations, exercises, etc.

The subject of this paper is the analysis of didactical texts with a focus on one or more mathematical topics - as a case, we consider Indonesian textbooks for grade 7, and the area of mathematics at this level which can loosely be referred to as proportion and ratio in arithmetic. Using this case, we propose a new methodological framework to analyse examples and exercises thoroughly. This framework is based on the anthropological theory of the didactic, and especially the notion of praxeology and praxeological reference model (see Barbé, Bosch, Espinoza, & Gascón, 2005).

The structure of the paper is as follows In Section 2, we present a selection of strongly related background literature for our case study, concerning student and teacher practices related to the proportion in arithmetic, textbook analysis at large, and research into textbook treatments of the proportion topic. In section 3, we introduce our theoretical framework for textbook analysis, based on the notion of praxeology of the anthropological theory of the didactic. In Section 4, we present the main result of the paper, namely a praxeological reference model for the topic of proportion, developed for and from a study of three Indonesian textbooks. As a supplement to the theoretical description of the model, illustrated by textbook excerpts, Section 5 contains a discussion of some methodological challenges and principles for applying the model, illustrated by concrete "limit" cases from textbooks. In Section 6, we show how the model may be used to produce a quantitative "profile" of the three textbooks; similar profiles could be made using the same model on other textbooks, possibly with a slight extension of the model. We discuss, in Section 7, this and wider perspectives of our methodology for producing explicit and systematic accounts of mathematical practices shown or elicited by a textbook within a given area.

## 2. Research background

In this section, we first review some of the main trends and methods available in recent research on mathematics textbooks, focusing on the precision with which topic is analysed. We then consider in more detail two recent studies on the proportion topic.

### 2.1Textbooks analysis

In a special issue of *Textbook Research in Mathematics Education* (vol. 45, issue 5, 2013), Fan et al. (2013) note the growth of research on mathematics textbooks during the past six decades; it is no longer a "new" field. Fan (2013, p. 773) considers that, in the wider perspective of improving textbooks or mathematics teaching, "it is only the first step to know what the textbooks look like, for example how a specific topic (e.g. algebra or geometry) is treated in a textbook or different textbooks, or how different types of problems are presented in a textbook or in textbooks in different countries". Indeed, (Fan, 2013, p. 774) also mentions that there seems to be a movement from "textbook analysis" towards "textbook research" which encompasses much wider empirical realms than the textbook itself (we could talk of a movement towards "zooming out"). At the same time, the first step may be far from completed - it concerns analytical research, based on solid methodological tools, on the finer details of the mathematical contents of the books. In fact, this paper begins from the premise that theoretical and methodological tools for such a *higher level* of granularity (that is, "zooming in") must be developed. Our analysis of mathematical contents in textbooks must be based on explicit models of such contents, rather than institutional point of view which is implicitly taken for granted.

As an example of research with this higher level of granularity, we refer to a study by Stylianides (2009) who developed an analytical approach to examine tasks (exercises, problems or activities) in American school textboooks for grade sixth, seven, and eight, considering both algebra, geometry and arithmetic. In this framework, Stylianides used 'providing proof' as one of task category and resulted that none of the exercises in the textbooks ask for "generic" (i.e. formal, "general") proofs, but instead asked students to provide various informal explanations, for instance based on a figure or computation. Stylianides' framework on reasoning and proving also played a significant role in a recent special issue of *International Journal of Educational Research* (2014, vol. 64, pp.63-148), focusing on special section: Reasoning and proving in mathematics textbooks: from elementary to the university level).

These categories are certainly specific to certain modalities of work in mathematics (argumentation, reasoning, proof) but they are completely generic with respect to the mathematical contents - the analysis works the same way for tasks on geometry and algebra (for example) and is largely insensitive to specific features of each of these content areas. In fact, concerning research on specific types of mathematical tasks in textbooks, we agree with González-Martín, Giraldo, and Souto (2013, p. 233) that the existing literature is extremely scarce.

In fact, our methodological approach has similarities to the one employed in the study by González-Martín et al. (2013), especially the use of the notion of praxeology to study tasks in textbooks; but the two methodologies also different, as we shall now explain. These authors investigated the case of the introduction of real numbers in Brazilian textbooks, based on a model which has, at its basis, rather broad classes of tasks for the students, such as $\mathcal{T}$: "Classifying a given number as rational or irrational". The broadness of this and other task classes considered in that paper stems from the multiplicity of techniques that may be used to solve a given task from this class. For example, for a task like deciding whether $5+\sqrt{3}$ is irrational or rational, textbooks provide a specific rule: 'the addition of rational and irrational number is irrational' which works here, if the solver knows that 5 is rational and $\sqrt{3}$ is irrational. The scope of this technique is quite limited (one needs only think of the case $\sqrt{0{,}1}$) and corresponds to a much narrower class of tasks than $\mathcal{T}$. The model still suffices to map out "large classes of tasks" which leads to remarkable characteristics of how the textbooks analysed treat the topic; but it does not exhaust the differences in terms of the precise technical knowledge which each of the books could develop among students. By contrast, our approach aims at classifying types of tasks in the precise sense of "tasks which can be solved by a given technique", and to draw up an explicit, precise model of the techniques.

## 2.2 Proportion in school textbooks

Students' and teachers' work with proportion and ratio (or proportional reasoning) is probably one of the most intensively studied topics in mathematics education research. In an early literature study, Tourniaire and Pulos (1985, p. 181) mention that "proportional reasoning has been the object of many research studies in the last 25 years". The authors give an interesting overview of research done during this period, which was largely dominated by cognitive paradigms of research; they also insist on the difficulty of describing explicitly the structure and boundaries of "proportional reasoning".

Research in the cognitive framework was, and is still, often based on test designs. These are of particular relevance to us because such designs sometimes indicate fairly detailed models of the mathematical components of the topic. For instance, to measure student difficulties with the different proportion type of tasks, (Hilton, Hilton, Dole, & Goos, 2013) designed a two-tier diagnostic instrument to measure the degree to which students master "proportional reasoning". However, the underlying reference model remains implicit in this and many similar studies: it seems that the authors take for granted that readers share the same idea about proportion or proportional reasoning; instead of definitions, the reader is left with the test instrument which, evidently, consists of examples of tasks, rather than explicit types of tasks described theoretically in terms of techniques. It cannot, thus, be used to classify tasks except if they are very similar to the test items, but it can serve as material for validating a given reference model in terms of whether it can classify the items.

In the literature, we find various useful theoretical distinctions of relevance to the theme of proportion, which have supported our model construction (Section 4). For instance, we note the four different kinds of ratio problems defined by van den Heuvel-Panhuizen (1996, p. 238): finding the ratio, comparing ratios, producing equivalents ratio, and finding the fourth proportional.

Considering textbook analysis, there are number of studies of proportion in textbooks based on broad models of students work with proportion. Dole and Shield (2008) developed a list of four "specific curricular content goals". Using these goals, the authors examined the extent to which these goals were pursued in two Australian textbooks. The authors later developed their model and extended the analysis to encompass five textbook series (Shield & Dole, 2013). Tasks and examples appear as illustrative cases of the analysis, but the corresponding content requirements (in terms of techniques) are not analysed.

We have also been inspired by a more fine-grained model,  developed by Hersant (2005) for the case of “missing number tasks”. Hersant developed a completely explicit model for the techniques identified in different programmes and corresponding textbooks. In terms of what we present in this paper, her model corresponds to a fine grained analysis of possible variations of a specific technique (the one called $\tau_6$ in section 5). Similar models were applied by García (2005) who focused on broader aspects of the transition between these arithmetic and algebra. Finally, Lundberg (2011) also focused on missing value tasks related to direct proportion. The studies of Hersant and Lundberg are based on the anthropological theory of the didactic, as the present paper, but consider only to illustrative cases while our model is used to characterize the arithmetical proportion topic as it appears in an entire textbook (cf. Section 6).

## 3. Theory and methodology

We now introduce our theoretical framework, based on the Anthropological Theory of the Didactic (ATD), in particular praxeological reference models and the levels of didactic co-determination. On this basis we introduce the context and methods of the present study.

### 3.1 Praxeologies

The basic idea of this study is to make full use the notion of *praxeology* from ATD,  proposed by Chevallard (1999). Praxeology means *praxis* and *logos*, to indicate that a praxeology is a model of some specific amalgam of human practice and knowledge. Concretely a praxeology is a 4-tuple  $(T, \tau, \theta, \Theta)$ where the four letters denote different, but closely related, components of the praxeology. While this notion is described in detail by several authors such as (Chevallard, 1999) and (Barbé et al., 2005) , it is so central to our work that we provide our own description here.

At the basis of a praxeology $(T, \tau, \theta, \Theta)$ we have a *type of tasks* $T$ that is a collection of tasks which can be solved by some *technique* $\tau$. Notice that $T$ and $\tau$ are in 1-1 correspondence: $T$ consists of the tasks which can be solved by $\tau$. Notice also that the term “task” in ATD simply means something humans can accomplish with a simple action (the technique); in mathematics, it could be some algorithm or other basic method. Since a praxeology is a model, it depends on the purpose of modelling what kind of human action it will be useful or feasible to distinguish as a technique; the theory does not provide any strict definition of what would count as a technique (and thereby, as a type of task). We note here that the main difference between our approach and the uses of ATD for textbook analysis provided by Lundberg (2011) and González-Martín et al. (2013) is the explicit definition of techniques (presented in Section 4), which enable us to work with types of tasks (in the proper sense of ATD, that is, defined by one technique) rather than the informal use of the term type of task as “a collection of tasks with a similar form and content”.

In many contexts (certainly those involving mathematical practice) it is essential to be able to describe and justify techniques. This leads to a “discourse about the technique” which is the element $\theta$ in the praxeology. Because $\theta$ represents “logos about techniques”, it is called a *technology* in ATD (not to be confused with every uses of the term). Finally, the “practical discourse” of how to do task (the technology) is complemented by a discourse about the technology itself, the *theory* $\Theta$. This discourse allows us to challenge, combine and explain the practical discourse independently from specific techniques; for instance, the problem of solving polynomial equations can be discussed at a theoretical level through definitions and existence theorems, and this discourse can then serve to relate, compare, explain and validate concrete techniques for solving more specific kinds of polynomial equations.

A *reference praxeological model* for some human activity is then simply an explicit description of praxeological elements $(T, \tau, \theta, \Theta)$ which we use as a reference for analysing the activity. The model can be more more or less detailed according to the purposes of our analysis.

### 3.2 Levels of didactic co-determination

The study of textbooks is full of indications of institutionally stable ways of organising the practice and knowledge which the books aim to engage the students with. First of all, the textbook will usually indicate the school type and age level it is meant for, as well as the discipline - for instance, one the textbooks analysed in our study (Nuharini & Wahyuni, 2008; Wagiyo, Surati, & Supradiarini, 2008; Wintarti et al., 2008) has the full title (in English translation): “Mathematics 1: concepts and applications for grade 7, SMP/MTs”. Here, SMP/MTs denote two kinds of junior high school in the Indonesian school system, “1” refers to the first year in junior high school, and “7” to the grade while counting also the preceding six years in elementary school. “Mathematics” naturally refers to the school subject which, in turn, can be seen to consist of several levels and elements that are apparent in Chapter and Section headings.

ATD provides a hierarchy of explicit *levels of didactic co-determination* to help explicate and examine these “layers” of organising and structuring the teaching of praxeologies in institutions, usually called *schools.* We do not use the whole hierarchy in this paper, but we will need to use the following levels precisely and coherently:

- The *discipline* is here the school subject mathematics (in Indonesian lower secondary school)
- The *domain* within mathematics is “arithmetic” (cf. Section 4). In general a domain is a larger part of a discipline which unifies a number of different theories.

- A *sector* is defined by a theory, unifying several praxeologies (sometimes called a *regional organisation*). The one considered in this paper is defined in Section 4 and concerns "proportion" of numbers, which appear also in many social practices outside the school.
- A *theme* is defined by a technology and thus unifies related techniques and types of tasks; it is located within a sector. For instance, "ratio and scale" indicates a discourse with the notions "ratio" and "scale" and central tools to describe and justify specific techniques of calculation within the proportion sector.

We notice that besides textbooks, these levels also appear more or less directly in national curricula of many countries, and while "mathematics" is a discipline in schools almost everywhere, the lower levels may display larger variation.

**3.3 Our context**

Several factors motivate the special interest of analysing and assessing textbooks for Indonesian schools, for instance:

- The sheer number of students who could, in a given year, be using a textbook (According to Statistic Indonesia (2013) there are 12.125.397 grade 7, 8, 9 students in Indonesia in 2013 ; all are taught in the same language and according to the same national curriculum)
- The fact that only 37% of the teachers who have the required education level (The World Bank, 2011) results in a dependency on textbooks by many Indonesian teachers.

Indonesia has nine years of general, compulsory education (6 years of elementary school for students aged 7-13, and 3 years of lower secondary school for age levels 13-16). All authorised textbooks are made available electronically and can be downloaded at www.bse.go.id.

In the Indonesian curriculum, students are supposed to learn proportion within the arithmetic domain during the first grade of lower secondary school. However, the curriculum does not specify the detailed contents of the sector "proportion". Thus, one might expect a large variation in how textbooks treat the theme. In this paper we analysed the proportion sector as it appears in the following three textbooks, which are the only textbooks which are both authorized for grade 7 in the year 2014 and digitally available : Nuharini and Wahyuni (2008), Wagiyo et al. (2008), and Wintarti et al. (2008). The digital (online) access of the three books means that they are widely used. These textbooks were all produced in 2008 at the occasion of a major curriculum reform.

### 3.4 Methodology

The way to construct and use a praxeological reference model needs further explanation. First of all, the model is not constructed independently from the material to be analysed, but it is constructed *along with the analysis* and serves, in the end, to make that analysis completely explicit. It should then also be *reproducible* in the sense that the same analysis would be made by other researchers who have familiarized themselves with the model.

Next, to analyse a sector we need to identify what part of the textbook it corresponds to. As Indonesian textbooks follow the national curriculum quite closely, it is easy to identify the parts of the textbooks which correspond to proportion. Then, within these parts of the books, we begin to analyse all *examples* to identify the *techniques* they present students with, and the corresponding types of tasks. The examples give us explicit information about the techniques which students may use when solving exercises. The exercises are then solved and analysed in terms of the types of tasks found in examples; if needed, new types of tasks are added to the model, in order to be able to classify and to describe all exercises precisely and objectively. We emphasize that the reference praxeological model is as much a result as it is a tool of our analysis.

## 4. Praxeological reference model for the sector of proportion

In accordance with the literature reviewed in Section 2, we consider *proportion* as concerned with numbers and quantities, thus belonging to arithmetic in the broad sense of "calculation with positive real numbers" (possibly with units and occasionally including also zero) in school and other social contexts. We note here that a *quantity* can be seen, abstractly, as a positive real number together with a unit, such as 0, 75 litre or 5 apples. Here, the unit (litre or years) corresponds to some measure that the number "counts". In the domain of *algebra, one* can consider magnitudes as products of numbers and unit symbols, but with the domain of arithmetic, units have to treat with more semantic than syntactic means of control; in particular, operations are done only with numbers, and the questions of units must be handled separately, with reference to the context of measurement.

In our reference model, *proportion* will actually be a *sector* within the *domain* of arithmetic. It is unified by a *theory* that keeps together the two *themes* which the sector consists of; each of the themes has their explicit technology, which in turn unifies and relates the *subjects* within the theme. We first describe the theory level of our model which, in fact, is quite distant from the texts we have analysed, but which is indispensable for the describing and applying the rest of the model (the themes and subjects) with precision. Then, we present two themes that provide types of tasks and techniques in English translation.

### 4.1 Basics of a theory of proportion

A systematic reference model requires precise notation and terminology. As researchers, we establish this from the basis, in our own terms (while it is be inspired from the literature reviewed above, especially (Miyakawa & Winsløw, 2009), the "principle of detachment" (Barbé et al., 2005) is a main point of ATD, to avoid whole sale assumption of established institutional jargon, or even of ideas and terms that are often taken for granted by scholars.

In the following we designate numbers or quantities by letters ($x$, $y_j$ etc.) to describe a theory which involves *only numbers and quantities* (and only occasional "letters" in their place, in the case of "unknowns" to be determined). In the rest of this section, letters are understood to represent positive real numbers or quantities.

**Definition.** Two pairs $(x_1, x_2)$ and $(y_1, y_2)$ are said to be *proportional* if $x_1 \cdot y_2 = x_2 \cdot y_1$; we write this in short as $(x_1, x_2) \sim (y_1, y_2)$. More generally, two *n*-tuples $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are said to be proportional if $(x_i, x_j) \sim (y_i, y_j)$ for all $i, j = 1, \dots, n$ ; we then write $(x_1, \dots, x_n) \sim (y_1, \dots, y_n)$.

It is easy to prove that $\sim$ is an equivalence relation on $\mathbb{R}_+^n$ for all *n* = 2,3,... (one can make use of P1 below). A number of other useful properties of this relation are listed below, where, for the sake of brevity, we just formulate the results for 2-tuples:

- P1. If we define the *internal ratio* of a pair $(x_1, x_2)$ as $\frac{x_1}{x_2}$, then $(x_1, x_2) \sim (y_1, y_2)$ is logically equivalent to equality of the *internal ratios* $\frac{x_1}{x_2}$ and $\frac{y_1}{y_2}$.
- P2. Similarly, $(x_1, x_2) \sim (y_1, y_2)$ holds if and only if the *external ratios* $\frac{x_1}{y_1}$ and $\frac{x_2}{y_2}$ are equal (notice that external ratio concerns *two* tuples, while internal ratio depends only on one).
- P3. If $(x_1, x_2) \sim (y_1, y_2)$ and $x_1 < x_2$, then $y_1 < y_2$.
- P4. If $(x_1, x_2) \sim (1, r)$ if and only if $\frac{x_2}{x_1} = r$ , that is, if and only if $x_2 = r \cdot x_1$.

We finally note that for 2-tuples $(x_1, x_2)$ and $(y_1, y_2)$, the property $(x_1, x_2) \sim (y_2, y_1)$ is sometimes called *inverse proportion*; when it holds, we say that $(x_1, x_2)$ and $(y_1, y_2)$ are *inverse proportional*. This does correspond to a relation on 2-tuples which, however, is not an equivalence relation (it lacks transitivity); also it does not have natural generalisation to *n*-tuples. It is, nevertheless, as the definition also shows, closely related to proportion (sometimes called "direct proportion" to distinguish it from inverse proportion).

With this theoretical basis of the sector we can now describe the rest of our reference model, consisting of two themes, each constituted by a number of types of tasks.

**4.2 Theme 1: Ratio and scale**

Property P4 above deals with the special case of proportion where one of the tuples is of the form $(1, r)$. This case is closely linked to a technology involving *ratio* and *scale*. Both terms refer to the number *r* in P4 (and thus a property of a single pair of numbers); we use *scale* for the special cases where *r* or 1/*r* is an integer, and ratio for the general case. In any cases, when *r* is a fraction of integers *m*/*n*, the notation *m*:*n* is often used, as in the alternative formulation $x_2: x_1 = m: n$ of the characteristic property in P4. In Table 1, we present three tasks $(\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3)$ that exemplify the three types of tasks in the theme ratio and scale.

| | |
|---|---|
| $\mathrm{t}_1$ | The price of eggs was Rp. 10.000,00 per kg. But the price of egg increased 6:5 from the original price. What is the current price of egg per kg?<br>Answer:<br>Current price: original price = 6:5<br>Current price = $\frac{6}{5} \times \text{Rp.}\,10.000{,}00$<br>=Rp. 12.000,-<br>(Nuharini & Wahyuni, 2008, p. 148) |
| $\mathrm{t}_2$ | A mother gives R.p 5.000,00 for pocket money to a kid. $\frac{2}{5}$ of pocket money is used to buy stationery. How much pocket money is left ?<br>(Nuharini & Wahyuni, 2008, p. 147) |
| $\mathrm{t}_3$ | Ali saves Rp. 300.000,00 in the bank and Budi saves Rp. 450.000,00. Determine the ratio of Alis' saving and Budi's saving?<br>Answer:<br>ratio $= \frac{\text{Rp.}300.000{,}00}{\text{Rp.}450.000{,}00} = \frac{2}{3}$<br>(Wagiyo et al., 2008, p. 115) |

Table 1: Tasks that exemplify the types of tasks in the theme « ratio and scale »

In table 1, $\mathrm{t}_1$ and $\mathrm{t}_3$ are tasks that appear in an example in the textbook quoted. Thus, the technique can be read off from the quote. We found that although $\mathrm{t}_1$ and $\mathrm{t}_2$ look quite similar at first, they are different because the given ratio should be applied differently (multiply or divide) and it is a real point that students should distinguish and choose among those options. The task $\mathrm{t}_3$ is clearly different as the students are asked to compute the ratio. We note that tasks involving scale (as defined above) are not included in the table, and could look as follow (this is an example of a task of the same type as $\mathrm{t}_1$).

> A map has a scale 1: 2.000.000 and distance between A and B on the map is 3,5 cm.
> Determine the real distance from A to B.
> (Wagiyo et al., 2008, p. 113)

Based on analysing these and many other tasks occurring in the textbooks, we defined three types of tasks ($\mathrm{T}_1 - \mathrm{T}_3$), and the corresponding techniques($\tau_1 - \tau_3$), as shown in Table 2; the connection between tables 1 and 2 is, naturally, that $\mathrm{t}_\mathrm{i}$ is of type $\mathrm{T}_\mathrm{i}$ (i=1,2,3).

| Type of task | Technique |
|---|---|
| $\mathrm{T}_1$: Given $x_1$ and $r$, find $x_2$ so that $(x_1, x_2) \sim (1, r)$. | $\tau_1$: $x_2 = r.x_1$ (multiplying by the given ratio) |
| $\mathrm{T}_2$: Given $x_2$ and $r$, find $x_1$ so that $(x_1, x_2) \sim (1, r)$. | $\tau_2$: $x_1 = x_2/r$ (dividing by the given ratio) |
| $\mathrm{T}_3$: Given $x_1$ and $x_2$, find $r$ so that $(x_1, x_2) \sim (1, r)$. | $\tau_3$: $r = x_2 \,/\, x_1$ (finding the ratio) |

Table 2: Types of task related to ratio and scale.

### 4.3 Theme 2: Direct and inverse proportion

In theme 1, the tasks really involve only one tuple; the implicit tuple (1, *r*) is either completely identified with one number (the ratio). We now proceed to a theme which is unified by a technology on certain relations between *two* tuples (most often, but not always, 2-tuples); these can either be *directly* or *inversely* proportional; both relations have important and common examples in real life (e.g. s. Here, we identified four types of tasks. As before, we first give characteristic examples for each of them (Table 3).

| | |
|---|---|
| $\mathrm{t}_4$ | The order of numbers in a proportion must be correct. Indicate for each statement if it is false:<br>a. My age: father's age = 4:1<br>b. Population of Jakarta: population of Bandar Lampung = 1:10<br>c. Toni's age: Toni younger brother's age = 3:2<br>(Wagiyo et al., 2008, p. 116) |
| $\mathrm{t}_5$ | In Bu Ina's grocery, the price of a package containing 2 kg of sugar is Rp. 9.400,00 and the price of a package containing 5 kg of sugar is Rp. 22.750,00. Which package is cheaper? What would you do to solve that problem?<br>(Wintarti et al., 2008, p. 194) |
| $\mathrm{t}_6$ | The price of 2 m fabric is Rp. 45.000,00. How much does 10 m fabric cost?<br>Answer:<br>The price of 2 m fabric is Rp. 45.000,00.<br>So, the price of 1 m fabric is $= \frac{Rp.\ 45.000{,}00}{2} = Rp. 22.500{,}00$.<br>Thus, the price of 10 m fabric is: $10 \times \mathrm{Rp.}\, 22.500{,}00 = \mathrm{Rp.}\, 225.000{,}00$<br>(Wagiyo et al., 2008, p. 120) |
| $\mathrm{t}_7$ | A package of candies was distributed to 20 children, so that each child receives 10 candies. How many candies would each child receive if the same package of candies were distributed to 50 children?<br>Answer:<br>20 children= 10 candies<br>50 children= n candies<br>Based on inverse proportion, one gets $\frac{20}{50} = \frac{n}{10} \Leftrightarrow 50 \times n = 20 \times 10 \Leftrightarrow n = 4$<br>(Wagiyo et al., 2008, p. 124) |

Table 3: Tasks that exemplify the types of tasks in the theme « direct and inverse proportion »

The corresponding types of tasks are shown in Table 4. Notice that the technique $\tau_4$ can be justified by property P3 (proportional tuples have the same order relations). As regards direct and inverse proportion, these four types of tasks exhaust almost all exercises and examples in the three textbooks; the exceptions and limit cases are discussed in the next section.

| Type of task | Technique |
|---|---|
| $T_4$: Given numbers $a, b$ and given that $x > y$ are relations with $a, b$. Can it be true that $(x, y) \sim (a, b)$? | $\tau_4$: The answer is yes only if $a > b$. |
| $T_5$: Given $(x_1, x_2)$ and $(y_1, y_2)$, compare internal ratios | $\tau_5$: Calculate $\frac{x_1}{x_2}$ and $\frac{y_2}{y_2}$, and compare. |
| $T_6$: Given $(x_1, x_2)$ and $y_1$ find $y_2$ so that $(x_1, x_2) \sim (y_1, y_2)$ | $\tau_6$: Calculate $y_2 = \frac{x_2 \cdot y_1}{x_1}$. |
| $T_7$: Given $x_1, x_2, y_1$ find $y_2$ such that $(x_1, x_2)$ and $(y_1, y_2)$ are in inverse proportion | $\tau_7$: Calculate $y_2 = \frac{x_1 \cdot y_1}{x_2}$. |

Table 4: Types of task related to "direct and inverse proportion"

## 5. Methodological remarks

In this section, we discuss some methodological challenges we encountered with the above model, above all tasks which we found hard or impossible to classify with it. These occur in four main groups.

### 5.1 Combination with techniques from other sectors

Many exercises contain more than one question and each of these can be a task, or a combination of tasks, in the sense of ATD. In order to relate "old knowledge" with the knowledge taught in a given chapter, exercises may draw on other sectors besides that of the chapter. Specifically, when analysing exercises from a chapter on proportion, some of the techniques required to solve the exercise may come from other sectors and even domains; we then simply disregard this part in our analysis. However, sometimes the two techniques (one from the sector we study, one from without) may be rather difficult to separate, or we need to make strong assumptions to classify the tasks. We found two such cases in the three books: one exercise (Nuharini & Wahyuni, 2008) in which students need to use knowledge about similar triangles (and then solve a task of type $T_1$), another one in which substantial modelling needs to be done from a described situation before one gets to an inverse proportion problem (of type $T_7$). Our model can only be used to account for the proportion part of these exercises.

## 5.2 Combinations of two techniques can replace a third

In some cases, a technique is equivalent to the combination of two other techniques. Here is a typical case of a problem for which both the simple technique and the combination appear quite naturally, taken from an example in a textbook (Table 5)

| |
|---|
| A car needs 3 litre of gasoline to go 24 km. How many kilometres can the car reach with 45 litres of gasoline? |

Table 5. A task with combination of two techniques can replace a third (Nuharini & Wahyuni, 2008, p. 152)

There are three known numbers (3, 24, and 45) and students are asked to find one unknown number. The textbook demonstrates two solutions to the problem above (see Table 6).

Solution:

**1st approach:**

With 3 litres of gasoline, a car can go 24 km, thus 1 litre gasoline can reach = $\frac{24}{3}$ km = 8 km

The distance that can be reached with 45 litres of gasoline is 45 x 8 km= 360 km

**2nd approach:**

| gasoline | Distance |
|---|---|
| 3 liter | 24 km |
| 45 liter | $x$ |

$x = \frac{45}{3} \times 24\ km = 36\ km$

Thus, the distance that can be reached with 45 litres of gasoline is 360 km

Table 6: two solutions to the same problem (Nuharini & Wahyuni, 2008, p. 152)

In the first solution, the authors are using $\tau_3$ to find the ratio. Then, the answer can be found by multiplying the result with the ratio, following $\tau_1$. In the second solution, the technique $\tau_6$ is used. Thus, the simple technique $\tau_6$ is in fact shown to be equivalent to a combination of two techniques $(\tau_3 + \tau_1)$. Both approaches result in the same answer, however, in the first solution, some extra information is produced, namely the distance which the car can run on one litre of gasoline, while this is not asked for in the problem itself. In view of the form of the question (three given numbers, one to be found), we decided to count this task only as belonging to $\mathrm{T}_6$ and to treat similar exercises in the same way. Even though students can develop their reasoning by using $\tau_3 + \tau_1$, we have classified this task in $\mathrm{T}_6$, based on the simplicity of $\tau_6$ that would make it a more likely choice for students, in comparison to the more complicated one $(\tau_3 + \tau_1)$.

### 5.3 Combinations of two techniques

For other - much rarer - problems, it is actually necessary to combine two techniques. Table 7 shows an instance, which is based on the same notation as the case considered in Section 5.1.

| | |
|---|---|
| Determine $: y: z$ . | |
| a. | $x: y = 1: 2$ and $y: z = 3: 4$ |
| b. | $x: y = 2: 3$ and $y: z = 4: 5$ |

Table 7. A task with combination of two techniques (Wagiyo et al., 2008, p. 123)

This problem requires that one combine the ratios of two couples which are related to each other because the second element of the first couple is identical to the first element of the second couple. For instance, to solve task ‘b’, one can first use $\tau_3$ and then $\tau_6$, as follows:

$(2, 3)\sim(1, r)$ gives $r = \frac{3}{2}$ $(\tau_3)$ ;

$\left(\frac{3}{2}, y_2\right)\sim(4,5)$ gives $y_2 = \frac{\frac{3}{2}\cdot 5}{4} = \frac{15}{8}$ $(\tau_6)$. So $x: y: z = 1: \frac{3}{2}: \frac{15}{8}$.

Unlike the case considered in section 5.2, the task cannot be solved directly by one of the simple techniques of the model, so the use of two techniques is actually needed. We classify this problem as containing a task of type $T_3$ and a task of type $T_6$.

### 5.4 Non-classified problems

In the three textbooks chapters selected as explained in Section 3.4, we encountered only three exercises which we could not at all classify according to the model. Two of them do not belong the domain of arithmetic (no numbers are given), and the third, the exercise belongs to the domain of algebra, more specifically to the sector of solving equations involving proportions). One can also say that these tasks concern the level of technology or theory of arithmetic tasks (for instance the first is related to identify phenomena relatively to types of tasks in arithmetic). Thus, these three exercises are simply not classified, even if they may indeed be considered of relevance to the solution of more concrete tasks actually belonging to arithmetic praxeologies.

## 6. An application: a quantitative survey of textbooks

We now present the result of applying the reference model to the three textbooks. In the parts of the three textbooks that were identified with the sector “Proportion”, we found in total 30 tasks located in examples, and 276 tasks located within exercises. For each book, we first classified the tasks that occurred within examples (Table 8) and then the tasks within exercises (Table 9). Most tasks located in exercises are of a type already located in examples; in this case, we classified the task as belonging to that type (with exception of the case mentioned in Section 5.2, where both a simple technique and a combination of techniques were demonstrated in an example). All three textbooks have exercises with tasks that cannot be solved by techniques demonstrated in a worked example within the book (and hence appear in Table 12 but not in Table 11). These tasks tend to be exceptional and some of them gave rise to specific (new) types of tasks in the reference model ($T_4$ and $T_5$).

| | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ |
|---|---|---|---|---|---|---|---|
| Wagiyo et al. (2008) | 1 | 1 | 8 | 0 | 0 | 5 | 1 |
| Wintarti et al. (2008) | 1 | 1 | 2 | 0 | 0 | 2 | 0 |
| Nuharini and Wahyuni (2008) | 2 | 0 | 2 | 0 | 0 | 3 | 1 |
| | 4 | 2 | 12 | 0 | 0 | 10 | 2 |

Table 8. Types of tasks in the examples, numbers of occurrence.

| | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ |
|---|---|---|---|---|---|---|---|
| Wagiyo et al. (2008) | 14 | 2 | 62 | 4 | 4 | 26 | 18 |
| Wintarti et al. (2008) | 11 | 6 | 40 | 0 | 1 | 18 | 9 |
| Nuharini and Wahyuni (2008) | 7 | 8 | 24 | 0 | 0 | 10 | 12 |
| | 32 | 16 | 126 | 4 | 5 | 54 | 39 |

Table 9. Types of tasks in the exercises, numbers of occurrence.

The most eye-catching in these two tables is the similar pattern we find in the two textbooks: the sector “proportion” is, essentially, constituted by five types of tasks ($T_1$, $T_2$, $T_3$, $T_6$, and $T_7$) which account for 90% of the examples and 81% of the exercises. These dominant tasks have numerous occurrences in exercises and appear also as examples.

Many Indonesian teachers follow the textbooks closely when structuring and carrying out their teaching. One could therefore expect that these five dominant types of tasks capture most of the “realised” curriculum in Indonesian schools, as far as proportion is concerned. However, in the national curriculum for lower secondary school, there is no detailed discussion on how proportion should be taught and certainly nothing as precise as these types of tasks is even mentioned. Nevertheless, our analysis of textbooks (in this case, three state authorized textbooks used in almost every school) reveals these five types of tasks as a national “profile” of the proportion sector within arithmetic. While this profile cannot be traced to the curriculum, it seems to be well rooted in the didactic tradition of the school institution, which is especially carried and continued by textbooks.

## 7. Discussion

As illustrated by the short quantitative overview of the three textbooks, the praxeological reference model presented in Section 4 can be used to identify five dominant tasks which, together, form the core of the proportion sector in Indonesian school. At the same time, the model allowed us to single out a few exceptional types of tasks which complete the two themes of the sector and adds some autonomy to the student activity which the books can generate. We conjecture that different "exceptional" types of tasks may be found in other Indonesian textbooks (non-authorized or older) while the five dominant types would probably also dominate there. At any rate, both the similarity and differences in the mathematical core of the textbooks' treatment of the sector appears from a presentation such as given in Table 8 and 9.

In this paper, we have focused on types of tasks and techniques. In other words, we have not analysed corresponding technology or theory presented in the textbooks, which we will consider elsewhere; this will be of particular importance for analysing the connections with other domains, such as algebra and geometry. Similarly, we have not considered the ecological aspect of proportion, i.e. institutional conditions and constraints of Indonesian school, which are necessary to *explain* (rather than to analyse) the shape of the themes in the present textbooks, or to discuss alternative designs, *raison-d'être* of the themes, etc. Thus, this paper is far from exhausting the potential of text book analysis based on ATD. However, we claim that such an analysis will have to include, at its basis, an analysis of the granularity and precision demonstrated in this paper, and that our approach shows more generally that such a granularity with respect to the mathematical content of examples and exercises is indeed possible and useful in textbook analysis.

Our main point in this paper was, thus, to give a first demonstration of how the notion of praxeological reference model enables us to analyse the mathematical core of textbooks in a quite objective and detailed way, which could contribute to "common measures" for both comparative and historical studies of how a sector or theme appears in mathematics textbooks. As regards the practical level of exercises and examples, which is crucial to the mathematical activity it can support among students, teachers can use such a reference model to examine a textbook. For example, a teacher may compare the type tasks found in a textbook to those appearing in national examinations. Also for textbook authors, comprehensive analyses of themes as given in Section 6 may be useful to consider, in order to develop a more deliberate profile than what can be done by personal experience and more or less arbitrary variation of single types of tasks.

We acknowledge that the methodology proposed here only attends to certain specific aspects of textbooks, while leaving others untouched. It mainly focuses on mathematical themes, not - for instance - on the use of daily life contexts, style of presentation, or connections with other themes. It also does not question the ecology of the textbooks, for instance, the coherence or genesis of the national curriculum, or the conditions under which the textbooks are used in Indonesian schools.

For further research, it is also important to strengthen the reference model by applying it on different textbooks from different contexts (e.g. private textbooks or foreign textbooks). Including a wider array of empirical data, the reference model will not only have to be extended, but will also gain in solidity and use, for instance for comparative purposes. Finally, we currently work on extending the reference model to include themes from other related domains, such as similarity in plane geometry and linearity in algebra. This will enable us to identify actual or potential relations between the three domains, which are naturally important qualities in textbooks - in the absence of establishing explicit links between themes, they will tend to support the "thematic autism" (Barbé et al., 2005) which can be identified as one of the main challenges of school mathematics.

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## B. A reference model for analysing textbook treatment of similarity in plane geometry

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**Abstract**

The aim of this paper is to present a reference model for analysing how the topic “similarity” (or proportion, in the setting of plane geometry) is treated in lower secondary textbooks; it is based on a meticulous examination of six Indonesian textbooks. To construct the reference model, we use the notion of praxeology from the anthropological theory of the didactic. The reference model is applied to investigate explicit and implicit relations between the textbooks’ treatments of similarity (in geometry) and proportion (in arithmetic). The practice level of the model is also applied to provide a quantitative measure of how close the textbooks’ treatment of similarity is in the way the topic is assessed at the national examination at the end of lower secondary school.

Keywords: analysis textbooks, praxeological reference model, similar figures, proportionality

### 1. Introduction

Mathematics textbooks do not only present *knowledge* or “general facts”, but are also important tools for teachers to engage students in mathematical activities or *practice*; mathematics textbooks always include worked examples and exercises. They are an essential component for the teachers’ daily work with the textbook and so, in principle, for their choice of using it. To provide a useful and complete analysis of a mathematics textbook, the consideration of examples and exercises is therefore crucial. The precise analysis of practices is especially relevant when the textbook is used in contexts preparing for a centralized, high stakes examination. In such contexts, it is common that teachers concentrate students’ work on types of tasks which appear in the examination. One can then ask: to what extent does the textbook present a similar concentration? Does it explicitly emphasize examples and exercises which are closely related to the examination? To what extent does it enable student work on types of tasks which are not frequently appearing at the examination? To choose a textbook, teachers may rely on rather informal answers to these questions; a more precise tool is presented in this paper, for a specific topic (similarity in plane geometry).

At the same time, more global features and connections within the textbook are important, and should be related to the more local analysis of the practices which the textbook proposes to students. For instance, a central result on similarity of polygons concern the proportional relationship of corresponding sides; this could be explained with more or less explicit mention of the relation to the work done on ratio and proportion in arithmetic (usually in earlier grade). Explicit connections between different mathematical domains (here, geometry and arithmetic) are considered important to teach students that mathematics is a connected body of knowledge, and to avoid the tendency of "thematic autism" - that is, mathematics teaching that makes students visit one theme after the other, with no connection between them (Chevallard, 2006, 2012). We can make such a connection with the analysis of proportion in arithmetic in textbooks, already presented by Wijayanti and Winsløw (2015). As noted in that paper, textbooks at lower secondary level do not present much explicit theory; still, relations between the two domains may be identified both at the level of practice (e.g., exercises and techniques to solve them, c.f. Wijayanti, 2015) and at the level of more informal, expository discourse in the textbook. In this paper, we investigate the relations between similarity and proportion, as presented in the textbooks at both levels.

Concretely, in this study, we construct a reference model to analyse the treatment of similarity in Indonesian lower secondary school textbook. We show that this reference model can also be used to consider the extent to which the practice on similarity, proposed by the textbook, aligns with the items on similarity which appear at the national examinations of Indonesian lower secondary school. Finally, we discuss explicit and implicit relations, established in the textbooks, between similarity of plane figures on the one hand, and ratio and proportion on the other. In Indonesia, the last topic is treated in 7$^{th}$ grade and the first topic in 9$^{th}$ grade. So we look for these relations in the 9$^{th}$ grade textbooks' chapters on geometry.

This paper consists of seven sections. First, we present the necessary literature background on textbook analysis, especially studies of texts on geometry. Then, we also provide a short introduction to the anthropological theory of the didactic (ATD) as a new approach in textbook analysis, and formulate our precise research questions. In section 3, we explain the context and methodology of the study. In section 4, we present the reference model on similarity, as a main result of this study. Then, we apply the model to produce a quantitative comparison of task types of tasks in the textbooks analysed, and the Indonesian national examination (section 5). In section 6, we analyse the connections in the textbooks between similarity and proportion. In the final section, we present conclusive remarks related to the research questions, as well as perspectives for other applications of reference models of the type produced in this paper.

## 2. Background from Textbooks Analysis

Recently, the use of praxeological reference model has appeared as a new method to analyse mathematics textbook. For example, González-Martín, Giraldo, and Souto (2013) developed such a model to study the treatment of the real number system in Brazilian textbooks. A more fine grained analysis can be found in Hersant (2005), to study techniques for certain "missing value tasks" related to ratio and proportion, as they appear in textbooks for France lower secondary school. Hersant's reference model was developed by Wijayanti and Winsløw (2015) who focused on broader range of proportion tasks in the domain of arithmetic. As a result, they developed a larger reference model, consisting of seven types of tasks that can be applied to analyse the whole theme of "proportion" in Indonesian lower secondary textbooks (where the word theme is used in the sense of Chevallard, 2002b). They also noticed that some types of tasks are common and numerous in all the analysed textbooks, while others are rarer and appear just in a few of the textbooks. In this paper we examine how this method can be used for a related theme (similarity) in a different domain (geometry) and to what extent the textbooks analysed provide explicit links between the two themes (proportion and similarity).

Miyakawa (2012) presented a textbook study in the domain of geometry, which is also to some extent based on ATD. This study focuses on how lower secondary textbooks in Japan and France exhibit specific differences in the meaning, form and function of proofs in geometry. It is interesting that clear differences may appear between textbooks from different countries and be explained in terms of wider differences between them. A similar comparative study was done by Jones and Fujita (2013), on proof between British and Japanese textbooks.

Sears and Chávez (2014) studied the factors which seem to influence teachers' use of textbooks for geometry teaching in the US, for instance, what make them assign or not assign a given exercise as homework for students. Among the influential factors we find the teachers' perception of

- their students' competences for the given task, and
- the importance of the given exercise to prepare for high stakes assessments.

For instance, tasks involving proofs are rarely assigned because they are considered too difficult for most students and they are considered less important for external assessment purposes. This observation is related to one point in the present paper, namely the closeness of tasks in textbooks to tasks appearing in high stakes examination.

The often cited risk of school mathematics being disconnected (visit of different "monuments", cf. Chevallard, 2012) encourages many authors to investigate ways to strengthen the relations between different domains of school mathematics. For instance, García (2005) investigated the possibilities to establish relations between two themes in lower secondary school in Spain, namely ratio and proportion in the domain of arithmetic, and linearity on the domain of algebra. In the domain of arithmetic, proportion involves ideas like ratio and scale, which are also relevant for the context of linear functions.  However, García found that there is a poor connection between the two domains in textbooks for this level. This research inspired another point in the present paper, namely the use of praxeological reference models to investigate explicit or implicit connections in textbooks which go across domains.

At the end of this paper we consider, specifically, how textbooks relate the theme of proportion (in arithmetic, calculations involving scale and ratio) with the theme of similarity (in geometry). We consider both the extent to which explicit relations appear, and also the extent to which the geometrical theme of similarity goes beyond simply arithmetic computations. As pointed out by Cox (2013), many school exercises on similarity of geometrical Figures can be done by doing routine computations with given dimensions, without much consideration of the geometrical configurations at hand. She stresses that the tasks given to students play an important role for the extent to which they develop a "truly geometrical" knowledge of similarity, and investigates the use of tasks designed with this in mind Cox's study further motivated our systematic study of the types of tasks on similarity that appear in (a selection of) mathematics textbooks.

**Theoretical Framework and Research Questions**

Our approach to textbook analysis is founded on the notion of didactic transposition from ATD, the anthropological theory of the didactic (Barbé, Bosch, Espinoza, & Gascón, 2005; Chevallard, 1985, 1988; Chevallard & Bosch, 2014; Winsløw, 2011). This notion explains the exchange of knowledge between three institutional levels: the scholarly communities of mathematics (developing and maintaining scholarly mathematics), the level of "official school mathematics" (like ministerial committees, textbook publishers), and school mathematics in ordinary classrooms. We consider only the first two levels (and thus what is normally called external didactic transposition), but use a slightly more detailed model of "official school mathematics", as we distinguish three components that contribute to define it: the official curriculum, given some form of law or decree; the way school mathematics is presented in textbooks; and the official rules and practices for summative assessment (in our context, we have a national written examination in mathematics), see Figure 1. In the figure, we mean to indicate that textbooks are in principal designed to "deliver" the official curriculum, and the national examination is designed to assess the students' achievement of the curricular goals. At the same time, we want to study (in fact measure) the extent to which the exercise material in textbooks align with the assessment practices (concretely, actual exercises appearing the national examination), suggesting a more direct relation between official assessment practices and textbook design.

To analyse the transposition of specific knowledge and practice between these different institutional levels, we need an epistemological reference model, i.e. an explicit and independent model of the knowledge at stake, to avoid taking the viewpoint of some particular level and also to make our analysis completely explicit. We present such a model for the basic (practice oriented) components of similarity in section 4.

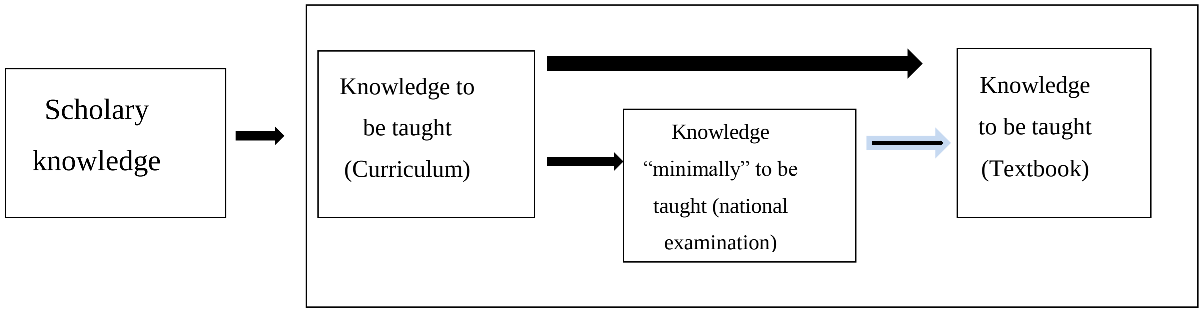



**Figure 1:** A detailed model of external didactic transposition

We use the notion of praxeology (Chevallard, 1999, 2002a; Chevallard & Sensevy, 2014; Winsløw, 2011) to describe mathematical practice and knowledge as our epistemological reference model. More details on this are available in the references just given, so we simply recall that a praxeology consists of four components: 1. a type of task (T); 2. a technique (τ); 3. a technology (θ); 4. a theory (Θ); thus a praxeology is a four-tuples (T/τ/θ/Θ), and be considered as a pair with two elements: praxis (T/τ), the practical block; and logos (θ/Θ), the theory block. We will use this notion to analyse the practice (examples, exercises) and the theory concerning similarity, as they appear in Indonesian lower secondary school textbooks.

Mathematical praxeologies in school do not only appear as isolated 4-tuples of the above kind. They are connected at different levels which ATD describe as discipline, domain, sector, theme, and subject (cf. the references above and Table 1). Here, the discipline is mathematics (as school subject), it consists of various areas of practice and knowledge like geometry and algebra (the domains); both disciplines and domains are, naturally, determined by the institution, although mathematics and its domain appear in very similar forms at schools across the world. The remaining levels are determined by praxeological levels: a sector is a collection of praxeologies unified by a common theory, etc. (see Table 1).

| Discipline: Mathematics | | | |
|---|---|---|---|
| Domain | | Arithmetic | Geometry |
| Sector | Theory | Proportions | Plane Geometry |
| Theme | Technology | Direct proportion: Ratio and scale | Similarity of polygons |
| Subject | Technique | $r = x_2/x_1$ | $\alpha = dist(Mx,My)/dist(x,y)$ |
| | Type of task | Given $x_1$ and $x_2$, find $r$ so that $(x_1, x_2) \sim (1, r)$. | Given two points and their images, find scale of magnification α. |

**Table 1**: Two related examples of sub disciplinary levels.

In this study, the relation between proportion and similarity goes across domains, as it concerns themes from the arithmetic domain and the geometry domain. It may, for this reason, be harder to establish. In general there is a tendency that theme, even within the same domain, remain relatively isolated and unconnected in school mathematics Chevallard (2012). Connections across domains could be even rarer, even if they are natural from a scholarly point of view. With this background we can now formulate our research questions for this study:

RQ1. What reference model can describe the theme of similarity as it appears in Indonesian textbooks?

RQ2. What can be said based on the reference model, on the extent to which the similarity theme in textbooks aligns with the national examination in Indonesia? (Here, types of tasks are the main unit of analysis).

RQ3. What connections between the themes of similarity (in geometry) and proportion (in arithmetic) are explicitly established in the textbooks? (Here, we consider mainly the praxis level)

## 3. Context and Methodology

Formal education in Indonesia is divided into three levels: primary school (grade 1-6), lower secondary school (grade 7-9), and upper secondary school (grade10-12). Students normally start in primary school at age 7. Mathematics is one of four subjects that Indonesian students have to pass with certain minimum marks at the final national examination of lower secondary school. This makes some schools offer additional hours for these four subjects, but also the regular teaching could be expected to focus rather much on the final exams towards the end of lower secondary school. We examine this hypothesis concretely by comparing the types of tasks appearing in textbooks with those appearing in the national exam.

The Indonesian government provides an online resource for teachers and students (the website of BSE: http://bse.kemdikbud.go.id/). On this website, one can download certain textbooks which have been authorized by the government for free. There are also textbooks published by private companies which are authorized for use in schools. But as these are not free as the texts on the BSE website, so the online textbooks are widely used.

In this study, we analyse relevant parts of the online textbooks for mathematics in grade 7 (a total of three textbook systems: Nuharini and Wahyuni (2008), Wagiyo, Surati, and Supradiarini (2008), and Wintarti et al. (2008)) and grade 9 (a total of six textbook systems: Wagiyo, Mulyono, and Susanto (2008), Agus (2007), Dris and Tasari (2011), Marsigit, Susanti, Mahmudi, and Dhoruri (2011), Masduki and Budi Utomo (2007), Djumanta and Susanti (2008)). To answer RQ2, we also analyse the national examination tasks in mathematics for the final exam in lower secondary school, in the period 2007-2015.

The first step in this textbook analysis is to identify chapters that treat similarity theme (typically, there is a chapter on "congruence and similarity"). Then, we analyse all textbook's examples and exercises on this theme. Concretely, we first consider examples to identify types of tasks and corresponding techniques which they present, and then identify types of tasks found in the exercise sections (assuming that techniques presented in examples are to be used when possible). We simply classify examples and exercises in types of task: whenever we meet a new one, the model is extended with that type of task. Thus, the model is fitted to the data (here, examples and exercises).

The resulting analysis gives a kind of profile of textbook, as one can expect that much of students' practice in actual teaching will be built around these types of tasks and techniques, along with elements of technology and theory provided in the main text of textbooks. Naturally, at this school level, theory is often "informal", so that even key notions like similarity may not be given a completely explicit definition. Even so, we try to describe technology that often is located in the reference epistemological model section. Additionally, the resulting analysis in the praxis level can be used to capture the national examination task related to similarity theme.

Pertaining to connection analysis, we firstly analyse a potential connection by observing the commonality of praxis level in both proportionality and similarity theme. Then, we consider an 'explicit' theory level by noticing the assertion of a 'closeness' of the two sectors and the use of 'proportionality' term in similarity. In the 'explicit' praxis level, we examine the direct explanation of an arithmetic technique in similarity. In this case, we focus more on similarity theme in 9$^{th}$ grade because it is more natural to recall knowledge from lower grade in higher grade rather than the other way around.

The core of reference model is, therefore, the types of tasks identified in examples and exercises. We use this categorization of tasks to make a quantitative analysis of "mathematical praxis" proposed by different textbooks (to investigate RQ1), and mathematical praxis required by the national examination (to investigate RQ2). Finally, we analyse the explicit connections which the analysed textbook chapters point out between similarity and proportion (to investigate RQ3).

## 4. Reference Epistemological model for similarity

### 4.1 Elements of the theory of similarity

In scholarly mathematics, similarity may be defined in different theories and levels of generality. Concerning similarity of any two subsets in the plane, one precise definition could be given based on transformations with specific properties. In school mathematics, however, similarity is usually defined more directly in terms of the Figures and their properties, and only for special cases like triangles and other polygons.

Here is a very common definition of similarity of polygons in secondary and even college level textbooks. (Alexander & Koeberlein, 2014, p. 218) Two polygons are similar if only if two conditions are satisfied:

S1. All pairs of corresponding angles are congruent

S2. All pairs of corresponding sides are proportional

For triangles, S1 and S2 are equivalent, and so similarity in this special case could be defined using any of them. One can also treat similarity of polygons in general through the special case of triangles, by dividing the given polygons into triangles ("triangulation"), and then similarity of the polygons depends on whether they can be triangulated into similar "systems of triangles" (a very difficult notion to make precise). We do not go further into this as triangulation techniques do not appear in the textbooks (See Euclid's elements of geometry in Fitzpatrick, 2008, p. 176). On the other hand, we will notice what relations the textbooks explicate between triangle similarity and the general case, and in what order they are presented.

A main challenge with the above definition is the meaning of "corresponding"; it could be formalized in terms of a bijection between the sides in the two polygons, satisfying that neighbouring sides are mapped to neighbouring sides etc.; it is an interesting task for undergraduate students to develop the details, which can also be found with various degrees of formalization in some textbooks (Lee, 2013, p. 213).

However, from a didactic point of view, such formalizations may not be relevant in lower secondary school. Here, the notion of "corresponding" will often appear as transparent, for instance when it is obvious from given Figures how sides and angles in two polygons correspond to each other. But, this apparent transparency is somewhat problematic: the "obvious" correspondence *requires* that the polygons are similar (so that we can pair congruent angles, and then the sides); but if similarity is to be checked by deciding if the S1 and S2 are satisfied, there is evidently a vicious circle (we can only use the conditions if the answer is yes).

In lower secondary school, even more informal definitions are common, involving visual ideas like “same shape” and carried by a large range of examples. If a semi-formal definition involving “corresponding” sides or angles is given, examples are also likely to be the only explanation of “corresponding”. The problem is that in these examples, one never sees “corresponding” angles that are not congruent, or “corresponding” sides that are not in equal ratio.

Corresponding angles can be described by ordering angles in both polygons and observe if the same angles appear. Then, corresponding sides can be paired by applying those corresponding sides one located between corresponding angles. Then, students can observe if the corresponding sides have equal ratios.

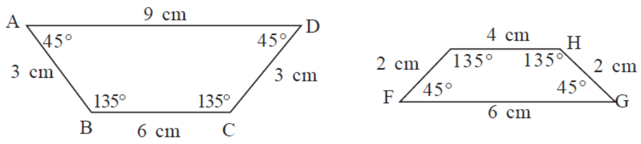
As we describe in the data analysis section, we analyse six Indonesian lower secondary textbooks. All textbooks give definition of similarity by using two components S1 and S2. To explain definition, the authors of five textbooks start with example of similarity in daily life situation, e.g. augmented photograph. Then, students are given two polygons and are asked to observe and/or to measure angles and sides in two given polygons. Then, students condition in the similarity definition. First, we can notice that students are taken for granted to understand the word ‘corresponding’. Second, students miss an opportunity to think about the meaning of definition by themselves.

We also noticed that all of textbooks treat similarity into two parts; polygon similarity and triangle similarity. Furthermore, polygon similarity always appears first. To connect similar polygon and similar triangle, the authors remain students about definition of polygon similarity: S1 and S2. For example, Dris and Tasari (2011) wrote that “Triangle is also a polygon, and then the definition of polygon similarity is also valid for triangle similarity”. The authors also point out that similar triangle is a special case, because they only need to apply either S1 or S2. Additionally, similarity of triangles can be proved by finding two proportional pairs of sides and two equal angles (S3).

We now proceed to build the reference model for types of tasks occurring the textbooks and at the national examination. The construction is based on our analysis of exercises and examples in the six textbooks of the study, as explained in the methodology section.

### 4.2 Practice blocks related to polygon similarity

We found four types of task in the textbooks, concerning polygon similarity. The description of each type of task is followed by the technique, and an example that is translated in English. Furthermore, there will be a discussion after each example is presented. Type of task 1 ($T_1$) is to decide if two given polygons are similar (Table 2). In the discussion we will also present some variations of $T_1$.




For the pair of Figures above, determine if they are similar or not?

(Masduki & Budi Utomo, 2007, p. 7)

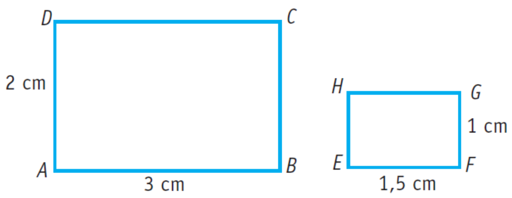
**Table 2:** Task related to $T_1$.

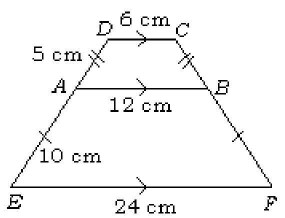
This type of tasks may be generalised, with some caveats:

$T_1$: given two figures of polygons P and Q, with given side lengths and given angles, determine if the two polygons are similar.

$\tau_1$: order the angles in P and Q from small too large (visual inspection), and see if the same angles appear. If so, verify S2 while regarding "sides between corresponding angles" as corresponding sides.

There are many variations of this type of task. "Variations"/special cases (e.g. where the "geometry" makes some of the definition superfluous). Some examples are given in Table 3.

<table>
<tr>
<td>

Determine if rectangle ABCD and rectangle EFGH above are similar or not?<br>(Marsigit, Susanti, Mahmudi, & Dhoruri, 2011, p. 28)</td>
<td>

Are ABCD and ABFE in the picture similar?<br>(Wagiyo, Mulyono, & Susanto, 2008, p. 4)</td>
</tr>
</table>

**Table 3:** variations of $T_1$.

In the first task in Table 3, students are given two rectangles, with measures of the sides. In this case, it is trivially true that all angles are congruent. To decide on S2, students can order the side lengths in (smaller, larger) for the two rectangles and compute the ratios of smaller sides. If they are equal, the rectangles are similar. While students determine congruent angles, using properties of rectangle in the first task, students in the second task can determine congruent angles, using similar symbols that are given in each side.

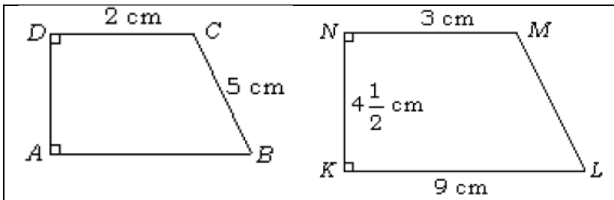
Tasks on two triangles with only angles or only side lengths given, do not belong to $T_1$. Thus, almost all tasks concerning similarity of triangles will not be of type $T_1$. In fact, the textbooks we consider all treat triangles apart from "general" polygons, which then have at least four sides.

Different from $T_1$ in which students are asked to decide on similarity, students in $T_2$ are given similar polygons with given angles. Then, they are asked to identify equal angles and corresponding sides. We can see the formulation of $T_2$ bellow.

$T_2$: given two similar polygons P and Q with given angles and given sides. Identify what angles and sides correspond to each other.

$\tau_2$: identify the corresponding angles by ordering them. Identify the corresponding sides as sides between corresponding angles.

In fact, the technique for $T_2$ given in most textbooks is less precise or explicit. Mostly, the authors just give the answer without any explanation to solve the task. In this type of task, we will also include triangles under polygons because the same technique applies to find equal angles and proportional sides. An example of $T_2$ can be found in Table 4.




Observe trapezoid ABCD and trapezoid KLMN.
Given ABCD and KLMN are similar.
a) Determine a pair of equal angles
b) Determine a pair of proportional sides
(Wagiyo et al., 2008, p. 8)

**Table 4:** Example of $T_2$.

Often, $T_2$ is followed by $T_3$ (see below) where students need some numbers to do calculation. $T_2$ is solved based on a visual representation. To solve $T_3$, students not only need a visual representation, but also a calculation technique (Table 5).

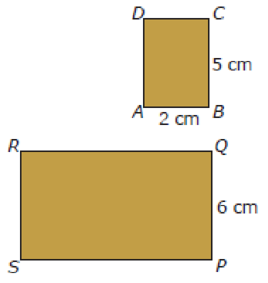
| | |
|---|---|
|  | If rectangle ABCD is similar to rectangle PQRS, determine the length of QR.<br>answer:<br>One condition of two similar Figures is to have proportional corresponding sides. So,<br>$\frac{AB}{PQ}=\frac{BC}{QR},\ \frac{2}{6}=\frac{5}{QR},\ 2QR=30,\ QR=15$<br>Thus, the length side of QR is 15 cm.<br>(Djumanta & Susanti, 2008, p. 6) |

**Table 5:** Examples of $T_3$.

The task in Table 5 asked students to find a missing side in one of two similar rectangles. Thus, it is guaranteed that the two rectangles have proportional corresponding sides. Firstly, students can use technique $\tau_2$ to identify corresponding sides. Secondly, students can compare the lengths of corresponding sides, using the formula $\frac{p_1}{q_1}=\frac{p_2}{q_2}$. Then, students can calculate the missing side by using the cross product technique from proportion in arithmetic: $p_2=\frac{p_1}{q_1}.q_2$. The task in Table 5 belongs to the following type of task:

$T_3$: given two similar polygons P and Q as well as one side $p_1$ in P and two sides $q_1$, $q_2$ in Q with $p_1$ and $q_1$ being corresponding, find the side $p_2$ in P that corresponds to $q_2$.

$\tau_3$ : calculate the missing side using $p_2=\frac{p_1}{q_1}.q_2$.

When two similar polygons are given, the authors also ask students to find a scale factor, as can be seen in Table 6.

$T_4$: given two similar polygons P and Q as well as sides $p_1, \ldots, p_n$ in P and $q_1, \ldots, q_n$ in Q. Determine the scale factor of P and Q.

$\tau_4$: $k$ is scale factor if $\frac{p_1}{q_1}=\frac{\cdots}{\cdots}=\frac{p_n}{q_n}=\frac{1}{k}$ (i.e. $k=\frac{q_i}{p_i}$ for any corresponding scales $q_i$ and $p_i$).

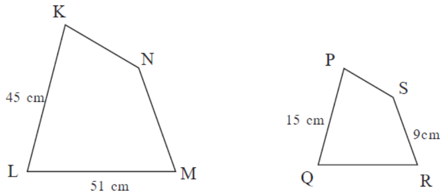
| |
|---|
| <br><br>Two similar quadrilaterals KLMN and PQRS are given. Determine the scale factor of between them.<br>answer:<br>Because KLMN ~PQRS, the ratio of corresponding sides is equal. It means that $\frac{KL}{PQ} = k$, with $k$ the scale factor. Given KL=45 cm and PQ=15 cm.<br>So, $\frac{KL}{PQ} = \frac{45cm}{15cm} = 3$. The, the scale factor is $k = 3$.<br>(Masduki & Budi Utomo, 2007, p. 12) |

**Table 6:** Example of $T_4$.

In $T_4$, Students are given two similar n-gons, where only two sides are known. The authors ask students to find the scale factor. First, students required to identify corresponding sides by using $\tau_2$ (or it can be seen from a figure visually correspond, as in Table 6). In order to get the scale factor, students need to compute $\frac{q}{p}$ for the pair ($p$,$q$) of corresponding sides. By convention, a scale factor is at least 1, so the technique also requires to take $p$ to be the larger of the two sides.

**Practice blocks related to triangles similarity**

The similarity of triangles involves a number of special techniques, and we found three types of tasks for this case. We now present them as above. In tasks of type $T_5$, students are given two triangles with given angles and are asked to decide if the triangles are similar; this is solved by verifying the pairwise equality of the given angles (possibly computing one or two angles using that the sum of angles in a triangle is $180^0$):

$T_5$**:** given two triangles S and T as well as $\angle a_1, \angle a_2$ or $\angle a_1, \angle a_2, \angle a_3$ in S and $\angle b_1, \angle b_2$ or $\angle b_1, \angle b_2, \angle b_3$ in T, determine if S and T are similar.

$\tau_5$**:** check if $\angle a_1, \angle b_1$, $\angle a_2, \angle b_2$, $\angle a_3, \angle b_3$.

An example can be seen in Table 7.

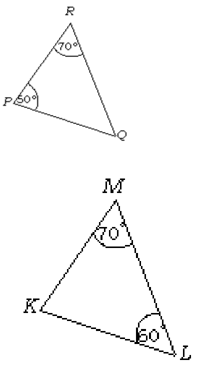
| | |
|---|---|
|  | Consider triangle PQR and triangle KLM from the figure above! Are $\Delta PQR \sim \Delta KLM$ ?<br>answer:<br>$\angle Q = 180^0 - \left(50^0 + 70^0\right) = 60^0$<br>$\angle K = 180^0 - \left(60^0 + 70^0\right) = 50^0$<br>Thus, $\Delta PQR \sim \Delta KLM$<br>because the corresponding angles are equal (Wagiyo et al., 2008, p. 13) |

**Table 7:** example of $T_5$

In $T_6$, students are also asked to decide on similarity. However, in this type of task, students are given the sides of two triangles. Students can order the sides from the shortest one to the longest one. Then, they can check if the corresponding sides of the two triangles have proportional ratio:

$T_{6:}$ given two triangles S and T as well as sides $s_1$, $s_2$, $s_3$ in S and $t_1$, $t_{2,}$ $t_3$ in T, determine if they are similar.

$\tau_6$: order the sides in the two triangles ($t_1 \le t_2 \le t_3$ etc.) and check if $\frac{t_1}{s_1} = \frac{t_2}{s_2} = \frac{t_3}{s_3}$.

An example of $T_6$ can be found in Table 8.

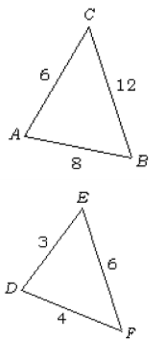
| | |
|---|---|
|  | Consider $\Delta ABC$ and $\Delta DEF$ on the picture above!<br>Are $\Delta ABC \sim \Delta DEF$ ?<br>answer:<br>Proportion of the shortest lengths from two triangles is $\frac{AC}{DE} = \frac{6}{3} = \frac{2}{1}$. Proportion of the longest lengths from two triangles is $\frac{BC}{EF} = \frac{12}{6} = \frac{2}{1}$. Proportion of the third length from two triangles is $\frac{AB}{DF} = \frac{8}{4} = \frac{2}{1}$.Then, $\frac{AC}{DE} = \frac{BC}{EF} = \frac{AB}{DF} = \frac{2}{1}$.<br>Thus, $\Delta ABC \sim \Delta DEF$ .<br>(Wagiyo, 2008, pg. 12) |

**Table 8:** Examples of $T_6$

We also found type of task that asks students to decide on similarity for two triangles. Students are given two corresponding sides and one corresponding angle that is located between these sides. To decide similarity of triangles, students can compare the corresponding angles. Additionally, students need to compare the ratios of the pairs of corresponding sides.

$T_{7:}$ given two triangles S and T as well as sides $s_1$, $s_2$, and $\angle a_1$ that is located between $s_1$ and $s_2$ in S and sides $t_1$, $t_{2,}$ and $\angle b_1$ that is located between $t_1$ and $t_2$ in T. Determine if they are similar.

$\tau_7$: check if $\angle a_1 = \angle b_1$ 1 and if $\frac{s_1}{t_1} = \frac{s_2}{t_2}$.

An example of $T_7$ can be seen in the Table 9.

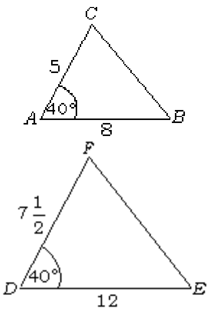
| | |
|---|---|
|  | Consider figure $\Delta ABC$ and $\Delta DEF$. Is $\Delta ABC \sim \Delta DEF$ ?<br>answer:<br>$\angle A = \angle D$<br>$\left.\begin{array}{l} \frac{AC}{DF} = \frac{5}{7\frac{1}{2}} = \frac{2}{3} \\ \frac{AB}{DE} = \frac{8}{12} = \frac{2}{3} \end{array}\right\}$ two sides that enclose the same angle.<br>Thus, $\Delta ABC \sim \Delta DEF$ (Wagiyo, 2008, pg. 13) |

**Table 9:** Example of $T_7$

Finally, the last task requires $\tau_3$ and some additional algebraic reasoning to identify a missing side in two similar triangles (Table 10).

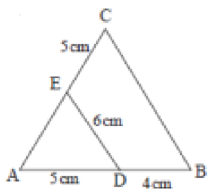
| | |
|---|---|
| Given $\Delta$ ABC~ $\Delta$ ADE with DE//BC.<br>Calculate the length of AE.<br> | answer:<br>we can use proportion of two line segments in two similar triangles.<br>$\frac{AD}{DB} = \frac{AE}{EC}$<br>$\frac{5}{4} = \frac{AE}{5}$<br>$4 \times AE = 5 \times 5$<br>Thus, the length of AE is 6, 25 cm<br>Masduki and Budi Utomo (2007, p. 27) |

**Table 10:** Example of $T_8$

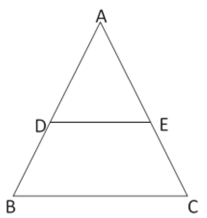


**Figure 1:** Figure of $T_8$

The technique applied for the question in Table 10 gives rise to

$T_8$: given the Figure 1 as shown with $\Delta$ADE~$\Delta$ABC, DE//BC, and given the length of three of four sides AE, AC, AD, AB. Find the remaining length.

$\tau_8$: use $\frac{AE}{EC} = \frac{AD}{BD}$ and isolate unknown side.

**4.3 Methodological remarks**

In this section, we will discuss a few tasks from the textbooks which are not easy to categorize into the above types of task. In Table 11, students are given a pair of polygons without specific angles and sides and are asked to decide if they are always similar, maybe similar, or never similar. For example, an equilateral triangle always has the same sides. Since equal sides are enough to prove that two triangles are similar, two equilateral triangles are always similar, etc. Students need to produce ‘cases’ and use more theoretical knowledge on polygon to solve this task which is difficult to formulate into explicit technique. Thus, we decide to eliminate this task.

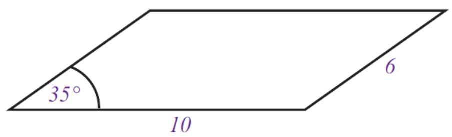
| |
|---|
| Consider the statements below. Write B if the statement is always true, K if the statement is sometime true and S if the statement is always wrong.<br>Two parallelograms are similar<br>Two equilateral triangles are similar<br>Two rhombuses are similar<br>Two pentagons are similar<br>Sulaiman et al. (2008, p. 7) |

**Table 11:** A task without numbers

In Table 12, students are given only one polygon. Then, they are asked to find another polygon which is similar to the given polygon. We consider leaving this isolated task because this task does not appear in the five others. The task can be formulated as followed and the example can be seen in Table 12.

T: given a polygon P as well as $\angle a_1,...,\angle a_n$ in P and sides $p_1$, …, $p_n$ in P. Construct another polygon Q which is similar to P.

$\tau$ : choose a scale factor k to find the sides $q_1 = k.p_1,...,q_n = k.p_n$ if $\angle b_1,...,\angle b_n$ in Q are the same angle as $\angle a_1,...,\angle a_n$ in P.

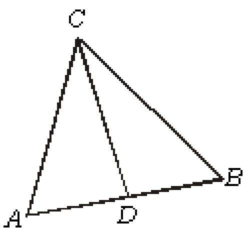



Construct two other parallelograms that are similar to the given parallelogram.

(Agus, 2007, p. 7)

**Table 12:** A task that is only found in one textbook.

The next example of unclassified task is when students need more than one prior knowledge that is found in only one textbook (Table 13).




Consider $\Delta$ ABC. CD is a bisector line, prove that $\dfrac{AC}{BC} = \dfrac{AD}{DB}$.

Wagiyo et al. (2008, p. 19)

**Table 13:** A task that needs more than one prior knowledge.

To solve the task in Table 13, students need to know what a bisector line is, and a visual imaginary to construct two similar triangles. Students also need a parallel line concept to prove similarity. In this case, similarity is not only connected to other topic, but also as a way to make a new theory $\frac{AC}{BC}=\frac{AD}{DB}$. However, we leave this task unidentified as it is only found in one textbook. Thus, this task is not very representative of public textbooks.

## 5. Quantitative survey of types of task in textbook and in the national examination

We want to emphasize that Table 14 shows only the numbers of task related to similarity. For each of the six textbooks, we classified all tasks related to similarity into the eight types of tasks ($T_1$, …, $T_8$) that we discussed above. In total, we analysed 422 tasks. In six textbooks, some of the tasks do not belong to these eight main types ('unidentified' in Table 14). These 'unidentified' tasks are typically more theoretical (cf. section 5.3).

| | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ | $T_8$ | Unidentified task |
|---|---|---|---|---|---|---|---|---|---|
| Wagiyo et al. (2008) | 10,4 | 16,5 | 49,6 | 0,9 | 7,8 | 4,3 | 3,5 | 3,5 | 5,2 |
| Agus (2007) | 11,7 | 8,8 | 41,1 | 0 | 0 | 0 | 17,6 | 0 | 20,6 |
| Marsigit et al. (2011) | 8,6 | 10,0 | 48,6 | 1,4 | 15,7 | 10,0 | 4,3 | 0 | 1,4 |
| Dris and Tasari (2011) | 13,8 | 7,5 | 65,0 | 1,25 | 5,0 | 1,25 | 0 | 0 | 6,25 |
| Masduki and Budi Utomo (2007) | 1,7 | 8,5 | 49,0 | 5,0 | 13,5 | 12,0 | 1,7 | 1,7 | 5,0 |
| Djumanta and Susanti (2008) | 3,1 | 14,0 | 53,1 | 7,8 | 6,3 | 3,1 | 3,1 | 0 | 9,4 |

**Table 14:** A quantitative survey of textbooks analysis (percentage of all tasks).

As we can see in Table 14, there is a considerable variation in how frequently the types of task appear in each textbook. Having such a picture may orient teachers in their choice of textbook. For example, teachers who focus on students' getting a broad knowledge of the theme can choose a textbook that has all types of tasks. Teachers who focus more on advanced task can consider textbooks that have more theoretical tasks (unidentified task).

| | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ | $T_8$ |
|---|---|---|---|---|---|---|---|---|
| 2008 | | | 2 | | | | | |
| 2009 | | | 2 | | | | | |
| 2010 | | | 1 | | | | | |
| 2011 | | 1 | 1 | | | | | |
| 2012 | | | 2 | | | | | |
| 2013 | | 1 | 1 | | | | | |
| 2014 | | 1 | 1 | | | | | |
| 2015 | | 1 | 1 | | | | | |

**Table 15:** A quantitative survey of national examination.

We also notice that in both examples and exercises, there are a few dominant types of task, namely $T_3$ and $T_2$. This observation aligns with the national examination tasks in which only $T_3$ and $T_2$ appear (Table 15). It is interesting to note that even though the curriculum does not prescribe specific types of task, textbooks seem to align with the national examination to some extent when it comes to the types of task they give priority. So, students (and teachers) can rely on the textbook to emphasize preparation for the national examination. At the same time, most of the textbooks cover a broader range of techniques than required at the examination.

## 6. Connections in the textbooks between similarity and proportion

The analysis and comparison of textbooks obviously cannot limit itself to the analysis of practices in independent sectors and domains. Connections between domains are crucial for the students to learn mathematics as a coherent discipline. If we compare the practice blocks of proportion in arithmetic (Wijayanti and Winsløw , 2015) with the practice blocks of similarity in geometry (section 5.2), we notice that proportion and similarity share similar techniques:

$T_3$: Given two similar n-gons P and Q as well as one side $p_1$ in P and two sides $q_1$, $q_2$ in Q with $p_1$ and $q_1$ being corresponding, find the side $p_2$ in P that corresponds to $q_2$.

$\tau_3$: calculate the missing side using $p_2 = \frac{p_1}{q_1}.q_2$.

This "missing side ($T_3$)" practice is very similar to a practice block which Wijayanti and Winsløw (2015) found to be dominant in Indonesian textbooks' treatment of ratio and proportion in arithmetic (7th grade), namely the following:

T: given a pair of numbers $(x_1, x_2)$ and a third number $y_1$ find $y_2$ so that $(x_1, x_2) \sim (y_1, y_2)$

$\tau$ : calculate $y_2 = \frac{x_2.y_1}{x_1}$.

The two types of tasks have almost the same technique. The difference is that in the tasks of the similarity sector, figures are used to present the numbers coming from dimensions in figures, while in the proportion tasks; students are given numbers using quantities like price, weight, length and so on. In both cases, students often have to infer from the situation - geometrical or quantitative - that similarity, respectively proportionality, can be assumed.

The treatment of arithmetic in 7th grade also includes a type of task which is very similar to $T_4$ above, namely the following (Wijayanti & Winsløw, 2015):

T: given $x_1$ and $x_2$, find $r$ so that $(x_1, x_2) \sim (1, r)$.

$\tau$ : $r = x_2 / x_1$ (finding the ratio).

The closeness of techniques from proportion and similarity suggest that it is didactically useful to build connections between them. However, we suppose that these potential connections might not be established for students, unless textbooks point them out explicitly, either by the text declaring a 'closeness' of the two sectors in general, by the use of specialized terms like 'proportion' in geometry, or directly at the time of explaining techniques such as the above.

As regards connection at the level of sectors, we focus on statements in the chapter on similarity that refer explicitly to the sector of proportion in 7$^{th}$ grade arithmetic such an explicit statement in three textbooks (Djumanta & Susanti, 2008; Dris & Tasari, 2011; Marsigit et al., 2011). Here is one example:

Masih ingatkah kalian tentang konsep perbandingan yang telah dipelajari di kelas VII? Konsep perbandingan ini harus kalian pahami terlebih dahulu sebelum mempelajari konsep kesebangunan bangun datar, karena pada pembahasan konsep kesebangunan akan berhubungan dengan perbandingan.

Dris and Tasari (2011, p. 1)

> 'Do you still remember about the proportion concept that you have learnt in the 7$^{th}$ grade? The proportion concept is needed before we learn about polygon similarity, because similarity relates to proportion' (translated in English).

Declarations such as above merely indicate the existence of some connection between sectors. However, it says nothing about what this connection means. To clarify this situation, we will also elaborate more on the sector level. Proportion as a sector can be divided into two themes, namely ratio (scale) and proportion (Wijayanti & Winsløw, 2015). The appearance of these themes in similarity would mean that part of the technology of these themes is invested in the textbooks' discussion (technology) of similarity techniques. Concretely, we have looked for the use of specialized terms like 'scale' and 'proportion'. We found that all of the textbooks use the term of 'proportion' in defining similarity. For example,

Dua segitiga dikatakan sebangun jika sisi-sisi yang bersesuaian sebanding atau sudut-sudut yang bersesuaian sama besar.

Djumanta and Susanti (2008)

'Two triangles are similar if the corresponding sides are proportional or the corresponding angles are equal' (translated in English).

Secondly, we found the term 'scale' in five textbooks (Djumanta & Susanti, 2008; Dris & Tasari, 2011; Marsigit et al., 2011; Masduki & Budi Utomo, 2007; Wagiyo et al., 2008). In the example, scale is used to relate proportion and similarity:

> '*Different sizes of a picture* can be obtained by enlarging or reducing the original picture by a certain factor. Thus, *pictures of different size* have the same shape and proportional sides. The numbers of proportional sides is often found using a scale. In

> other words, *the enlarged (or reduced) picture* and the original picture are similar' (Transleted in English, Wagiyo et al., 2008, p. 1)

Of course, the most concrete and direct connection between the sectors occur at the level of practice, that is when techniques from proportion are used for solving tasks on similarity. However, no textbooks explicitly point out such a connection e.g. in examples of how to solve task on similarity. But two textbooks (Djumanta & Susanti, 2008; Dris & Tasari, 2011) suggest such connection by investing a section called 'prerequisite competences' before treating examples of similarity task; this extra section simply shows some examples of proportion task of the types treated in grade 7. This way the authors suggest (without further explanation) that certain 'familiar' tasks and techniques are useful to recall before approaching the new types of task.

## 7. Conclusion

This study shows one way in which the anthropological theory of the didactic can be applied to analyse textbooks. As the theory part in the lower secondary textbooks is limited, we can mainly observe how the notion of similarity is introduced and then worked within tasks. We found that the authors focus more on how to use similarity to solve tasks than on treating in the notion itself.

The chapter on similarity always start with an informal definition of what it means for two (general) polygons to be similar. This reflects, in fact, the order suggested by the curriculum. We constructed a reference model based on the most common task on similarity found in the textbooks. We found a few tasks that cannot be categorized using these eight types of task but these are only found in a few textbooks. Thus, our reference model captures essentially the practical blocks found in the analysed text. The eight types of task fall in two parts: polygon similarity and triangle similarity. We also found that the same types of task dominate in the textbooks and appear as dominative types of task in the national examination. Finally, we conclude that on reference model can help us to see a potential connection between proportion and similarity. A few books explicitly establish connection at the level of the sectors and at the level of subjects (individual technique), while all have at least some explicit connection at the level of themes (use of terms). As we observed in the introduction section, connections in the textbooks maybe important to help students to see mathematics as connected body of knowledge.

Above all, this study wants to show how the ATD framework can be used to do comparative textbooks analysis. In fact, this framework allows us to compare the student's activity proposed by textbooks in a somewhat neutral way based on epistemological analysis. As a result, teachers can also apply the results of this study to choose what textbook to use for students to relate to the praxeologies of similarity.

After analysing the organization of the practice blocks, the analysis of themes becomes possible. Again, the ATD approach helps the researchers to see the connection or disconnections in a more objective way. Finally, the reference epistemological model was used to analyse the textbooks in relating to official documents such as the national examination and curriculum. This shows the role the national examination plays supplementary guide line for textbook authors and teachers when it comes to the types of task that they give priority.

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## C. Two notion of 'linear function' in lower secondary school and missed opportunity for students' first meeting with functions

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**Abstract**

The notion of function is considered one of the most difficult parts of the common lower secondary curriculum. In this paper we discuss the potential role played by linear functions, invariably used as first examples of this new notion. As empirical basis we use a praxeological analysis of the function chapter in four Indonesian lower secondary textbooks. The main point of our analysis is that the class of functions of type f(x)=ax (where a is some given number) does not appear explicitly at the level of theory, neither in the sense of being given a name, or in the sense that properties of the class is studied. We discuss the implications of this for students' learning of the more general (theoretical) notion of function.

Keywords: functions, linear function, praxeology

### 1. Introduction

It is well known that Klein (1908, 2016) successfully proposed that functions should occupy a fundamental place in school mathematics, and more precisely that

> *We begin with the graphical representation of the simplest functions, of polynomials, and rational functions of one variable* (2016, p. 82).

This paper concerns the very first steps of this plan, in its current and potential form as it can be observed in textbooks - and, in particular, the "simplest functions" considered there.

As we shall see in more detail later, in lower secondary school, the notion of function is indeed introduced through the elementary example of first degree polynomials, that is functions of type $f(x) = ax + b$, where $a$ and $b$ are fixed numbers. And the graphical representation is immediately and centrally discussed; it also motivates that such functions are called *linear* in secondary level textbooks.

However, in more advanced (or, in Klein's terms, "higher") mathematics, "linear function" has a different meaning as well. The entry for "linear function" on Wikipedia (Linear function, n.d) reads:

In mathematics, the term **linear function** refers to two distinct but related notions:

- In calculus and related areas, a linear function is a polynomial function of degree zero or one, or is the zero polynomial.

- In linear algebra and functional analysis, a linear function is a linear map.

Thus, the definition which is relevant for “calculus and related areas” is the one given above. It is distinct, and in fact different, from the definition used in linear algebra:

**Definition (linear algebra)**. Consider a map $f: V \to W$ between two vector spaces V and W over the scalar field K. We say that f is *linear* if the following two properties hold:

(L1) $f(x+y) = f(x) + f(y)$ for all $x, y \in V$

(L2) $f(cx) = cf(x)$ for all $x \in V$ and all $c \in K$.

We will need the following, which is well known and easy to prove (only (D)⇒(E) is slightly non-trivial - notice that in the special case of $= \mathbb{R}$ , we have also $K = \mathbb{R}$):

**Theorem.** For a function $f: \mathbb{R} \to \mathbb{R}$ the following conditions are equivalent:

(A) $f$ is a linear map

(B) We have $f(x) = f(1)x$ for all $x \in \mathbb{R}$

(C) There is $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$

(D) $f$ is continuous and satisfies (L1)

(E) $f$ satisfies (L2).

We notice also (L1) is *not* sufficient to ensure linearity of a function $f: \mathbb{R} \to \mathbb{R}$ (See for instance Anderson, 1979); but (L2) suffices.

From the condition (C) we also see that the linear algebra notion gives a more restricted class of functions than the definition used in secondary level textbooks.

In this paper, we shall follow the secondary mathematics terminology, so that a *linear function* $f: \mathbb{R} \to \mathbb{R}$ is any function given by $f(x) = ax + b$ for some numbers *a* and *b*[1]. This also gives the simplest class of polynomials (first degree), as prescribed by Klein, and thus a natural first step towards the classes of functions studied in the Calculus.

In the next two sections, we consider more closely the implicit and explicit roles, in the lower secondary curriculum, of the function class given by the conditions (A)-(E) above. Our main

[1] This is different from the choice made in De Bock, Neyens, and Van Dooren (2016), namely to use the word affine for this function

point will be that this function class and its properties are not explicitly treated at this level, and that - given students' familiarity with phenomena and problems which such functions can be used to model - this implies a loss of significant potentials to support the first introduction to the notion of function.

## 2. Proportion functions

We shall follow Van Dooren, De Bock, Hessels, Janssens, and Verschaffel (2005) and use the term *proportion function* for a function of the form $f(x) = ax$. Roughly speaking, their study demonstrates how students at in lower secondary school tend to use "proportion models" (based on multiplication) even for word problems which call for different models, including linear but non-proportional ones. The "over-use" of proportional models is ascribed to the place tasks requiring such models occupy in the primary and lower secondary curriculum - not as explicit functions (indeed, functions appear relatively late in the curriculum) but as calculations techniques for exercises in which some numbers are given and others are to be found.

The most important of these techniques concern missing value tasks; an systematic study of how these appear in Indonesian textbooks was presented in Wijayanti and Winsløw (2015). These tasks come in various types, but the most common form gives a context which allows to assume the relationship $\frac{a}{b} = \frac{c}{d}$ for four numbers *a*,*b*,*c*,*d* out of which three are given and the fourth is to be computed. Students can solve such tasks by using some "cross product technique"; it invariably asks to multiply the "corresponding" known quantity by a ratio of other known quantities, for instance (to find *c*): $c = \frac{a}{b} \cdot d$ . We note that this technique can also be expressed based on a proportion function $f(x) = \frac{a}{b}x$ which, given a number *d*, computes the "corresponding" number *c*. Let us consider a typical exercise:

If 2 bag can load 4 kg rice, and how much can 5 bags load ?

The unknown weight that 5 bags can load is, with the arithmetical technique, computed as

$$\frac{4\text{ kg}}{2\text{ bags}} * 5\text{ bags} = 10\text{ kg}$$

while the function approach takes the ratio (4/2 = 2) as the constant to multiply the given number of bags ($x$) with, in order to get the weight which that number of bags can load:

$$f(x) = 2x.$$

In such tasks, the "constant which defines $f$" (here 2) can be interpreted using condition (B) in the theorem: namely, it is the value of the unit (here, the weight loaded by one bag).

Proportion functions can also be connected to core material from the lower secondary geometry curriculum. For instance, Wijayanti (2016) found that cross product techniques also appear as central techniques in textbook exercises on similar polygons, and these techniques can be expressed (just as above) in terms of proportion functions which, for two similar polygons *A* and *B*, compute the length $f(x)$ of a side in *B* which corresponds to a given side *x* in *A*. Again the constant defining *f* is the value of the unit, namely the length of a segment in *B* corresponding to a unit length in *A*.

In both cases (as well as in other types of task related to proportion and ratio) it is interesting to consider the meaning of the two linearity conditions, (L1) and (L2). Here, (L1) often has a natural and evident meaning. For instance, in the case of weight held by rice bags, it seems clear that the weight held by 2+5 rice bags must be the weight held by 2 bags, plus the weight held by 5 bags; or that if a side in polygon *A* is divided into two parts of length 2 and 5, then the length of the corresponding side in *B* can be computed as the sum of lengths of segments which correspond to the two parts. (L2), by contrast, may not be so evident. But as continuity can usually be assumed in practical contexts, the theorem above means that "additive functions" are in practice the same as proportion functions".

The above remarks should be sufficient to convince the reader that a separate, explicit discussion of proportion functions and their properties (such as those listed in the Theorem) would be able to draw on, and formalize, considerable parts of lower secondary school students' previous knowledge. By contrast the more general case (of linear functions) relates to much less, essentially the equation of straight lines (if these are studied before functions, which is not always the case as we shall see). For the first meeting with functions, and an experience of functions as meaningful and useful generalization of familiar knowledge, this more general class of "examples" thus presents itself as much less potent. We have illustrated these remarks in Figure 1, where each box corresponds to a more or less heavy "theme" in the curriculum. Jumping over the "proportion function theme" (marked grey), or merely visiting it briefly as an example of linear functions, leads to cutting off the part of the curriculum relying on functions from very important previous themes. Knowledge established there may then subsist and continue to appear in unwanted forms, as shown by the work of Van Dooren et al. (2005) especially if, as is often the case, students do not really become familiar with the independent curriculum (marked by dashed box).

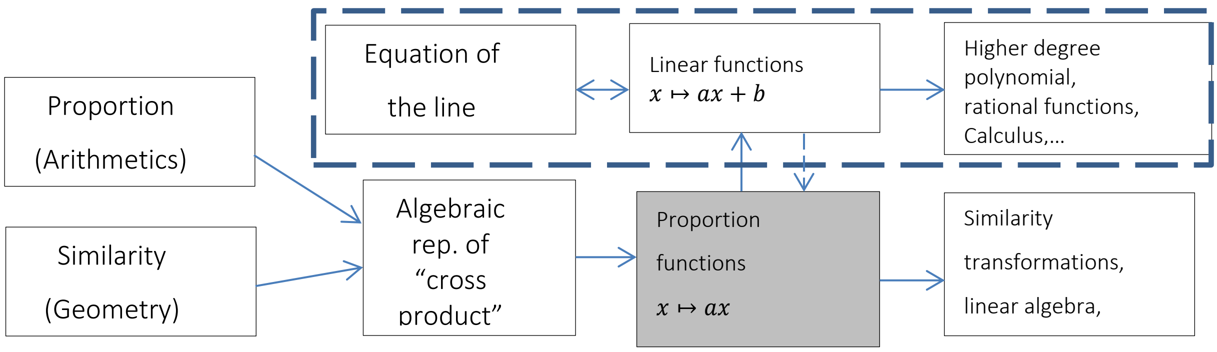


Figure 1. Potential links among themes related to functions

We shall now take a closer look at what is actually the case in a sample of textbooks from Indonesia which are representative of the curriculum in that country (and probably others).

## 3. Linear and proportion function in lower secondary textbooks

It is required for every public school in Indonesia to provide textbooks for students. To implement this regulation, the government provides a website (http://bse.kemdikbud.go.id/) where students and teachers can download online textbooks for free. Additionally, schools can also use public funding to buy printed copies of those online textbooks. The combination of affordable price, limited educational funding, and government regulation makes most public school use these online textbooks. We focus on the online Indonesian lower secondary textbook for grade 8 (13-15 years old), which is when functions are introduced in the centrally mandated course of study; there is always a chapter entitled “Functions” or similar, and this is what we analyse in this section. Four textbooks has been considered (Nuharini and Wahyuni (2008); Marsigit, Erliani, Dhoruri, and Sugiman (2011); Agus (2008); Nugroho and Meisaroh (2009).

Our analysis is based on praxeologies in the sense of the anthropological theory of the didactic (ATD); cf. (Barbé, Bosch, Espinoza, & Gascón, 2005; Chevallard, 1985, 1988; Chevallard & Sensevy, 2014; Winsløw, 2011). This entails analyzing the textbooks in terms of two interrelated levels : the logos or theory level (theoretical explanations, reasonings, definitions etc. which appear in the text) and the praxis level (the practices which students are induced to and which the theory levels explains and justifies). The praxis level consists of types of tasks ($T$) and their corresponding techniques ($\tau$), i.e. methods to solve tasks of a certain type; the logos level contains to levels, technology ($\theta$) which is the discourse directly pertaining to explain and justify particular techniques, such as the cross product technique considered in the previous sections; and theory ($\Theta$) which frames and justifies such a discourse, e.g. a definition of what it means for a function to be linear. Together, the practice block $(\mathrm{T}, \tau)$ and a corresponding logos block $(\theta, \Theta)$ - which may contain several entries for each four variables - constitute a praxeological organisation. Each of the boxes in Figure 1 can be further modelled in terms of praxeological organisations, and a textbook can be analysed in terms of such a model (as explained and exemplified in Wijayanti, 2016; Wijayanti & Winsløw, 2015).

The previous sections indeed contain theoretical elements about proportion functions, namely a definitions and a theorem; also, we have provided some technological discourse in the discussion of common praxis related to proportion and similarity and considered some of its relations with the theoretical elements. Normally, the technology and the theory about linear functions in lower secondary textbooks will be more informal than what we presented, which is more likely to appear at university level. Major potential gap (or missing link) between earlier parts of the primary secondary curriculum and the new theme on Functions, suggested by the global model in Figure 1, will thus be situated more precisely in the extent to which praxis and logos blocks on Functions are explicitly related to formerly established praxis and logos blocks, for instance on similar polygons or proportion problems. We observe what classes of functions are explicitly defined (theory), what praxis blocks are proposed through examples and exercises, and what technology is offered to support the practices. In particular, we are interested in the properties of functions which are explicitly discussed at the level of theory, and how they appear in tasks. We also analyse some typical tasks that involve linear functions. These tasks can be located in the exercise or in the example; we categorize the tasks in as types of task defined by a common technique. Many techniques are just "shown" by examples and reappear implicitly in exercises drawing on those techniques, which is why this work is analytical and not just observational. We will use this analysis of the praxis level as supplementary data to answer questions about the theory level.

### 3.1 Linear function in textbook (theory level)

We now analyze four Indonesian textbooks from theory level perspective. In three of them we find the explicit use of the term "linear function", and all of them provide multiple examples. Only two textbooks provide a formal definition of linear functions. For example:

> A linear function is a function $f$ on the real numbers that is given by $f(x) = ax + b$, where $a, b$ are real numbers and $a \neq 0$. (Marsigit et al., 2011, p. 51)

Two other textbooks give a slightly more informal definition, with an attempt to suggest that there are other kinds of functions which will be studied later:

> In this chapter, the functions that you will learn about are just linear functions, that is $f(x) = ax + b$. You will learn about quadratic function and other polynomial functions in the later classes (Nuharini & Wahyuni, 2008, p. 44).

Interestingly, one textbook does not even introduce the 'linear function' term, but as the other books, it explicitly states the relation of these functions to the equation of the straight line:

> The equation $f(x) = 2x + 1$ can be changed into the equation $y = 2x + 1$. The equation also can be seen as straight line, why is it? The equation $y = 2x + 1$ is called a straight line equation. (Nugroho & Meisaroh, 2009, p. 50)

Besides the explicit link with the geometric theme of straight line equations, all text books eventually proceed to quadratic functions. None of them give an explicit definition of proportion functions (even with other names) or point them out as a special case of linear functions.

All textbooks include, in the Function chapter, some discussion of the abstract notion of function, based on naïve set theory. This generally includes mentioning and illustration of the notions of domain, codomain, and range. Then, students are also introduced to the notion of relation and how it differs from the more specialized notion of function. Besides examples of functions given by formula (e.g. $f(x) = 2x + 1$), students are presented with four other ways to represent a function: Venn diagrams with arrows between them, Cartesian graph, tables, and sets of ordered pairs. In some of these cases, linear functions are considered on restricted domains, such as all integers or a finite set of them. On the other hand, linear functions are neither connected to examples of earlier praxis with proportion and similarity, nor discussed in terms of functional equations such as (L1) and (L2) which are, of course, also not valid for all linear functions.

### 3.2 Linear function in textbooks (praxis level)

We now proceed to classify, into types of tasks, the praxis blocks actually proposed to students in relation to linear functions. Before stating the results of our analysis of all exercises in the books, we consider an example to explain how the analysis was done:

> Given function $f: x \to 2x - 2$ defined on integer numbers. Determine a. $f(1)$, b. $f(2)$, (Agus, 2008, p. 30)

Here, students are given the formula defining function on integer numbers, and they are asked to find the image of two integers under *f*. As variant of this kind of task, students are asked to decide their own integer numbers to find the image of function. Sometimes they are also asked to graph the image of the function, or to represent it in terms of ordered pairs of numbers. In any case, the task is classified as being (or at least containing tasks) of the type

$T_1$ : given a function $f\ (x)$ on integers, find the image of function at specific integers.

$\tau_1$ : replace variable in the function expression and calculate the result.

Variants of $T_1$ include students being given two integers $m$ and $n$, and are asked to compute $f(m) + f(n)$.In the next type of tasks, students have to do "the inverse" of $T_1$ , namely solve the equation $f(x) = y$ with respect to $y$:

$T_2$: Given the closed form expression $f(x) = ax + b$ (where $a$ and $b$ are given), as well as the image of $f$. Determine the domain of $f$.

$\tau_2$ : For each value $d$ in the image, solve $ax + b = d$ with respect to $x$. The domain is then the set of all solutions obtained.

In contrast from $T_1$, $T_2$ are asked student to determine domain of function f(x) and image and expression of f(x) given. A simple algebra is needed to manipulate the equation that is resulted from function and the image of f(x). An example can be seen as follows

> Function $\mathrm{h(x) = x \rightarrow 7x + 6}$. Jika $\mathrm{h(c) = 27}$. Then determine the value of c …
>
> a. 5 c. 3
>
> b. 4 d. 2 (Marsigit et al., 2011, p. 63)

In $T_3$, some values of a linear function are given at certain points, and students are asked to determine the correct function expression from a list; the technique $\tau_3$ is to evaluate the expressions at the given points and identify the one whose values match the given values. For example:

> The price of a pencil is Rp. 1.200,00, the price of two pencils is Rp. 2.400,00, and the price of 5 pencils are Rp. 6.000,00. Which of the following functions describe this?
>
> a. $f : x \rightarrow 1200x$
> b. $f : x \rightarrow 2400x$
> c. $f : x \rightarrow 1000x + 200$
> d. $f : x \rightarrow 1300 - 100$
>
> (Marsigit et al., 2011, p. 63)

Of course, the above exercise is both overdetermined (three values are given, while two would suffice) and underdetermined (it is not stated that the function must be linear, however students only know that type of function). A more classical (and difficult) variant, then, is:

$T_4$: For a linear function $x \mapsto ax + b$, where $a$ *and* $x$ are given, along with one value of $f$ at a point. Determine (the expression defining) $f$.

$\tau_4$: If we are given that $f(s) = t$, solve $as + b = t$ with the respect to the parameter ($a$ or $b$) which was not given. Insert it into $f(x) = ax + b$.

This types of task can also be varied a bit, as in the following exercise (which also contains two tasks of type $T_1$):

> Given $f\ (x)\ =\ (x + a) + 3$ and $f\ (2) = 7$. Determine a. The function $f(x)$ , b. the value of $f(-1)$, c. the value of $f(-2) + f(-1)$. (Nuharini & Wahyuni, 2008, p. 45)

In $T_5$, students are asked to substitute an algebraic expression into a function f(x):

$T_5$: given the algebraic expression of a linear function f (x) and another algebraic expression $E$ (depending on x, so $E = E(x)$). Compute $f(E(x))$.

$\tau_5$ : substitute the algebraic expression in to a function f (x), and simplify the result, for instance collecting alike terms if needed.

Here is a case of a worked example of this type of task:

> Given the function $f\ (x)\ =\ 2x - 1$, determine: a. $f(x + 1)$; b. $f(x^2)$ (Nugroho & Meisaroh, 2009, p. 41).

The above types of task are all fairly standard and most appear with several exercises to train the technique. There are very few examples of more theoretical exercises, concerning properties (e.g. functional) of functions, rather than just calculations. Here is one of those rare examples which at least hold some potential in that direction:

> Given the function $f(x)\ =\ 2x$ defined on real numbers. Determine if $f(-x) = -f(x\ )$. (Nuharini & Wahyuni, 2008, p. 47)

We notice, though, that nothing is said about the meaning or importance of the "property" (in this case, *f* being an odd function, the generalization to (L2), or the like). So even this example does not really go much beyond unmotivated algebraic manipulation, let alone add to students' specific knowledge about functions or some class of functions.

## 4. Discussion

The preceding analysis indicates some striking characteristics of how lower secondary students first meet the notion of function in Indonesian schools, and probably other countries as well: as a relatively new set of tasks which are mostly of algebraic nature, besides the multiple representations which are involved in some tasks (with the crucial graphical one being, at least in

Indonesia, mainly worked with in a separate chapter on line equations). Invariably, functions are, at the first meeting, almost exclusively *linear* functions, and although proportion functions appear as examples, they are never named, and there is no explicit mention on the relations to previous themes in the curriculum - especially similar figures in Geometry, and proportion and ratio in Arithmetic, which however concern essentially proportion (rather than general linear) functions. The few apparent exceptions, like the task of type $T_3$ cited above (on prices of pencils), are not related to the past theme of proportion at the level of theory, which would require both an explicit treatment of (and probably name for) the class of proportion functions, and a set of convincing examples of how functions cannot only formalize but also unify and facilitate the mathematical modelling of phenomena which exhibit constant relative growth (e.g. distance under constant speed, cost under fixed unit price, etc.).

The overall tendency is that the theme mostly has links to future curriculum, namely, towards important topics such as polynomial equations, the algebraization of Geometry via Cartesian diagrams, and further function classes which will become important in upper secondary school. These are of course themes which the students do not yet know and so the relations which textbooks can make between them and the new theme of (linear) functions are inevitably implicit, or of little use if explicit.

In most books, linear functions are discussed to some extent and related to first order equations and straight lines in coordinate systems, but unlike the modelling of constant growth and similarity, the notion of function does not add much to these themes (while the diagrammatic representation is of course crucial to the students' later praxis with functions). Indeed, at the praxis level, linear equations is a stepping stone towards quadratic equations which is a main topic toward the end of secondary school, and linear functions form the first step in a series of new functions, with linear functions playing a key role in Calculus. We also note that the seven types of tasks, studied in Function chapter, mainly focus on algebraization.

## 5. Conclusion

This paper has pointed out one problem (the lack of solid relations between the praxeologies which constitute students' first meetings with functions, and praxeologies which are already familiar to students) and a possible solution: developing an intermediate theme on proportion functions, preceding and preparing the study of linear functions. This proposal can be further motivated by recent empirical research (De Bock et al., 2016) showing that most of students are able to recognize characteristic of representation of proportional, inverse proportional, affine function, but this study still does not answer question on why students have difficulty to distinguish these functions. Here, not only a "name" is needed, but also a thorough study of important properties of proportion functions (like additivity, relation between domain and range, multiple representations etc.) and how these properties can be interpreted in modelling contexts which the students have worked with for years. Teachers and curriculum authors should, themselves, become more familiar with functional properties of proportion functions, as stated in the introduction (Theorem), as a rich source of problems and points to study within this theme.

It is well known that functions constitute a delicate abstraction which many students do never really come to grips with, while they may still succeed relatively well with routinized algebraic tasks as the types presented above (and, of course, more advanced ones in upper secondary Calculus). The reason why the "stepping stone" of proportional functions is underdeveloped in most curricula could well be due, in part, to a lack of terminology, given than "linear" is associated with "lines" and thus reserved to the more general case of functions $x \mapsto ax + b$ (with two parameters).

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## D. Linking proportionality of arithmetic, algebra and geometry domain in Indonesian lower secondary textbooks

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**Abstract**

The aim of this study is to apply the anthropological theory of the didactic (ATD), mainly using the notion of praxeology, on textbook analysis. Our focus is the treatment of proportion in arithmetic, algebra and geometry domains in Indonesian lower secondary textbooks. The results show that ATD can be used to build a reference model to analyse the type of tasks proposed in textbooks. Furthermore, the analysis is useful to capture a link between these three domains.

Keywords: link, arithmetic, algebra and geometry domain

The use of the anthropological theory of the didactic (ATD) brings a new approach in terms of epistemological mathematics perspective in analysing textbooks. The aim of this study is to investigate the description of proportion in arithmetic, geometry, and algebra domains in Indonesian lower secondary textbooks. We can say that proportion has always been part of the mathematics to be taught. At the most elementary level, it appears just after the introduction of the four arithmetic operations, with the classic technique of the "rule of three". It is part of geometry in the treatment of similarities of plane figures. In today's curriculum, it also appears as an example of elementary function (the linear or proportional function), in the domain of algebra or functional relationships. It is one of the few cases where the same concept and some related techniques appear in different school mathematical domains with not always the same designation or mathematical environment. Moreover, this research also aims to investigate how these three mathematical organisations which appear in three different domains are connected to each other in textbooks.

The next section discusses the problems and research questions on how textbooks are used in Indonesia secondary schools. We also discuss related papers focusing on textbook analysis using ATD in section 2. Then, the methodology used is described in section 3 and followed by a discussion on context and data analysis. The results are divided into two categories: a discussion about the praxeological analysis of the three domains; a discussion about how these three domains are related based on the praxeological analysis performed.

## 1. Textbooks use in Indonesia

The fact that 48 110 859 students from 7-18 years old went to school in 2013 (Statistic Indonesia, 2013) constituted a huge need of textbooks in Indonesia. These conditions also motivate a competition among publishing companies to focus on textbooks at school level as a marketing target. Furthermore, students' cultural and institutional diversity encourages government to initiate an authorized organization that is called *badan standar national pendidikan* (BSNP, national education standard organization). One of the responsibilities of this organization is to evaluate textbooks' feasibility. Moreover, it has been found that only 37% of teachers have a bachelor degree as required qualification to teach (The World Bank, 2011). Even though textbooks are not the only resource used to teach, the lack of teacher qualification might be a reason for a teacher to depend on a textbook. This dependency, together with the authorization process, turns Indonesian textbooks into an object of study of special interest.

There are many authorized textbooks for teachers (and schools) to choose. Do they really assure students' needs? What kind of didactic approach do they use? In this paper, we try to propose a textbook analysis based on praxeology organizations.

This research will only focus on the way textbooks deal with the notion of proportion that can be defined as a relationship between two quantities. Here, quantity can be conceived as amount, length, volume, weight, etc. Thus, two quantities ($Q_1$, $Q_2$) are said to be proportional if there exists a correspondence between them that, given any two elements $q_{1.1}$, $q_{1.2}$ in $Q_1$ and the corresponding $q_{2.1}$, $q_{2.2}$ in $Q_2$, we have $\frac{q_{1.1}}{q_{1.2}} = \frac{q_{2.1}}{q_{2.2}}$.

In Indonesian lower secondary schools, proportion is located in the arithmetic domain and it includes quantities such as time, distance, weight, price, etc. In this case, the proportion term often appears as a subsection of the chapter "ratio and proportion". Furthermore, proportion can also be found in the geometry domain to describe a relation between two figures: similarity. In the algebra domain, proportion could be pictured as a relation between two variables that often appear as a linear function and the equation of a straight line containing the origin. However, in this last case, the name "proportion" rarely appears.

The dissemination of proportion into different domains shows how school mathematics is a connected body of knowledge even if it does not appear as such to the students, who are invited to visit a theme after the other without needing to relate them. Chevallard (2012) described this condition as a *visiting monuments* experience, in which students have to enjoy mathematics without knowing its *raisons d'être* from the past knowledge.

Proportion is a vintage discussion in mathematics worlds, but it is still used in today's curriculum in some countries, for example Indonesia. This research focuses on proportion in Indonesian lower secondary textbooks from grade $7^{th}$, $8^{th}$, and $9^{th}$ (12-15 years old). Based on this background, we will formulate two research questions for this study:

1. How can the mathematical praxeologies related to proportion in arithmetic, algebra and geometry domains be described?
2. What are the explicit connections among these three domains?

## 2. Textbooks analysis research using the ATD

Textbook analysis using ATD has been discussed in some domains in relation to proportionality. Hersant (2005) conducted a research about how missing value tasks are discussed in six different periods from 1977-2004. She found six different techniques that were presented differently in each period of time, for example rule of three, unit value, ratio, etc. We can say that she considers a point-praxeology, that is a praxeology build around a single type of tasks, through different periods of time. In the geometry domain, a research was conducted by Miyakawa (2017) considering mathematical proof in plane geometry as a subject taught in Japan and France. He found that the nature of geometry proof in Japanese textbooks is influenced by Euclidean geometry while it is affected by the emphasis of the transformational geometry from the *New Math Era* in the French case. Research by Miyakawa is about geometry proof tasks focusing on plane geometry focuses on a regional praxeology formed by different types of tasks and techniques, organised around different technologies but sharing a common theory. Even though the two focus research are different in terms of the "size" of the praxeology, they present an example of textbooks analysis focusing on a single type of task: rule of three and proof.

In an analogous way, González-Martín, Giraldo, and Souto (2013) conducted a research about how Brazilian high school textbooks introduce real numbers. By using the praxeological analysis, they distinguish six different types of tasks. Additionally, the results also revealed two dominant tasks in textbooks: (1) students are asked to determine if a given number is rational or irrational, (2) students are asked to find a finite decimal approximation from a given irrational number (application of rational number). These authors present a number of types of tasks that can be analysed using the same technological elements, which still belong to a local praxeology.

A narrower research on how the notion of praxeology is used to analyse different types of tasks in textbooks was conducted by Wijayanti and Winsløw (2015) who found two themes in the arithmetical domain that were divided into seven specific types of tasks. Additionally, this research also showed that there were some dominant tasks that appeared in the textbooks. The fact that there is a dominant task in the textbooks confirms a tendency among authors to use particular types of task. Again these authors confirmed that the notion of praxeological organization in ATD can be used not only to analyse a specific type of tasks or technique, but also to analyse many types of tasks in the textbooks, that is, to move from point to local praxeologies.

Generally, textbooks include many topics that students have to face in a particular period of time. Naturally, they are related to each other. However, research on how these topics link to each other has not been conducted yet intensively. An remarkable study on how proportion in algebraic and arithmetic domain can be linked was carried out by García (2005). He found that the connection between the arithmetic and algebraic domains was quite poor in Spanish mathematics education at lower secondary level. Even though García (2005) mainly focuses on one local praxeology, his work gives a picture on how ATD was used to describe a regional praxeology, thus capturing the link between the arithmetic and the algebraic domain. These results encourage us to expand the link among three domains in proportion.

## 3. Methodology

In this research, we will focus on how the praxeological organization of proportion in arithmetic, algebra and geometry domains is treated and how the praxeological analysis can be used to see the link between these three domains in textbooks. To begin with, we will propose the notion of didactic-codetermination. However, this research focuses on the specific levels that can be described using the elements of mathematical praxeologies: "sectors" that can be characterised by a shared theory ($\Theta$); "themes" that are defined by a shared technology ($\theta$), and "subjects" that correspond to one or few techniques ($\tau$) and types of task ($T$). For example, proportion in geometry domain is called similarity. This similarity sector can be defined by polygon similarity. Additionally, some types of task and techniques can be constructed related to polygon similarity (see table 4).

This textbook analysis research is located in the knowledge to be taught in the didactic transposition process. However, this analysis does not try to approach the bridge between the knowledge to be taught and the knowledge actually taught or between the knowledge to be taught and the scholarly knowledge. Even so, the analysis of a whole sector of the knowledge to be taught can reveal important characteristics, not only of the textbooks considered but of the didactic transposition process itself.

**Context and data analysis**

The data for our analysis are online textbooks (http://bse.kemdikbud.go.id/) of lower secondary school from chapter ratio and proportion from grade 7 (Nuharini & Wahyuni, 2008a; Wagiyo, Surati, & Supradiarini, 2008; Wintarti et al., 2008),, function chapter from grade 8 (Agus, 2008; Marsigit, Erliani, Dhoruri, & Sugiman, 2011; Marsigit, Susanti, Mahmudi, & Dhoruri, 2011; Nugroho & Meisaroh, 2009; Nuharini & Wahyuni, 2008b)and similarity chapter in ninth grade (Agus, 2007; Djumanta & Susanti, 2008; Dris & Tasari, 2011; Marsigit, Susanti, et al., 2011; Masduki & Budi Utomo, 2007; Wagiyo, Mulyono, & Susanto, 2008) that are mainly from the 2006 curriculum. We develop a description of the mathematical praxeologies in each domain. We focus on examples and exercises to identify the types of task and techniques. Even though the theory level is limited, we try to highlight the informal technological elements that are presented in the textbooks or, at least, need to be mobilised to solve the types of tasks using the proposed techniques. In this theory analysis, we also analyse textbooks from the 2013 curriculum.

In relation to the second question, we try to analyse the connexion of types of task among the three domains. Then, we focus on the technology elements by classifying the similar notions of "proportion" used. Lastly, we observe a relation at the level of the theory. Here, we concentrate on how the term of proportion is used /and explained in the upper grade. For example, how is the proportion term from the 7$^{th}$ grade introduced when working with linear functions (8$^{th}$) and similarity (9$^{th}$)?

## 4. Results

This section will describe the mathematical praxeologies that appear in each domain (section 5.1.). Additionally, in section 5.2., we will discuss how these three domains are connected.

### 4.1 Mathematical praxeologies of proportion in each domains

*Mathematical praxeologies of proportion*

In arithmetic, proportion is defined as two pairs $(x_1,x_2)$ and $(y_1,y_2)$ that satisfy the relation $x_1 \cdot y_2 = x_2 \cdot y_1$ and is often symbolised as $(x_1,x_2) \sim (y_1,y_2)$. More generally, two $n$-tuples $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ are said to be proportional if $(x_i, x_j) \sim (y_i, y_j)$ for all $i, j = 1, \ldots, n$ (Wijayanti & Winsløw, 2015).

In the textbooks, the introduction of proportion starts with the definition of a ratio between two similar quantities. Then, it is developed with comparison of two ratios from similar quantities or different quantities. Quantities that are normally used are length, mass, time, etc. and it is generally wrapped with daily life situations. The authors also introduce the algebraisation of the proportion technique, that is, the calculation with expressions of the form $\frac{x_i}{x_j} = \frac{y_i}{y_j}$.

| | |
|---|---|
| Question:<br>A car needs 3 litres of gasoline to get 24 km. How many kilometres can the car reach with 45 litres of gasoline?<br>[Translated from Indonesian] | Sebuah mobil memerlukan 3 liter bensin untuk me-nempuh jarak 24 km. Berapa jarak yang ditem-puh mobil itu jika meng-habiskan 45 liter bensin? |
| Solution:<br>3 litres of gasoline can reach 24 km, thus 1 litre gasoline can reach $= \frac{24}{3}$ km = 8 km. Distance reached with 45 litres of gasoline : 45 x 8 km= 360 km<br>[Translated from Indonesian] | **Penyelesaian:**<br>*Cara 1*<br>3 liter bensin menempuh jarak 24 km, sehingga 1 liter b<br>menempuh jarak = $\frac{24}{3}$ km = 8 km.<br>Jarak yang dapat ditempuh dengan 45 liter bensin = 45 × 8 km = 360 km. |

*Table 1.* Tasks that exemplify type of task in arithmetic proportion (Nuharini & Wahyuni, 2008a, p. 152).

The example of table 1 illustrates the type of tasks $T_6$. Wijayanti and Winsløw (2015) found seven types of tasks that are divided into two themes: 1. ratio and scale, and 2. direct and inverse proportion (table 2).

| Ratio and proportion |
|---|
| $T_1$: Given $x_1$ and $r$, find $x_2$ so that $(x_1, x_2)\sim(1,r)$.<br>$T_2$: Given $x_2$ and r, find $x_1$ so that $(x_1, x_2)\sim(1,r)$.<br>$T_3$: Given $x_1$ and $x_2$, find $r$ so that $(x_1, x_2)\sim(1,r)$. |
| Direct and inverse proportion |
| $T_4$: Given numbers $a, b$ and given that $x > y$ are relations with $a, b$. Can it be true that $(x, y)\sim(a,b)$?<br>$T_5$: Given $(x_1, x_2)$ and $(y_1,y_2)$, compare internal ratios.<br>$T_6$: Given $(x_1, x_2)$ and $y_1$ find $y_2$ so that $(x_1, x_2)\sim(y_1, y_2)$.<br>$T_7$: Given $x_1, x_2, y_1$ find $y_2$ such that $(x_1,x_2)$ and $(y_1, y_2)$ are in inverse proportion. |

*Table 2.* Type of tasks related to proportion

*Mathematical praxeologies of linear function*

Proportion in algebra appears as a relation between variables: the linear function. In this case, a linear function is defined as a polynomial function of degree zero $f(x) = ax$. This definition does not include any reference to linear maps, that is, functions satisfying the relationships $f(x + y) = f(x)$ and $f(ax) = af(x)$.

In all examined textbooks, linear function is located in a chapter called “Functions”. The authors discuss function by presenting a set of data in an arrow diagram (or Cartesian diagram) that fulfil the function condition. Then, the authors describe the definition of function such as “function (mapping) from a set A with set B is a particular relation where one value from the set A associates with exactly one value from the set B” (Nuharini & Wahyuni, 2008b, pp. 37, transleted in English). The formalisation such as the use of the ‘$f$’ notation and the notions of domain, codomain and linear function are presented right after the definition. In terms of the linear function definition, we found that textbooks mention a different form of linear function which is $f(x) = ax + b$. Most of the tasks use numbers and variables. However, we found a few numbers of linear functions tasks that use quantities.

Wijayanti (2017) identified five types of tasks in the textbooks regarding linear function that are presented in table 3. For example, a specimen of $T_1$ can be the following:

Given function $f\colon x \to 2x - 2$ defined on integer numbers. Determine a. $f(1)$, b. $f(2)$ (Agus, 2008, p. 30).

| | |
|---|---|
| $T_1$ | Given a function $\boldsymbol{f(x)}$ on integers, find the image of the function at specific integers. |
| $T_2$ | Given the closed form expression $f(x) = ax + b$ (where $a$ and $b$ are given), and given $f(x_0)$ for a given $x_0$, find $x_0$. |
| $T_3$ | Some values of a linear function are given at certain points, and students are asked to determine the correct function expression from a list. |
| $T_4$ | For a linear function $x \mapsto ax + b$, where $a$ and $x$ are given, along with one value of $f$ at a point. Determine (the expression defining) $f$. |
| $T_5$ | Given the algebraic expression of a linear function $f(x)$ and another algebraic expression $E$ (depending on $x$, so $E = E(x)$). Compute $f(E(x))$. |

*Table 3.* Type of tasks about linear functions

We decide to consider that these type of tasks correspond to "linear function" ($f(x) = ax$) even though the definition is different, because all the types of task found can be applied to the linear function form.

*Mathematical praxeologies of similarity*

Proportion in geometry connects with the notion of "similarity" of figures. We can summarize that the six textbooks considered use the same definition about similarity. Two polygons $P$ and $Q$ as well as their angles $\angle a_1, \ldots, \angle a_n$ in $P$ and $\angle b_1, \ldots, \angle b_n$ in $Q$ and sides $p_1, \ldots, p_n$ in $P$ and $q_1, \ldots, q_n$ in $Q$ are defined as similar if $\angle a_1 = \angle b_1, \ldots, \angle a_n = \angle b_n$ so that $\frac{p_1}{q_1} = \frac{\cdots}{\cdots} = \frac{p_n}{q_n}$.

In all examined textbooks, discussion about similarity always goes together with congruence and is related to length as quantities. Additionally, the authors used "scale" to introduce similarity. Since the cross product technique is also found in this domain, the use of unknowns and equations are also central technical elements in similarity, even if they do not explicitly appear at the technological level. Furthermore, we found that algebra manipulation is also used to solve some types of task, e.g $T_8$ : find a side of a triangle similar to another one (see table 5).

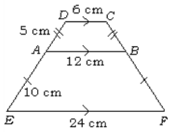
| | |
|---|---|
|  | Are figures *ABCD* and *ABFE* in the picture similar? (Wagiyo, Mulyono, et al., 2008, p. 4) |



*Table 4.* Example of similarity task

We identify in textbooks eight types of tasks that can be divided into two themes, namely polygon similarity and triangle similarity (Wijayanti, 2016). The eight types of tasks can be seen in table 5. And the example of $T_1$ can be seen in table 4.

| Polygons similarity |
|---|
| $T_1$: given two figures of polygons *P* and *Q*, with given side lengths and given angles, determine if the two polygons are similar.<br>$T_2$: given two similar polygons *P* and *Q* with given angles and given sides, identify what angles and sides correspond to each other.<br>$T_3$: given two similar polygons *P* and *Q* as well as one side $p_1$ in *P* and two sides $q_1$, $q_2$ in *Q* with $p_1$ and $q_1$ being in correspondance, find the side $p_2$ in *P* that corresponds to $q_2$.<br>$T_4$: given two similar polygons *P* and *Q* as well as sides $p_1, \ldots, p_n$ in *P* and $q_1, \ldots, q_n$ in *Q*, determine the scale factor of *P* and *Q*. |
| Triangles similarity |
| $T_5$: given two triangles *S* and *T* with two or three angles known, , determine if *S* and *T* are similar.<br>$T_6$: given two triangles *S* and *T* as well as sides $s_1$, $s_2$, $s_3$ in *S* and $t_1$, $t_2$, $t_3$ in *T*, determine if they are similar.<br>$T_7$: given two triangles *S* and *T* as well as sides $s_1$, $s_2$, and $\angle a_1$ that is located between $s_1$ and $s_2$ in *S* and sides $t_1$, $t_2$, and $\angle b_1$ that is located between $t_1$ and $t_2$ in *T*. Determine if they are similar.<br>$T_8$: given a figure like Figure 1 with $\boldsymbol{\Delta}$ADE~$\boldsymbol{\Delta}$ABC, DE//BC, and given the length of three of four sides AE, AC, AD, AB. Find the remaining length. |

*Table 5.* Types of tasks about similarity in the geometry domain

**4.2 Relations among the three domains**

Firstly, we will discuss a connection in the level of the subject; especially we will present a commonality in the types of tasks and techniques. From the description above, we can notice that there are two types of tasks that share similar techniques. In ratio and proportion, we will find $T_6$ 'given $(x_1, x_2)$ and $y_1$ find $y_2$ so that $(x_1, x_2) \sim (y_1, y_2)$' and in similarity we will find $T_3$ "given two similar polygons *P* and *Q* as well as one side $p_1$ in *P* and two sides $q_1$, $q_2$ in *Q* with $p_1$ and $q_1$ being in correspondence, find the side $p_2$ in *P* that corresponds to $q_2$".These two types of tasks can be solved using $\boldsymbol{y_2 = \frac{x_2 . y_1}{x_1}}$. Thus, this condition is a potential fact to connect proportion and similarity.

Furthermore, we found a task in linear function that corresponds to the "missing value problem":

Diketahui harga sebuah pensil Rp1.200,00, harga 2 pensil Rp2.400,00, dan harga 5 pensil Rp6.000,00. Fungsi yang menunjukkan pemetaan tersebut adalah ....

a. f : x → 1200x

b. f : x → 2400x

c. f : x → 1000x + 200

d. f : x → 1300 – 100

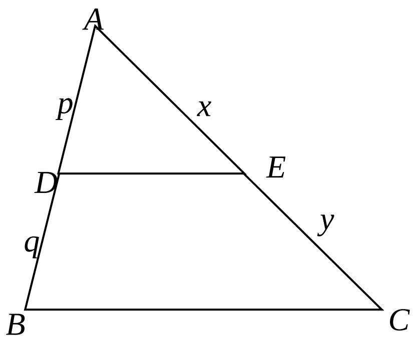
(Marsigit, Erliani, et al., 2011, p. 63)

The price of a pencil is Rp. 1.200,00, the price of two pencils is Rp. 2.400,00, and the price of 5 pencils is Rp. 6.000,00. Which of the following functions describe this? a. … [as above]. (Transleted in English )

Based on the example above, we see that a typical proportion task also appears in both linear function and similarity themes. This situation can be used as evidence that the three themes are connected at the level of the praxis.

Secondly, we will focus on the theoretical and technological elements that are used in the three domains. Here, we observe on how proportion is used in similarity and linearity. We find proportion term is used to define similarity, even though they do not explain how proportion can be connected to similarity. Meanwhile, we do not find that the term proportion is used in linear function.

However, even if the theoretical blocks of the praxeologies is not explicitly presented, we can infer some of their elements. In table 6, we see an example of how Thales theorem is discussed in similarity. Here, students are given a figure with $\Delta$ADE~$\Delta$ABC, DE//BC and given the four sides AE, AC, AD, AB. Then, they are asked to find the proportion of the segments of the two lengths of triangle ABC ‘ $\frac{p}{q} = \frac{x}{y}$ ’.




Given $BC \parallel DE$. The length of $AD = p$ and $DB = q$. $\Delta ADE$ and $\Delta ABC$ are similar. Find the proportions of the segments of the two length of triangle ABC.

$$\frac{AD}{AB} = \frac{AE}{AC}.$$

$$\frac{p}{p+q} = \frac{x}{x+y}.$$

$$p\,(x + y) = x\,(p + q)$$

$$px + py = px + qx$$

$$py = qx$$

Therefore, the proportions of the segments of the two lengths of triangle ABC are: $\frac{p}{q} = \frac{x}{y}$ (Sulaiman et al., 2008, p. 12).

*Table 6.* Thales theorem proof

In order to solve the question above, students need to have prerequisite knowledge not only about similarity but also about quantity, variable, proportion, etc. Interestingly, some of this prior condition can also be found in linear function and proportion (table 7). Therefore, this evidence shows that these three domains are linked at the theoretical level.

Table 7 presents some of the elements that we consider as part of the theoretical level of the observed praxeologies. In this case, we notice that the word “proportion” is introduced with and similarity. For example, one textbook uses proportion term to define proportionality “Two triangles are similar if the corresponding sides are proportional and the corresponding angles are equal” (Djumanta & Susanti, 2008). There is even a book that declares explicitly that proportion is a prior knowledge needed to learn similarity. Conversely, we cannot find explicitly the term “proportion” in the introduction of linear functions.

| **Domain** | **“Arithmetic”** | **“Algebra”** | **“Geometry”** |
|---|---|---|---|
| **Sector** | “Ratio and proportion” | “Function” | “Similarity” |
| **Theme** | Quantities<br>(mass)<br>Units (kg)<br>Decimal<br>numbers<br>Ratio (scale)<br>Direct proportion<br>Indirect<br>proportion<br>Augmentation<br>Reduction<br>Table | Quantities<br>(mass)<br>Units (kg)<br>Real<br>numbers<br>Set of<br>numbers<br>Variable<br>Equation<br>Relation<br>function<br>Linear<br>function<br>Domain<br>Codomain<br>Range<br>Table<br>Graph<br>Cartesian<br>diagram<br>Arrow<br>diagram<br>Pair orders | Quantities<br>(width, length)<br>Units (cm, km)<br>Decimal<br>numbers<br>Unknown<br>Equation<br>Ratio, scale<br>proportion<br>Polygon<br>Congruent<br>Augmentation<br>and reduction<br>Parallel line<br>characteristics<br>Pythagoras<br>theorem<br>Thales theorem<br>Angles<br>Sides<br>Figures |

*Table 7.* Theoretical elements of proportion, linear function, and similarity

## 5. Discussion

The study shows that praxeology organizations can be used to describe three important sectors of Indonesian school mathematics as it appears in textbooks: proportion, linear function and similarity. Furthermore, these praxeologies are useful to analyse the link between the three domains. It is found that there are at the same time some explicit connections and implicit relations. For example, we can find explicit connections at the theory level. Even though it is presented without explanation, one textbook clearly states that proportion is linked to similarity. However, many other possible theoretical links remain implicit. An example can be seen in the way textbooks share a type of tasks and some technologies without having a clear statement about their connectivity. Moreover, the fact that these three domains share some types of tasks and technologies (ratio and proportion, quantities, equation etc.) show that there is a possible link among domains (using type of tasks or techniques or others), but this link is not developed in textbooks. Additionally, it is also interesting to know the reason why this lack of explicit connectivity appears in the textbooks. The absence of connection in lower secondary textbooks raises a question on what proportion is about or what proportion is made of. These questions correspond to the analysis of the process of didactic transposition and the description of school mathematics that is important for further research.

Regarding the teacher's point of view, the description of the types of tasks and techniques carried out in each domain might help them have a vivid description and common measure of the examples and exercises that are located in textbooks. Thus, the praxeological analysis also appears to be an efficient tool for teachers. Additionally, the example of connecting domains can be used to identify the quality of textbooks or, at least, their "mathematical coherence". We acknowledge that we have mainly focused on the level of the *praxis* when analysing textbooks praxeologies. This decision is influenced by the fact that lower secondary textbooks are much more explicit in the praxis level elements than in the theory ones. However, it is also necessary to broaden our perspective to the theory level, for instance by using more advanced textbooks, teachers' and students' interviews or direct observation of teaching processes.

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## E. Didactic transposition phenomena through textbooks: the case of proportionality

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**Abstract**

Disconnection of proportion, linear function and similarity in school mathematics (as they sometimes appear in different domains like arithmetic or measure, relationships or algebra and geometry) raises a question on what "proportionality" is about and what is taught under this label. We use a research methodology based on the Anthropological Theory of the Didactic, particularly the process of didactic transposition in the analysis of textbooks, to explain the evolution of the *knowledge to be taught* around proportionality, paying special attention to the effects of the New Math reform. What happened to be an important mathematical organisation at the core of the traditional secondary school mathematics, the "Theory of Ratios and Proportions", was intended to be completely replaced by the new language of sets, variables and functions. After the failure of the New Math reform, some elements of the classical curriculum were partially restored: the notions of ratio and proportion, the problems of "rule of three", etc. This gave rise to a hybrid organisation where proportions coexist with equations and functions, and where the notion of quantity have trouble in finding its place. This didactic transposition phenomenon seems similar to the one found by Gerickle and Hagberg (2010) in natural science education, where different historical models are used in parallel in textbooks to describe gene function.

Keywords: proportion, didactic transposition, New Math reform, functions, quantities.

## 1. Introduction

Proportionality or proportional reasoning has been the object of numerous investigations in mathematics education since the 1970s (Noelting, 1980; Tourniaire & Pulos, 1985; Behr, Harel, Post & Lesh, 1992; Lamon, 2007; Fernández, Llinares, Van Dooren, DeBock, Verschaffel, 2012). Most of the research on this subject deals with students' difficulties with proportional reasoning, especially to discriminate multiplicative from additive situations. Some authors also pointed out the phenomenon of overgeneralization of linear models (or illusion of linearity) leading students to solve problems with two related quantities using a proportional relationship, independently of the nature of the quantities or relationships.

Even if many investigations focus on the students' performances when solving problems in a rather isolated way, some of them also refer to the way proportionality is taught at school, to explain the students' strategies. For instance, (Fernandez et al, 2012) interpret some of the differences found in the development of proportional reasoning from primary to secondary school students –and also between Spain and Belgium– by the way proportion is taught at these levels and by its location in the curriculum:

> The observed trends in the development from additive to multiplicative reasoning occur at different ages in each country. Differences in mathematics education traditions are likely to explain this phenomenon. Generally speaking, the teaching of proportionality comes later in the Spanish curriculum, less attention is paid to discriminating between proportional and non-proportional situations, and the "technical" rule of three algorithm (cross product) takes a more prominent role as the way to solve standard proportional problems. (Fernandez et al, 2012, p. 431)

These authors also summarise how research literature provides curriculum and teaching indications about how to deal with proportionality:

> In that sense, Kaput and West (1994) suggest that curricula should also present additive and multiplicative situations jointly to give students opportunities to exercise in discriminating between them, and to focus their attention on the nature of covariation relations. Probably, the use of more realistic situations can also help students to identify and recognize the multiplicative relationships in proportional situations and discriminate them from additive ones. The concrete representations that support and extend students' natural build-up reasoning patterns rooted in counting, skip counting, and grouping should be emphasized (Kaput and West 1994). Tables and graphs representing proportional and other relationships between quantities in real contexts may further help students to understand these relationships (Lamon 1995). (Fernandez et al, 2012, p. 433)

Given the increasing importance attributed to the way mathematics is taught to explain the students' performances, some investigations from different countries have started focusing on the analysis of textbook material related to proportionality. For instance, Shield and Dole (2002, 2013) analyse different Australian textbooks to explore how knowledge connections and proportional reasoning are promoted. Their work is followed by Ahl (2016) in the case of Sweden textbooks. Da Ponte and Marques (2011) present a description of proportion tasks in mathematics textbooks for middle school students of Portugal, Spain, Brazil, and USA. The French case is analysed in detailed by Hersant (2005), covering a large historical period, from the beginning of the 19 century to the current days. Enlarging the mathematical domain considered, in her analysis of Indonesian textbooks, Wijayanti (2015, to appear) includes the treatment of the theme of proportion in geometry (similarity), arithmetic (ratio and proportion), and algebra (linear functions), and examines the way these themes are linked to each other.

What we can see in these works is a movement from the study of the students' performances and difficulties in solving problems involving (or not) proportional relationships between quantities, to the study of what is called, in the theory of didactic transposition (Chevallard, 1985, Chevallard and Bosch, 2014) the *knowledge to be taught* – or, according to the TIMSS framework, the *intended and potentially implemented curriculum*. We can say, at the risk of generalizing, that the mathematical knowledge to be taught at lower secondary school usually includes proportionality (or proportion) in three different domains: its more direct presence is in arithmetic with the theme of ratios and proportions and the related theme of percentages; it also appears as the first and simplest relationship between quantities in the form of the function of proportionality or linear function; it is finally also part of the geometrical work with the study of similarities between figures.

The theory of didactic transposition points to the fact that the mathematical bodies of knowledge – or mathematical organisations, to take the term introduced by Freudenthal (1972) – that are proposed in curricula and appear in greater detail in textbooks are not "natural" but the result of social constructions. These productions are elaborated throughout a process of transformations and adaptations from their origin in the "scholarly institutions" where they are produced, to the school institutions where they end up being taught and learnt. The study of these transformations is important to know the constraints under which teaching processes take place, since the result of the didactic transposition process shapes in many ways the specific mathematical activities that teachers propose to their students.

In the case of proportionality, considering the effects of the didactic transposition process leads us to formulate the following research questions:

1. What is taught and not taught in relation to proportion in today's lower secondary education? How the current organisation of mathematical content around proportion can be explained in terms of the didactic transposition process?
2. What didactic transposition phenomena explain the current situation and the difficulties to change them?
3. Are there other similar didactic transposition phenomena, in mathematics or other disciplines (e.g. science)?

It is clear that textbooks are good empirical material to approach the knowledge to be taught (together with curricula, official accompanying documents, etc.) and to identify possible didactic transposition phenomena related to the choices made in selecting the bodies of knowledge that are currently taught – and discarding some possible others. To analyse the knowledge to be taught, it is important to look at the whole process of didactic transposition, also in its historic dimension. Thus, we propose a research methodology to analyse textbooks in the perspective of didactic transposition (following Kilpatrick (1992) and mainly Chevallard (1985), Chevallard and Bosch (2014), Bosch and Gascón (2006) and Barbé, Bosch, Espinoza, and Gascón (2005)) to show a specific didactic transposition phenomena that appears in the case of proportionality and that presents some similarity with another one found in science education by Gericke and Hagberg (2010) in the case of genetics.

The methodology followed will be the one proposed by the Anthropological Theory of the Didactic (Chevallard, 2006) based on the praxeological analysis of didactic transposition processes through the construction of a previous reference epistemological model. The empirical material used will be some current textbooks from Indonesia and Spain (and others countries indirectly from other authors' research: Australia, France, Portugal, Brazil and US) and some old textbooks and mathematics books from Spain, France, the UK and the USA.

We will first present a literature review about textbook analysis and proportion to stress the specificities of the didactic transposition analysis. We will then present the main keys of the methodology followed and provide a sample of the evidence found, before proposing a discussion and concluding remarks.

## 2. From textbooks analysis to didactic transposition

We will begin this literature review section with a discussion about general textbooks analysis in proportion and narrow it down to connecting it to didactic transposition phenomena.

The analysis of textbooks is considered by some authors as part of the research methodologies in mathematics education and even as a whole domain in itself. Fan, Zhy and Miao (2013) present a survey of research that explicitly focuses on mathematics textbooks, to identify future directions in this field of research. They note that textbooks analysis and comparison attracted most researchers' interest in this field. The focus is usually put in how different mathematics content have been treated in textbooks. Concerning the mathematical domain, normally only a small unit of analysis is considered and less attention is given regarding connection between topics. A deeper study about e-textbooks is suggested by Gueudet, Pepin, Restrepo, Sabra, and Trouche

(2016) who proposed a framework to analyse the level of connectivity in e-textbooks, including connections between different topic areas. This situation shows that 'connection' in textbooks analysis is gaining interest from researchers.

As mentioned before, research on school textbooks regarding proportionality is mainly addressed to analyse the type of tasks that are proposed as teaching and learning activities. Assuming the strong influence of textbooks in teaching practices – especially at secondary school level – this kind of research intends to better understand the opportunities and limitations that teachers and students can get. Ponte and Marques (2011) have analysed how proportion is introduced and developed in selected mathematics textbooks for middle school students in Portugal, Spain, Brazil and USA. Following the OCDE framework, they categorized proportion tasks according to their cognitive demand considering three dimensions: reproduction task, connection task and reflection task. The authors provide an interesting description of how proportion is introduced in each textbook and its mathematical environment:

> In the Portuguese textbook the study of the concept is based on the previous study of rational numbers. The same happens with Spanish textbook that also builds on the previous work on percent. Meanwhile, in the Brazilian and American textbooks, the study of proportion is based in the work on rational numbers, equations, patterns, and regularities, in line with the NCTM's (2000) recommendations." (p. 7)
>
> As a starting point for the study of proportion, the Portuguese textbook presents the concept of ratio. Then, it addresses the notion of proportion and the fundamental propriety of proportions, and, afterwards, presents percents, scales, and direct proportion. The Spanish textbook for grade 7 approaches proportion using proportional series to discuss the relationship of proportional magnitudes, then presents direct and inverse proportion, and, finally, revisits the concept of percent in a deeper way. The Brazilian textbook grade 6, in a first chapter, presents ratio, percent, and change in proportional magnitudes, alluding to inverse proportional magnitudes. In a subsequent chapter, the textbook relates proportion with geometry, presenting the topics of scales, proportional rectangles, the fundamental property of proportions, and similar triangles. Finally, the American textbook introduces in chapter 24, the concepts of ratio, rate, proportion of similar figures and scales, and shows how to construct circular graphs and compute interest rates. Next, in chapter 25, it revisits the concept of percent and enlarges it. (pp. 7-8)

> In one case (Portugal) the study of proportion is based in the study of rational numbers, in another (Spain) is based on rational numbers and percents and in another cases (Brazil, USA) it is based in the study of rational numbers, equations, and patterns. In the Portuguese, Brazilian, and American textbooks the notion of ratio plays in important role, but it does not happen in the Spanish textbook that uses the notion of "proportional series". Only two textbooks mention the "fundamental property of proportions". (p.10)

In what concerns the "cognitive demand of tasks", the authors note that "their distribution is similar (with emphasis in connections) and the structure is also similar (with emphasis on closed tasks)" (p. 10).

Focusing on the students' understanding of proportion, Dole and Shield (2008) analysed Australian eight grade mathematics textbooks, using specific curriculum content goal and Vergnaud's (1983) framework of multiplicative structures (a more revised categorization can be found in Shield and Dole (2013) and Ahl (2016)). The results reveal that textbooks provide more calculation procedures rather than focusing on proportional reasoning and multiplicative thinking, what the authors consider as useful to distinguish proportional to non-proportional situations. They also state that numbers and variables as the main part of calculation procedures tend to replace the meaning of proportional reasoning in which recognizing the situation, including quantity characteristic, is important. Furthermore, Dole and Shield (2008) also categorize the relations across proportional topics (ratio, rate, scale, similarity). They find that some textbooks explicitly attempt to link ratio and rate but avoid using 'scale factors' (which is learned before) in similarity, and that proportional relationship is not only limited to proportions and similarity, but is also connected to linear functions.

In a study about functions, Mesa (2004) analysed proportional tasks in 24 middle school textbooks from 15 countries and found that only 10% of the tasks suggest social data or physical phenomena, including proportion cases, whereas the rest suggest symbolic rule, ordered pairs or a controlling image. She added that the limited number of tasks in function using social data or physical phenomena support the disconnection between proportion and functions.

Using the notion of praxeology within the Anthropological Theory of the Didactic (ATD), some recent research by Wijayanti (2017a), who analysed Indonesian secondary school textbooks, showed that proportion, similarity and linear function are connected at the practice level: the typical task called "forth proportional numbers" appearing more or less explicitly in the three sectors. However, the connection remains implicit at the theory level. For example, in order to proof that two triangles are similar, students need to have proportion, quantity, and numbers which are also a prerequisite knowledge in proportion and linear functions, but the connection between these notions is not stated. Therefore, as Wijayanti (2015) said in a previous work, the connection in practice level is didactically useful, however this implicit connection might not be established for students unless they pointed the link explicitly. Table 1 shows the variety of notions used by textbook authors in these three domains.

| **Domain** | **"Arithmetic"** | **"Algebra"** | **"Geometry"** |
|---|---|---|---|
| **Sector** | "Ratio and proportion" | "Function" | "Similarity" |
| **Theme** | Quantities (mass)<br>Units (kg)<br><br>Decimal numbers | Quantities (mass)<br>Units (kg)<br>Real numbers<br>Set of numbers<br>Variable<br>Equation | Quantities (width, length)<br>Units (cm, km)<br>Decimal numbers<br>Unknown<br>Equation |
| | Ratio (scale)<br>Direct proportion<br>Indirect proportion<br>Augmentation<br>Reduction | Relation function<br>Linear function<br>Domain<br>Codomain<br>Range | Ratio, scale proportion<br>Polygon<br>Congruent<br>Augmentation and reduction<br>Parallel line characteristics<br>Pythagoras theorem<br>Thales theorem |
| | Table | Table<br>Graph<br>Cartesian diagram<br>Arrow diagram<br>Pair orders | Angles<br>Sides<br>Figures |

*Table 1.* Theoretical elements of proportion, linear function, and similarity (Wijayanti, 2017a)

The referred research on textbooks analysis and proportionality is mainly concerned by providing a frame of analysis of what we can call the 'quality' of the mathematical tasks (or praxeologies) proposed as part of the knowledge to be taught. In some cases, this 'quality' is defined as the capacity to 'promote proportional reasoning' according to five keys of proportional reasoning skills identified from research on teaching and learning proportionality (Dole & Shield, 2008; Shield and Dole, 2013; Ahl, 2016). In another, it appears in terms of the 'cognitive demand, structure and context' of the tasks presented in textbooks (Da Ponte and

Marques, 2011). In the last one (Wijayanti, 2017a), it can be interpreted as the existence of connections between praxeologies at the different levels – practical and theoretical.

In the perspective of the didactic transposition, the question that arises is not only what the knowledge to be taught is made of – and which are its richness, incompleteness and limitations, but *where does it come from*. In other words, the question is about how the different pieces or bodies of knowledge that conform the knowledge to be taught have been selected, the origin of these pieces and the way they have been newly structured in the current school mathematics organisations. Instead of taking the knowledge to be taught as a given and analyse its properties, qualities and limitations in order to improve them or help teachers and students take the best of it, the question we wish to address in this paper is why the knowledge to be taught is at it is, what choices explain its being as it is and the processes through which it has been elaborated.

For this it is important to take into account that the didactic transposition is not only a process involving different institutions, but that it is also happens in different time periods. For example, Hersant (2005) conducted a research about the treatment of proportionality in the French compulsory education, focusing on the particular type of tasks of the forth proportional quantity (simple rule of three) and considering five periods from 1887 to 2004 according to the different reforms that have taken place in France. As results, we can see different approaches of the teaching of proportionality in each period, at both the levels of practice (types of problems and techniques) and theory (conceptual environment, connections and validation).

For example, proportion was introduced as a relation between two quantities in the early period 1887-1923: two quantities are defined as proportional when, if one is multiplied or divided by 2, 3, etc., then the other is also multiplied or divided by 2, 3, etc. This definition supports the technique known as "reduction to the unit", which is mainly based on an oral discourse accompanied by a sequence of divisions and multiplications: if 5 pencils cost 3€, then 1 pencil costs 5 times less, so (3/5)€, and 8 pencils cost 8 times more, so 8x(3/5)€ = (24/5)€ = 4,8€. Quantities, ratios and proportions appear as important notions to justify the use of the technique, its development to more sophisticated forms, like the direct writing of a proportion: $5:8 = 3:x$, if $x$ is the price of 8 pencils, and its resolution using "cross products": $x = 3\cdot 8/5$, etc.. The whole mathematical organisation can be easily expanded to cover problems of inverse and complex proportionality.

In the next period (1923-1945), the domain of proportionality is enlarged to include the study of physical systems where "quotient quantities" are involved (density, scale, velocity) and "proportionality is not considered anymore only from the analogical viewpoint (quantities that vary in the same ratio) but also from the analytical viewpoint (relationship between measures of quantities).” (Hersant, 2005, p. 10, our translation). Then, in 1945-1969 there is a progressive algebraization of the topic and the notion “function of proportionality” appears for the first time at secondary level, even if the techniques based on proportions and rules of three prevail.

The New Math reform (1969-1977) represents a huge change in the considered mathematical organisation. Proportionality disappeared from secondary education and, at primary level, proportional quantities were replaced by maps or relationships between sets (or “lists”) of numbers. What remains of the fourth proportional problems is addressed by techniques based on numerical tables and the use of linear properties between the rows of columns of the table, later deriving to cross-products.

The period 1977-1985 corresponds to a “counter-reform” and a return to the teaching of proportionality, but the role of quantities does not come back with the same strength. Proportionality is defined as a relationship between finite sequences of measures and the use of numerical tables persist. Since then, as summarized by Hersant (2005, p. 21, our translation): “From 1978 on, the return to the study of “concrete” situations implicitly induces a return to the notion of proportional quantities. However, the work carried out does not really affect proportional quantities but rather sequences of measures of proportional quantities, as can be seen with the use of tables”. The mathematical organisation that remains has been called an “epistemological heterogeneity” (Comin 2002, in Hersant 2005, p. 21) with a mixture of notions and expressions coming from different stages of the evolution of mathematics: ratios, quotients, rational numbers; proportions and equations; tables of proportionality tables, numerical relationships and functions; etc.

Our research relies a lot on Hersant’s study in terms of what we can call “time transposition”, even if we are not considering a precise lens as hers. In contrast, we are also including in our analysis the evolution of the scholarly knowledge in relation to the evolution of the knowledge to be taught. And we will not only focus on proportion, but also try to find a read thread between proportion, similarity and linear function. Additionally, we will use textbooks from different countries. In summary, we are presenting an analysis of the didactic transposition process of proportion by combining time dimension and two institutions: the scholarly and the school one, as can be seen in table 1.

## 3. Methodology: the analysis of the didactic transposition process

Our analysis of the didactic transposition process in the case of proportion will not be as detailed as the one presented by Hersant (2005) because we are not limiting it to a single country nor to a single type of tasks. Our unit of analysis will be broadly defined as the mainstream of the knowledge to be taught in relation to proportionality, at least considering the countries from which we have direct or indirect information (Indonesia, Spanish, French, Portuguese and English speaking countries, etc.). We will take into account the first step of the process of didactic transposition (Chevallard, 1985): the one involving the scholarly knowledge and the knowledge to be taught. And we will only consider three main periods of time that we put in correspondance with three main types of curriculum organisations. We will name the first one "classical mathematics" or "classical period", using the distinction proposed by the mathematician Atiyah (2002) when he presented his vision on how mathematics world is changed around 20th century:

> Let me look at the history in a nutshell: what has happened to mathematics? I will rather glibly just put the 18th and 19th centuries together, as the era of what you might call classical mathematics, the era we associate with Euler and Gauss, where all the great classical mathematics was worked out and developed. You might have thought that this would almost be the end of mathematics, but the 20th century has, on the contrary, been very productive indeed. (Atiyah, 2002, p. 665)

The second period corresponds to the New Math (or Modern Math) reform that took place in between 1960-1980 depending on the countries. This international reform that affected many countries was led by the conviction that mathematics has to act as a driving force in the development of hard sciences and of human and social sciences as well, in citizens' daily lives, and, beyond that, in the modernization of society and particularly at school (Gispert, 2014, p. 238). For example, Euclidian geometry and calculus were no longer taught as such but replaced with algebra of sets, functions or maps, probability theory, and statistics.

The era that began with the "modern mathematics" reform was abandoned in the early 1980s in favour of a teaching method that, envisioning mathematics in the diversity of its applications, placed the accent on problem solving and favoured "applied" components of the discipline page (Gispert, 2014, p. 239). On the one hand, the didactic use of sets raised many issues related to linguistic coherence. On the other hand, it is likely that many teachers felt very uncomfortable with a mathematical content they had not previously experienced and simply tried to reproduce in class (Ausejo & Matos, 2014).

Because the process of didactic transposition is usually represented graphically as a horizontal process (see table 1), we will introduce the historical component by categorizing classical mathematics, math reform, and counter-reform as a vertical transposition.

For the first question, we will use the methodology of didactic transposition to show that the mathematical organisations taught under the labels of ‘proportionality’, ‘ratio and proportions’, etc. appear as an ‘assemblage’ of mathematical elements (concepts, techniques, etc.) that come from different historical periods in the development of mathematics. For this, we need a *reference epistemological model* (Bosch & Gascón, 2006) to describe the proportionality approach in terms of praxeologies.

This reference epistemological model (REM) needs to cover all the notions and relationships that appear in the different periods of time. Our main tool for the REM will be the notion of function between two quantities *A* and *B*. It is presented in Annex 1. To answer the second question, we look at the curricular constraints that we attribute to the didactic transposition processes. Additionally, we will look at these constraints in relation with other sectors, e.g. similarity and (linear) function. In the last phase, we will relate our findings to a similar didactic transposition phenomenon that has been pointed out in the field of science education.

| Years | Scholarly knowledge | → | Knowledge to be taught |
|---|---|---|---|
| Classical math ↓ | | | |
| New Math reform | | | |
| “Counter-reform” | | | |

Table 1. Time dimension in the didactic transposition and focus of the study

## 4. The case of proportionality: a complex didactic transposition process

The following discussion relies on data of books from different periods distinguishing “scholarly” books and “textbooks” even if in practice there is a mixture. The aim is to show the complexity of the didactic transposition process. We categorize mathematicians (e.g. Newton, Euler, Huygens, Leibniz, Laplace, etc.) as scholarly knowledge. While in knowledge to be taught we use school textbooks. Regarding transposition in time, we consider three main periods: classical math, new math and after new math (or current) eras.

### 4.1 Classical mathematics

### 3.1.1 Scholarly knowledge

In the classical mathematics, ratio and proportions are an important tool in all mathematical domains. Mathematicians describe the (mathematical, physical and social) world in terms of ratios and proportions between quantities. Proportions appear as the "basic language" to work with relationships between arithmetical, geometrical and physical entities. For instance, Newton's second law of motion was formulated in 1687: "The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed". Two centuries later, it is still the main language of mathematicians and physicists, as shows the following excerpt from Fourier and Freeman's book *The Analytical Theory of Heat*:

> The increments of volume of bodies are in general proportional to the increments of the quantities of heat which produce the dilatations, but it must be remarked that this proportion is exact only in the case where the bodies in question are subjected to temperatures remote from those which determine their change of state (Fourier & Freeman, 1878, p. 28)

Proportion is not only found as language, but also it is discussed as a chapter title. Let us consider, for instance, Euler's *Elements of Algebra* (first published in German in 1765 and translated into English from its French version in 1828). This book was presented as ”an Elementary Treatise, by which a beginner, without any other assistance, might make himself complete master of Algebra" (p. xxiii). It can be considered as an interface between the scholarly and the school knowledge, since it is authored by one of the greatest mathematicians but is proposed as a learning tool.

The book content is organized in two parts: the ”Analysis of Determinate Quantities" covering from the arithmetical operations with numbers to the resolution of algebraic equations with one unknown, and the ”Analysis of Indeterminate Quantities" addressing more complex equations with more than one unknown. The first part is divided into 4 sections, from the ”Methods of calculating Simple Quantities" with elementary arithmetics, till the resolution of algebraic equations. In this part, the whole section III (50 pages) is devoted to the theme ”Ratios and Proportions". It contains 13 chapters:

SECTION III.

*Of* Ratios *and* Proportions.



Figure *1*. Table of contents of Euler's book *Elements of Algebra* (1828)

This is the structure of what we will call the classical organisation around proportionality, that will be the basis of many of the mathematical organisations found in textbooks for many years, till the arrival of the New Math reform in the 1960s. The definition of different types of ratio (arithmetical and geometrical)[2] are provided, together with their main properties ad transformations. Meanwhile, other necessary notions are provided, like the greatest common divisor to simplify, add or compare different geometrical ratios. A rather developed work with the transformation of geometrical proportions follows, with some proper terminology (the "extremes" and "means" of the proportion, the first and last member, etc.) and their use to solve practical problems about different social and commercial situations (see annex 1).

It is important to notice that, in the section devoted to "the Rules of Proportion and their Utility", the author says:

> This theory is so useful in the common occurrences of life, that scarcely any person can do without it. There is always a proportion between prices and commodities; and when different kinds of money are the subject of exchange, the whole consists in determining their mutual relations. The examples furnished by these reflections will be very proper for illustrating the principles of proportion, and shewing their utility by the application of them (p. 155)

[2] The difference between "arithmetical" and "geometrical" ratios and proportions has now disappeared from our mathematical culture. At a disadvantage, since it permits to distinguish two modalities of growth that still cause many difficulties: what are now called "additive" and "multiplicative" growths.

Follows a variety of examples about money exchanges, including discount, and the elaboration of the "double rule of three" to solve them.

This brief description aims at providing some evidence to the fact that ratio and proportions – and what we newly call "proportionality" – played a central role in the mathematics of the classical period. It appeared as the main tool to describe relationships between quantities, in a similar way as we use functions today.

### 3.1.2 Knowledge to be taught

When we turn to the knowledge to be taught at this same period, we find an important mathematical organisation called the Theory of Ratios and Proportions. This organisation usually appears in Arithmetic books in a simplified version, with few theoretical elements and many practical cases to be solved with different versions of the rule of three (see annex 1). It is also afterwards treated in Algebra books to provide a more general presentation, using letters and further developing the theoretical elements (properties of the proportions, etc.). The theory of ratios and proportions was a core part of the classical mathematics related to the arithmetic and algebraic work. This kind of content organization is then found in many school and college textbooks of arithmetic and algebra during the end of the XIX century and the beginning of the XX, till the New Math reform.

We are taking as example *The principles of arithmetic* and *The principles of algebra* from Hotson (1848). These are textbooks for college students that present a very typical organisation if we compare with the old French and Spanish textbooks of the same period (Hersant 2005, Bosch 1994). The importance of the theory of Ratio and Proportion is clear according to the place it occupies, as we can see in its Table of Contents:

- Definitions (p. 1)
- Vulgar fractions (pp. 2-28)
- Decimal fractions (pp. 29-46)
- Compound numbers and the tables (pp. 47-73)
- Ratio and proportion and their practical applications (pp. 74 – 94)

Figure 2. Table of contents of section "Ratio and Proportion" (Hotson, 1848)

A high school textbook from the U.S. written by Colburn (1862) also presents Ratio and Proportion of one of the important chapters, after the introduction of numbers, fractions and their operations:

XII. RATIO AND PROPORTION.



Figure 3. Table of contents of section "Ratio and Proportion" (Colbur, 1862)

Additionally, a series of school textbooks from primary, elementary, intermediate, and advance lesson education were written by Herz (1920). Proportion is not explicitly discussed in the textbooks, but ratio appears as one of 20 chapters of the intermediate book:

RATIO



Figure 4. Table of contents of section "Ratio and Proportion" (Colbur, 1862)

Even if the notion of proportion is not introduced, fourth proportional problems appears using a rather original technique based on multiplying the fourth number given by a ratio, in a strategy that seems rather close to a functional modelling:

**Exercise 31—Written.**

1. **If 6 pencils cost 12 cents, what will 12 pencils cost? Compare what you want to buy with what was bought. Will they cost more or less money? If they will cost more, the larger number leads in the ratio, if they will cost less the smaller number leads. The ratio then is? They will cost how many times 12 cents?**

Figure 5. Example of resolution of a proprotionality problem. Herz (1920, p. 47)

As was common in this kind of problems, the assumption of the proportional relationship was always taken for granted. At most, the justification one could expect consists in an oral discourse of the sort: “2, 3, etc. times more pencils will cost 2, 3, etc. time more cents, so pencils and cents are proportional quantities and…” This kind of explanation (or “reasoning”, as was then called) will disappeared in the modern forms of proportionality that still persist today.

In summary, what we can call the classical organisation of proportionality can be described in praxeological terms as follows:

- The practical block contains three important types of problems and the corresponding techniques: direct, inverse and compound proportions or rules of three. These problems are then completed with a variety of applications to commerce and trade: money exchange, interest, discount, stocks, bankruptcy, partnership, etc. The techniques used to solve the problem may vary from one educational level, period of time or author to another, but they are all based in the consideration of a co-variation between two quantities and a ratio preservation (see annex 1).
- The technical block contains a technological discourse called the “Theory of ratios and proportions” where several properties of ratios and proportions are introduced and used to compute with proportions in a similar way as we operate with equations nowadays: exchange the middles or the extremes, multiply or divide the first (or last) two terms by the same number, multiply the middles and the extremes, add the corresponding terms, etc. The theory is based in implicit properties of quantities and variation and, of course, all the operations with numbers that are supposed to be previously established.

Usually the notion of ratio and proportion and the simplest techniques of rule of three (mainly the reduction to the unit and sometimes the “cross product” one) are introduced at the primary level in arithmetic courses, while the technology of the ratios and proportions treated at a general level corresponds to a higher level and can appear in algebra textbooks. Even if we are not developing this point here, ratios and proportions also play an important role in the third domain of the mathematics to be taught: geometry (see annex 2).

### 3.2 The New Math reform

As Bjarnadóttir (2014, p. 442) mentioned, the rule of three that existed mainly since the beginning of mathematics, survived well till the twentieth century, but was practically eradicated by the school mathematics reforms of the 1960s. The New Math reform that took place in many countries during the 1960s and 1970s completely transformed the global organisation of the mathematics to be taught, at both primary and secondary level. Kilpatrick (1997, p. 87) depicts the origin of the reform in the following terms:

> Most of the new math reformers believed that the school mathematics program had become so entangled in senseless jargon and was so out of step with mathematics as then taught in the university that it needed a complete overhaul. The language of sets, relations, and functions would provide not only a more coherent discourse in the mathematics classroom but also a more meaningful structure for learning. Students would be drawn to mathematics by seeing how it fit together and, in particular, how the great ideas of modern mathematics brought order into the chaotic curriculum of literal numbers that were not really numbers at all, a notation for angles that did not distinguish between an angle and its measure, and operations on numbers that included turning them over, bringing them down, canceling them, and moving their decimal points around.

In terms of the didactic transposition process (Chevallard, 1985), the New Math reform can be explained in terms of "compatibility" between the knowledge to be taught and the scholarly knowledge:

> On the one side, the taught knowledge […] should be seen, by "scholars" themselves, as *close enough to the scholar knowledge*, so as not risking the mathematicians' denial, which would undermine the legitimacy of the social project, socially accepted and supported, of its teaching. On the other side, and at the same time, the taught knowledge should appear as *far enough from the "parents'" knowledge* […], that is, from the knowledge banalised in society (and especially banalised by school!). [... When] the taught knowledge becomes old in relation to society; a new relationship *shortens the distance with the scholarly knowledge*, the one of the specialists; *and put the parents at*

> *a distance*. This is the origin of the didactic transposition. (Chevallard, 1985, pp. 26-27, our translation)

At the end of the 19$^{th}$ century, after what Atiyah characterised as classical mathematics, the language of functions started replacing the one of ratios and proportions. Mathematics worked with functions, variables, sequences, etc. and they would soon start talking about structures. In what concerns proportionality, the classical organisation of ratios and proportions was considered as obsolete, since functional relationships could perfectly replace it. This is what happens in today's scholarly mathematics. For instance, if we search in the Encyclopaedia of Mathematics[3], no entry is found under the label "proportion". Only a short definition of "Arithmetic proportion" is proposed, as:[4]

> *Arithmetic proportion:* An equation of the form a−b = c−d, where a, b, c, d are given numbers. An arithmetic proportion is also called a difference proportion.

The word appears in the entry "Arithmetic"[5] to explain Euclid's work, even if is assimilated to "the theory of fractions"!

> Books 7–9 of Euclid's Elements (3rd century B.C.) deal exclusively with arithmetic in the sense in which the word was employed in ancient times. They mainly deal with the theory of numbers: the algorithm for finding the greatest common divisor (cf. Euclidean algorithm), and with theorems about prime numbers (cf. Prime number). Euclid proved that multiplication is commutative, and that it is distributive with respect to the operation of addition. *He also studied the theory of proportion, i.e. the theory of fractions*. Other books of his treatise include the general theory of relations between magnitudes, which may be considered as the beginning of the theory of real numbers (cf. Real number).

Therefore, if we look at the new mathematical curriculum that was elaborated, the organisation around proportion has disappeared. The table of contents of a book like (May, 1959) addressed to high school students and teachers, proposes a sequence of chapters titled 1. Elementary Algebra, 2. Elementary Logic, 3. Elementary Theory of Sets, 4. Plane Analytic Geometry, 5.

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[3] https://www.encyclopediaofmath.org

[4] https://www.encyclopediaofmath.org/index.php/Arithmetic_proportion

[5] https://www.encyclopediaofmath.org/index.php/Arithmetic

Relations and Functions, 6. Numbers (!), 7. Calculus, 8. Probability, 9. Statistical Inference and 10. Abstract Mathematical Theories. Of course proportionality appears in chapter 5, with the study of linear functions. Its treatment is, however, rather extreme, but will illustrate how radical the reform was:

> When in scientific discourse it is said that "$y$ varies directly with $x$" or that "$y$ is proportional to $x$", the meaning is that there exists a number $m$ such that $(x, y) \in mI$ whenever $x$ and $y$ are corresponding values of the variables […]. For example, for constant speed, distance is proportional to time. Letting $y$ = distance and $x$ = time, $y = mx$ for some $m$. Here $m$ = the speed. If we know one pair of values, we can calculate $m$ and so determine the function. We call $m$ the *constant of proportionality.* Suppose that we know that when the time is 6 sec the distance is 25 ft. Find the function and determine the distance when time is 10 sec. (May, 1969, p. 271)

When we look at other textbooks like (Murphy, 1966), we see that proportion is not treated at all. It is only used in geometry to define similar triangles: "Two triangles are similar if corresponding angles are congruent and corresponding sides are proportional" (p. 183). And a similar situation appears in (Peterson ,1971).

### 3.3 After the New Math (today's mathematical organisations)

As Kilpatrick (2012, p. 569) says:

> In no country did school mathematics return to where it had been before the new math movement began […] [M]any of the ideas brought into school mathematics by the new math have remained. For example, textbooks still refer to sets of numbers and sets of points. Much of Euclidean geometry has been replaced or overlaid by ideas from transformation geometry, coordinate geometry, and vector geometry, with coordinate geometry appearing early. Pupils encounter and solve inequalities along with equations. Numbers are organised into systems that have properties, such as the distributive property, that can help simplify computations. […] Terms such as *numbers, numeral, unknown, inverse, relation, function,* and *graph* are given reasonably precise definitions and used to clarify notions of quantity, space, and relationships.

However, the counter-reform also restored many (old) praxeological elements and inserted them into the renewed mathematical organisation. In the case of proportions, the old techniques of the rule of three and their theoretical environment reappeared, but have now to coexist with the praxeologies structured around the notion of functions and equations. The mathematical "ecosystem" is not the same anymore. In particular, the role played by quantities in the classical mathematical has now been replaced by a rigorous construction in terms of real numbers and functions of numerical variables that, even if it is not explicitly introduced at school, remains in the background of the whole curriculum organisation. As Hersant (2005) noticed, since 1977, the return to the study of concrete situations did not mean the return of a really work on proportional quantities but rather on sequences of measures that end up being only numbers.

We are not entering here in a detailed analysis of today's knowledge to be taught around proportionality – which might vary from one period of time to another, from one country to another and even from one textbook to another (Ponte & Marques, 2011). We will only state – and leave it as a working hypothesis – that we are in front of blurred or hybrid organisations made of pieces taken from different eras, mixing up elements of different praxeologies that maintain redundancies and some incoherence in the kind of tools used. For instance, functions or relationships can be introduced to define proportionality, but then only the old techniques of the rule of three (more or less modernised in terms of tables and cross-products) are used to solve the problems.

A very radical example of this situation can be found in a grade 7 Spanish textbook of the period just after the New Math reform, where a chapter about "The Linear Function" is immediately followed by a chapter about "Ratios and proportions" without any mention of functions in the second case and only the following section in the first one (Martínez, 1974, p. 135, our translation):

> *Linearity and proportionality*
>
> Given two distinct numbers $x_1$ and $x_2$ and their images by a linear map $f$ of coefficient $a$. Let us put $y_1 = f(x_1)$ and $y_2 = f(x_2)$. We have $y_1 = ax_1$ and $y_2 = ax_2$.
>
> The *quotients* or *ratios* are $y_1/ax_1$ and $y_2/ax_2$ are equal to $a$.
>
> The identity of two ratios $y_1/ax_1 = y_2/ax_2$ is called a *proportion*.
>
> We then have the following result: *two numbers from the starting set and their images form proportion*.

In the chapter called "Application of linear functions to problem solving", the notions of quantity and those of directly and inversely proportional quantities are introduced, and all problems solved using the traditional direct and inverse rules of three. The language of functions only appears in two occasions, playing a purely formal theoretical role. In the first, after presenting a problem about a bag of 60kg of coal costing 45 pesetas, and asking for the price of a bag of 180kg of coal, the authors state that the goods and the price are quantities related in such a way that to each weight only one price is associated and that when multiplying the number of kilograms (not the *weight*) by 2, 3, etc. the price is also multiplied by 2, 3, etc. They then define:

> Two quantities are directly proportional when to each quantity of one corresponds only one quantity of the other and, moreover, when multiplying the first by a number, the second is multiplied by the same number.
>
> Between both quantities there exists a *linear map*.
>
> This linear map is *bijective*, and it is an *isomorphism*.
>
> In effect:
>
> $f(60 + 120) = f(180) = 135$ ptas.
>
> $f(60) + f(120) = 45 + 90 = 135$ ptas.
>
> Then $f(60 + 120) = f(60) + f(120)$
>
> *The homologue of the sum is equal to the sum of homologues.*

And a similar statement is made in the case of inversely proportional quantities by establishing that in this case there exists "an isomorphism between *M* and 1/*M'*, the elements of 1/*M'* being the inverse of the elements of *M'*." However, the list of practical problems included in these chapters are only solved using techniques based on the rule of three: the tabulation of the data given in two columns and the effectuation of cross-products to find the unknown quantity, the order of the elements depending on the type of proportionality (direct or inverse) established between any pairs of quantities.

This is in our opinion a paradigmatic case of two different mathematical organisations that are only juxtaposted, that will coexist during a certain period of time before being mixed in different ways and degrees. The New Math reform tried to replace the old language of ratios and proportions by the language of functions, without really succeeding in it. In fact, functions are still not being widely considered as a tool to solve problems; they are rather approached as an object to describe than an instrument to use. It is curious, for instance, that the language of functions – and their symbolism – that we as didacticians use to describe the mathematical organisations around proportionality are still not seen as basic enough to become part of the students' elementary praxeological equipment.

As we have done to develop the REM, Fernandez et al (2010, p.4) also used the functional symbolism to explain the types of problems and strategies considered in their research:

> Strategies influenced by multiplicative approaches are based on the properties of the linear function: $f(a + b) = f(a) + f(b)$ and $f(ma) = mf(a)$ taking into account the transformations between or within ratios [...] [T]he use of erroneous proportional strategies in nonproportional tasks can be exemplified by the following non-proportional problem that is modeled by $f(x) = x + b$, $b \neq 0$ (additive situation) [...]

However, they certainly do not expect the students use these functional tools to solve the proposed problems!

## 5. Discussion

### A hybrid organisation

The situation described in the previous section can be summarized in table 3.

| **Period of time** | **Scholarly knowledge** | **Knowledge to be taught** |
|---|---|---|
| Classical math | Proportions as the main tool to describe and establish relationships between quantities (numerical, geometrical and abstract - algebraic) | Ratios and proportions<br>(proportionality between quantities) |
| New math | Functions as the main tool to describe relationships between variables. The construction of the set of real numbers avoids the use of the notion of quantity, which is relegate to its use in sciences | Sets, maps, functions (only abstract numbers, no quantities) |
| Counter reform (current situation) | | Proportionality between numerical variables (coexistence of ratios and proportions with linear functions) |

Table 6. Summary of the didactic transposition process related to proportionality

In short, we can say that mathematics (from Euclid till Euler or even later) ratios and proportions was a central issue in the old days. Then, with the introduction of functions, it became an auxiliary notion and finally disappeared from the scholarly mathematical kingdom. (The situation is not the same in sciences where there still a practical use of proportions and ratios, especially via the presence of quantities and the dimensional analysis.) In today's textbooks, proportionality, ratios and proportions (even in geometry) are defined in terms of numbers, not quantities. This is very different from what happened in the old Arithmetic and Algebra books where ratios, proportions, etc. were always defined as relationships between quantities.

A possible explanation is that with the New Maths reform, quantities were taken out of school mathematics in benefits of “real numbers”. Then “real numbers” became “numbers” but quantities did not come back to school mathematics. At secondary level, the notion of variable could play the role played by quantities in the classical mathematics. But at primary level, all relationships are defined among numbers, which introduces a lot of redundancies that are never explicitly treated: fractions and ratios, proportions and equations, proportionality and linear function, similarity and geometrical transformation. A similar situation appears with similarity in geometry: proportionality here affects “lengths” of sides of figures, but it is often defined among numbers.

In the school mathematical organisation before the New Math, ratios were always defined between two magnitudes of the same quantity (even if afterwards the computations are done with numbers). A “theory of quantities” is assumed in textbooks, even if it is not explicitly presented. The New Maths reform took quantities out of school mathematics in the benefit of sets of numbers, variables, relations and functions (or maps). In our current mathematical organisations, we can identify some “pieces” of the classic theory of ratios and proportions but they appear without quantities, only with numbers (at primary school) and eventually with variables. This last is the option taken by Wikipedia in its entry “Proportionality”:

> In mathematics, two variables are proportional if a change in one is always accompanied by a change in the other, and if the changes are always related by use of a constant multiplier. The constant is called the coefficient of proportionality or proportionality constant.
>
> If one variable is always the product of the other and a constant, the two are said to be directly proportional. x and y are directly proportional if the ratio y/x is constant.
>
> If the product of the two variables is always a constant, the two are said to be inversely proportional. x and y are inversely proportional if the product xy is constant.
>
> To express the statement "y is directly proportional to x" mathematically, we write an equation $y = cx$, where c is the proportionality constant. Symbolically, this is written as $y \propto x$.
>
> To express the statement "y is inversely proportional to x" mathematically, we write an equation $y = c/x$. We can equivalently write "y is directly proportional to 1/x".

> An equality of two ratios is called a proportion. For example, a/c = b/d, where no term is zero.

We can thus conclude that today's mathematical curriculum is made of *disconnected pieces,* as has been shown by Wijayanti (2017a) and others that come from mathematical organisations of different ages. We postulate that the ambiguous role played by quantities is one of the main reasons of the difficulty for functions to appear as a structuring tool.

**Quantities and proportion in the literature review**

Quantities have a fuzzy status in the scholarly knowledge, in spite of some isolated trials to mathematise them (Whitney, 1968). The set of real numbers seems to provide a rich enough environment for any elementary work of arithmetic, algebra, geometry and calculus to be carried out. However, the use of mathematics as models for the study of extra-mathematical situations is difficult without the treatment of quantities. The issue of quantities from research on ratio and proportion by Adjiage and Pluvinage (2007) found that systematically working the separation and the articulation between within the physical and the mathematical domain involved in ratio helps pupils to discern invariants and access the proportionality model in better conditions. However, they added that this is not guarantee that this 'ratio literacy' can increase student ability in the algebra domain. Therefore, the authors agree that quantity is important to learn proportion, but they are still questioning the importance of quantity in algebra. Another research (Sierpinska, 1992) shows that quantity is also important in understanding functions. However, the author stressed that to distinguish numbers and physical magnitudes is one thing, and to understand the relationship between a physical law and a mathematical function is another – and this seems to be a condition for making sense of the concept of function...

**A similar phenomenon in Genetics**

The evolution of didactic transposition ends up with a curriculum made of mathematical organisations that come from different historical ages. We then see knowledge organisations that correspond to different periods in the evolution of science and that have difficulties to coexist. The connections are not trivial: they need to "destroy" some objects (notions, techniques, etc.) that become redundant. And, as is the case with quantities in mathematics, they also need to create new objects in order to provide students with more powerful tools to carry out the mathematical activities they are assigned.

In the domain of science education, Gericke and Hagberg (2007) reviewed contemporary literature of history and philosophy of genetics. As a result, five different historical models of gene function were categorized: the Mendelian model, the Classical model, the Biochemical-Classical model, the Neo classical model and the Modern model. Then, Gericke and Hagberg (2010) used the models to identify the representation of the phenomenon of gene function in current biology and chemistry upper secondary school textbooks. They found that most textbooks used neoclassical models to describe gene function. Contrarily, very few modern models appeared. They also mentioned that this situation inhibits students' ability to improve integration of concepts and biochemical processes from molecular genetics with those of classical genetics. The results of this paper also show that often textbooks propose hybrids models elaborated from two or more historical models. They added that different ways of presenting gene function can lead to logical inconsistencies that introduce ambiguity into the definitions of gene function – and, thus, in the students' knowledge.

We consider that this phenomenon can be approximated to the one of the teaching of proportionality and propose as further research to try to establish more connexions between the two research lines.

## 6. Concluding remarks

In this paper we have presented an example of the use the methodology of didactic transposition analysis to include the historical perspective in textbooks analysis. The research relies on the use the praxeological frame to give a more precise account of the mathematical knowledge to be taught about proportionality in lower secondary school, the disconnections found and the complex origin of the different pieces of knowledge that conform the knowledge to be taught. Finally, as further research to develop, we point out the existence of a possible similar didactic transposition phenomenon to the one analysed by Gericke and Hagberg (2010) in the case of genetics. In our case, we have seen that proportionality remains as a piece of the current knowledge to be taught in spite of its disappearance from scholarly mathematics. Does this contradict the theory of didactic transposition? We postulate that one source of legitimacy comes precisely from mathematics education and, in particular, research on cognitive education (from the tradition of genetic psychology) that have been considering since the last 30 years "proportional reasoning" as one the most important mathematical school learnings. Another source of legitimacy might come precisely from the natural sciences, where the language of ratios and proportions (particularly supported by different kinds of quantities) is till used in the scholarly institution. It thus seems that the mathematicians' institution has been replaced by other "scholarly" institutions of another kind. However, our hypothesis of the central role of quantities in the difficulties to connect proportionality and functions and, furthermore, to give more efficacy to functions in the students' mathematical work raises the need of a real transformation of school mathematics where quantities had a role in relation to algebra and functions which is not the case nowadays. A development of the didactic transposition work is necessary and cannot be done individually by teachers. It certainly needs the cooperation of the whole mathematical and educational community, including mathematicians.

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**Annex 1. Reference epistemological model on proportionality**

We propose a description of the classical mathematical organisations around proportionality structured in terms of praxeologies: types of tasks, techniques, technological and theoretical elements. We are using the functional notation to describe the techniques, even if this formalism is not part of the techniques.

THEORETICAL AND TECHNOLOGICAL ELEMENTS of the classical Arithmetic

- *Quantities* are primary notions. A quantity corresponds to anything that varies and can thus be measured using a unit (a specific element of the quantity).
- Elements of a quantity can be added and multiplied by a scalar ("take 2, 3, 4 times more or times less"). As a consequence, the *ratio* between two elements of a quantity is always defined.
- A direct or inverse *proportional relationship* between two quantities is defined by the relationships between the ratios of correspondent elements: identity of the direct ratios in case of direct proportionality, identity of the inverse ratios in case of inverse proportionality.
- The explicit relationship (or function) between the two quantities is not known; only the relationship between the ratios of the elements (variations).
- In case of several quantities, the relation of proportionality (direct and inverse) can be generalized by multiplying the ratios (direct or inverse, depending on the case).
- We can formalize it as follows:

  $f: A \rightarrow B$ is direct-proportional if

  $$f(\lambda a) = \lambda f(a) \text{ for all } a \in A \text{ and any scalar } \lambda \qquad (1)$$

  $f: A \rightarrow B$ is inverse-proportional if

  $$f(\lambda a) = (1/\lambda) f(a) \text{ for all } a \in A \text{ and any scalar } \lambda \qquad (2)$$

  $f: (A_1, A_2, \ldots A_n) \rightarrow B$ is compound-proportional if

  $$f(\lambda_1 a_1, \lambda_2 a_2, \ldots, \lambda_n a_n) = \lambda_1^{\alpha} \lambda_2^{\alpha} \ldots \lambda_n^{\alpha} f(a_1, a_2, \ldots, a_n) \qquad (3)$$

  with $\alpha = 1$ in case of direct proportionality and -1 in case of inverse.

- Even if it is not usually explicitly stated, it can be deduced from (2) that $f$ is also additive:

  $$f(a + a') = f(a) + f(a')$$

  since $f(a + a') = (a + a')f(1) = af(1) + a'f(1) = f(a) + f(a')$

- It can also be deduced that $f(a) = ka$ for all $a \in A$ taking $k = f(1)$ (linear relationship). However this supposes that we are considering $a$ and $f(1)$ as numbers instead of quantities and, thus, dimensionless. In the classic Arithmetic, multiplication between quantities is not always defined, only when it can be interpreted in cultural terms.
- To these theoretical elements about quantities and relationships between quantities, needs to be added the rules, description and justification of the calculation with ratios and proportions, what is

called “The theory of ratios and proportions” and that we call the “algebra of proportions” since it consists in a set of written manipulations of proportions.

TYPE OF PROBLEMS

Given two quantities $A$ and $B$ with a direct-proportional relation $f$,

given $a$, $a' \in A$ and $b \in B$ such that $b = f(a)$, find $b' = f(a')$.

[Similar types of problems can be defined in case of inverse and compound proportionality]

TECHNIQUES

*In some cases, the computations are directly applied from the reading of the problem. In many others, the first element of the techniques is to organize the quantities given in a table with a blank or an* x *to indicate the missing value (any ordered table is possible):*

| *Table 1* | *Table 2* | *Table 3* | *Table 4* |
|---|---|---|---|
| $a$ — $b$ | $a'$ — $x$ | $b$ — $a$ | $x$ — $a'$ |
| $a'$ — $x$ | $a$ — $b$ | $x$ — $a'$ | $b$ — $a$ |

*Note that the technological elements about quantities are only applied to derive an identity between ratios (a proportion) from the table. Once the proportion is written, its terms are interpreted as numbers (dimensionless quantities) and, thus, any calculation is possible.*

REDUCTION TO THE UNIT:

$f(a) = b \Rightarrow f(1) = (1/a)f(a) = b/a$ (“the image of 1 unit is $a$ times less the image of $a$”)

$b' = f(a') = f(a' \cdot 1) = a' f(1) = a' \cdot b/a$ (“the image of $b$ units is $b$ times more the image of 1”)

*Observation: The technical tools are mainly discursive and written computations at the end of the so called “reasoning”.*

SIMPLE RATIO:

In case we can easily find a □ such that: $a' =$ □$a$ (“$a'$ is □ times $a$”)

We obtain $b' = f(a') = f($□$a) =$ □$f(a) =$ □$b$ (“the image of □$a$ is □ times the image of $a$: □ times $b$”)

*Observation: This technique is not “institutionalized” but can be derived from the technological elements and applied in simple cases. It leads to the following one.*

PROPORTIONS (“internal ratio”):

$f(a')/f(a) = f(a' \cdot a/a)/f(a) = a'/a \cdot f(a)/f(a) = a'/a \Rightarrow b = f(a') = (a'/a) \cdot f(a)$

*Observation: This is the general technique. Because f preserves ratios, we can directly write the identity without calculating the value of the ratio (which is not always easy to find). Once the identity is established, we can calculate with the proportion and obtain the final answer.*

*The two final techniques do not correspond to the same arithmetical technology and theory since the identities (proportions) are directly written between the elements of different quantities. This supposes that the quotient of two quantities is always defined and that the multiplication between a quantity A and the quotient A/B of quantities A and B is always defined. These techniques appear when the tabular model is considered as dimensionless and the work is done directly with numbers instead of quantities.*

COEFFICIENT OF PROPORTIONALITY:

Because $f(x) = kx$, we have $b = ka$ and $b' = ka'$

Thus $k = a/b$ and we have $b' = (a/b)a'$

PROPORTIONS ("external ratio"):

Because $f(x) = kx$, we have $b = ka$ and $b' = ka'$ (*)

Thus: $b/a = b'/a'$ (= $k$) and we have $b' = (a/b)a'$

*OBS:* Dividing both equations (*) we obtain homogeneous ratios: $b/b' = (ka)/(ka') = a/a'$.

*PASSAGE TO INVERSE AND COMPOUND PROPORTIONALITY*

*All the techniques can be adapted to the inverse-proportionality case, just taking the inverse of the ratios.*

*The same can be done with compound proportionality by multiplying the diverse ratios (direct or inverse depending on the case). Notice than in this case because the multiplication is always between ratios, there is no problem with the quantities used. In the two final techniques, it supposes to work with the product of quantities (or numbers as measures of the quantities).*

**Annex 2. Excerpts of the book Callendar, B. F. (1836). Geometry. New-York: Francis.**

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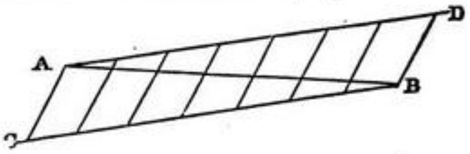
PROBLEM IV.

To divide a given straight line into any number of equal parts.

Divide the line A B into seven equal parts.



From the points A, B, draw the lines A D and B C, indefinitely, parallel to each other. On them, from A and B set off the number of parts required: join the opposite points, and it will divide the line A B as proposed.

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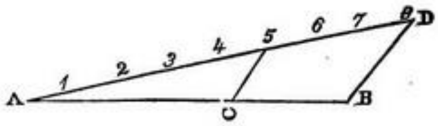
PROBLEM V.

To divide a line in any given proportion.

Divide A B in the proportion of 3 to 5.



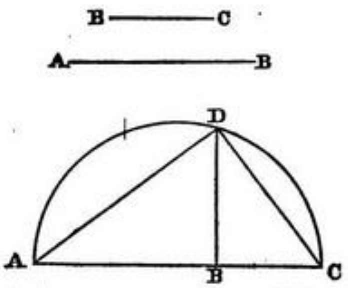
From the point A, draw A D of indefinite length. On it, set off eight equal divisions of indefinite length: join B with the eighth division, and parallel to it join C to the fifth division. Then A B will be divided in the proportion required.



PROBLEM VI.

To find a mean proportional to two given lines.

Find a mean proportional between the lines A B and B C.



Draw a line A C equal to A B and B C. From the centre of the line A C as diameter, describe a semicircle. At the point B draw a perpendicular B D and it will be a mean proportional between A B and B C.