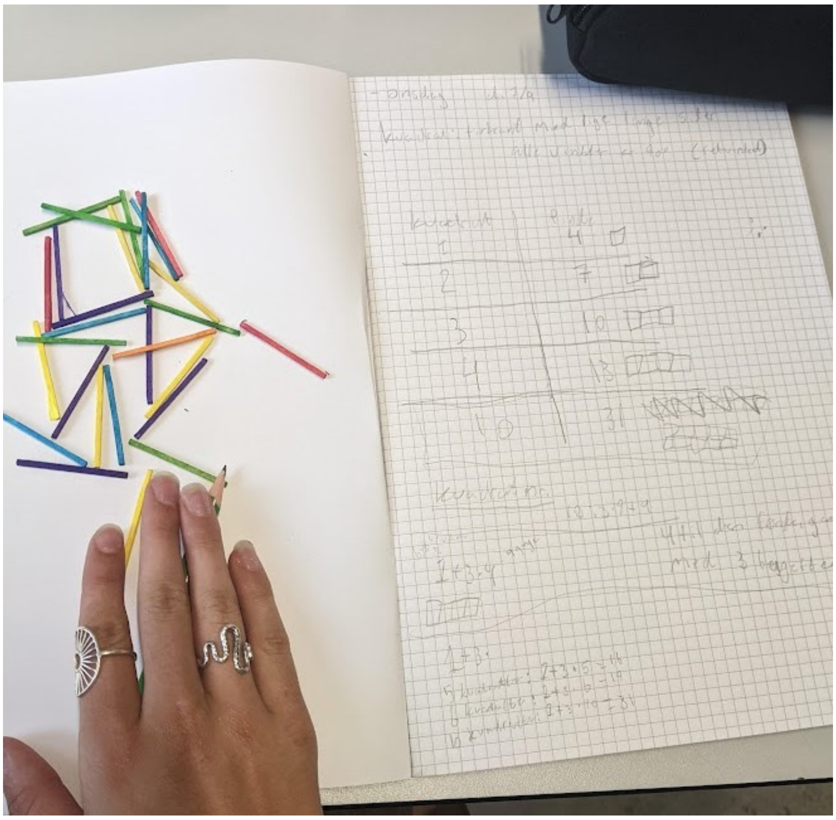
KØBENHAVNS UNIVERSITET
INSTITUT FOR NATURFAGENES DIDAKTIK



PhD Thesis

Pia Beck Tonnesen

The transition from arithmetic to algebra

Diagnosis-based experiment use of resources from Japan

Supervisor: Carl Winsløw

This thesis has been submitted to the PhD School of The Faculty of Science, University of Copenhagen 10. December 2024.

The transition from arithmetic to algebra
Diagnosis-based experiment use of resources from Japan

Doctoral Dissertation

Pia Beck Tonnesen

Department of Science Education
University of Copenhagen
Denmark

Supervisor Professor Carl Winsløw

To my family Frida, Jonas, Lasse and Allan

Table of contents

Acknowledgements

First of all, a big thank you to all the schools, mathematics tutors, mathematics teachers, and most importantly, the students who have contributed to this project. Your benevolence and input have been invaluable, and without them, this project would not have been realized.

I would like to extend my gratitude to Lene, Signe, Eva, and the rest of the green room and department 4262. Without your treatment, care, and support, this project would not have been completed.

My sincere thanks go to the Danish Council for Educational Research for providing me with the opportunity for three years of academic immersion, and to University College Copenhagen for supporting my PhD project. A special thanks to Docent Maria Christina Secher Schmidt, whose willingness to provide professional and personal sparring has been invaluable.

I am deeply grateful to my colleagues in the Department of Science Education, who, despite the challenges of the pandemic and the reforms, created the best framework for my PhD journey. Your professionalism, commitment, and care have been greatly appreciated. The mathematics education research group at IND – Carl, Britta, Mayu, Rongrong, and Derya – have been my 'research family', providing invaluable insights and perspectives through seminars, professional conversations, travels, and meetings.

My PhD program would not have been the same without my PhD office colleagues Derya, Sofie, and Jacob. Your help, encouragement, and good humor have made the long days at IND truly enjoyable.

Finally, to my co-supervisor Associate Professor Britta Eyrich Jessen thank you for your support, throughout the project. And to my supervisor, Professor Carl Winsløw, a heartfelt thank you for believing in me and my project, I will always be grateful for your unwavering support, your guidance, and your ability to instill hope, even at the darkest end of the tunnel.

Abstract

The starting point for this project was a longstanding public concern about the state of school mathematics in Denmark, compared with results from other countries, in particular in East Asia, and not least when focusing on arithmetic and algebra. Our first step was to develop a method for diagnosing the current "state" in a more comprehensive way than to simply look at students' results on tests and the obvious difference between school mathematics curricula in Denmark and in East-Asian countries like Singapore and Japan. The diagnosis is based on praxeological analysis in the sense of ATD (the anthropological theory of the didactic). It shows a gap between the official goals and the current teaching where algebra is not extensively taught as a modeling tool and students' success with associated tasks is similarly modest. The didactic transposition of school algebra appears rather fragmented, distributed over several years and without a core progression that is actually taught to, and learnt by, all students.

The diagnosis provides further motivation and context to the main goal of this thesis: investigating to what extent a research-based textbook material can support teachers (who do not habitually have access to such material) in the teaching of introductory school algebra. Concretely, we have investigated how Danish lower secondary school teachers use a translation of the relevant chapter from a Japanese textbook, with almost no instructions for how to use the text; and to what extent this use supports the students' transition from arithmetic to algebra. Both the differences in mathematical progression in the two curricula, and in habitual organization of didactic processes, lead to expect a number of obstacles to the implementation of the Japanese text in a Danish context. While these were also to some extent observed, we found a number of unexpected or at least non-trivial potentials of the use of the Japanese material.

Teachers successfully used the chapters' challenging "launch problem" to demonstrate school algebra as a modelling tool and to furnish a common example for several points in the chapter, and they were able to implement the focus on explaining and justifying important notational conventions (like the suppression of the multiplication symbol in most algebraic expressions). Concrete tasks and explanations in the text were generally used in the teaching without substantial obstacles. It appeared to be much less straightforward to realize the textbook's aims concerning theory (general principles and definitions). Despite these challenges, our data suggest that the teacher' didactical and mathematical profit from using the Japanese textbook went beyond what appeared directly in their first attempt to use it in teaching, so that later use could conceivably realize the potential of the text more fully.

Resumé

Udgangspunktet for dette projekt er en mangeårig offentlig bekymring over skolematematikkens tilstand i Danmark, specielt indenfor aritmetik og algebra, særlig når man sammenligner danske resultater i internationale undersøgelser med resultater fra især Østasiatiske lande. Første skridt i projektet var at udvikle en metode til at diagnosticere den nuværende »tilstand« på en mere omfattende måde, end blot at se på elevernes resultater i test eller på den åbenlyse forskel der er mellem skolematematikkens læseplaner i Danmark og i lande som Singapore og Japan. Diagnosen er baseret på praxeologisk analyse i betydningen ATD (den antropologiske didaktiske teori). Diagnosen viser en afstand mellem de officielle mål og den nuværende undervisning, hvor algebra ikke i udstrakt grad undervises som et modelleringsværktøj, og elevernes succes med tilknyttede opgaver er tilsvarende beskeden. Den didaktiske omsætning af skolealgebra er fragmenteret, da den er fordelt over flere år og uden tydelig progression eller kerneelementer, der læres af alle elever.

Diagnosen giver yderligere motivation og baggrund for denne afhandlings hovedformål: at undersøge, i hvilket omfang et forskningsbaseret lærebogsmateriale kan støtte lærere i undervisningen i skolealgebra. Konkret er der undersøgt, hvordan danske folkeskolelærere bruger en oversættelse af et relevant kapitel fra en japansk lærebog, uden lærerne fik nærmere instruktioner i brug af materialet, og i hvilket omfang denne lærebogsbrug understøtter elevernes overgang fra aritmetik til algebra. Forskellene i den matematiske progression i de to læreplaner gav anledning til at forvente en række forhindringer i anvendelsen af den japanske lærebog i en dansk kontekst. Disse forhindringer blev til en vis grad observeret, men der er også en række uventede fund eller ikke-trivielle potentialer i brugen af det japanske materiale.

Lærerne brugte med succes kapitlets udfordrende »åbningsproblem« til at demonstrere skolealgebra som et modelleringsværktøj og som fælles reference-eksempel gennem kapitlet. Lærerne implementerede også kapitlet fokus på at forklare og begrunde vigtige notationskonventioner, som f.eks. undertrykkelse af multiplikationssymbolet i de fleste algebraiske udtryk. Generelt blev materialets konkrete opgaver og forklaringer brugt i undervisningen uden væsentlige forhindringer. Det var dog tegn på at det er mindre ligetil at realisere lærebogens teoretiske mål (generelle principper og definitioner). På trods af disse udfordringer tyder vores data på, at lærernes didaktiske og matematiske udbytte af at bruge den japanske lærebog går ud over, hvad der fremgik direkte af deres første forsøg, så senere brug kunne tænkes at realisere tekstens potentiale mere fuldt ud.

List of Papers

Paper I:

Tonnesen, P. (in review). Diagnosing the state of lower secondary algebra. Revised version submitted April 2024 to *Recherche en Didactique des Mathématiques.*

Paper II:

Tonnesen, P. (2024a). A comparative study of didactic moments in a first chapter on algebra in Danish and Japanese middle school textbooks. *Hiroshima Journal of Mathematics Education 17*, 1- 24.
https://www.jasme.jp/hjme/download/2024/Vol17_01.pdf

Paper III:

Tonnesen, P. & Winsløw, C. (in review). Conditions and constraints for using a foreign textbook to support the transition for arithmetic to algebra. Submitted August 2024 to *Research in Mathematics Education*.

Paper IV:

Tonnesen, P. (2022). Diagnostic test tool based on a praxeological reference model to examine students' technical and theoretical algebra knowledge. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12).*
https://hal.science/hal-03745449/

Paper V:

Tonnesen, P. (2024b). The role of algebraic models and theory in Danish lower secondary school. In I. Florensa, N. Ruiz-Munzón, K. Markulin, B. Barquera, M. Bosch & Y. Chevallard (Eds.) *Extended Abstracts 2022: Proceedings of the 7th International Conference on the Anthropological Theory of the Didactic*. Birkhäuser. https://doi.org/10.1007/978-3-031-55939-6_52

1 Introduction

The thesis consists of five peer reviewed papers (of which two are still in second review), complemented by the present introductory text which provides

- an overview of central problems and ideas in the thesis (section 1)
- outline of the theoretical framework (section 2) on which the research questions (section 4) and the methodology (section 5) are based
- a systematic and relatively extensive review (section 3) of previous research in the areas related to the research questions, lending further motivation and context to these
- a comprehensive outline and discussion of the results presented in the five papers (sections 6 and 7)
- conclusions related to the overarching research questions of the thesis (section 8).

1.1 Motivation

The teaching of basic algebra at the lower secondary school level has a significant impact on students' subsequent educational prospects and constitutes a fundamental component of mathematics at the upper secondary school level (Grønmo, 2018). Analysis of data from international large-scale studies has revealed significant differences between countries when it comes to the outcomes of this and other crucial parts of school mathematics. The PISA 2018 study has shown that 14.8 % of Danish grade 9 students can be described as low achievers in mathematics (Christensen, 2019). These somewhat disappointing results do not only appear at such advanced levels, but can be traced back to primary school, with a declining trend.

According to the TIMSS study from 2019, there has been a significant decline in the mathematical skills of Danish fourth graders (aged 10-11) (Keldsen et al., 2021). Danish media echo and reinforce this narrative about unsatisfactory and declining mathematical skills among Danish students at all educational levels. For instance, it attracted significant public attention that more than 70% of first-year students at Copenhagen Business School failed a screening test on basic arithmetic and algebra (Ritzau, 2020). The PISA 2022 study led to headlines such as "Danish students have never performed worse in mathematics" (Ravn, 2023), "Danish 15-year-old have become “significantly” worse at reading and mathematics: Causes for concern” (Tofte, 2023) and “One in five Danish students do too badly in mathematics to become an engaged citizen” (Bjerril, 2024).

The widespread concern about students’ results in mathematics relates to a more or less explicit awareness of the functions of the discipline in the life of individuals, in education, at the labor market and in society. This motivates the political interest of studies such as TIMSS and PISA, confirmed for instance by the fact that PISA is organized and funded by the OECD, operating more generally with the idea of “knowledge economy” in which competences, and not least mathematical competences, are crucial currencies (cf. Gonczi, 2006). Subsequently, mathematics educators in Denmark and other countries have assumed the task of specifying and framing such competences in school settings. The version adapted for curriculum writing in the Danish context considers mathematical competence as follows: “to be able to act appropriately way in situations involving mathematics” (Niss & Jensen, 2002; Education, 2019). Furthermore, elaborate definitions of particular mathematical competences (such as “problem solving competence”) are more or less explicitly referred to in the current programs.

Non-withstanding the efforts to adjust Danish curricula to international trends of aligning educational and economical strategies, the aforementioned disappointments related to international measurements of students’ mathematical knowledge or competence have continued to occur. The government has appointed a wealth of committees and “expert groups” to try to provide advice on how to cope with this so-called “math crises”. Most recently, the Danish Ministry of Children and Education established a “mathematics expert group”, consisting of teachers from primary and lower secondary school, upper secondary schools, vocational education, university colleges and universities, including a few researchers in mathematics education. The relatively broad constitution of this group enabled a focus on transitions between different institutions and thus the way in which the “math problem” arises and accumulates across the entire educational system. The remit of the expert group was to identify the key challenges and a gross catalogue of recommendations stretching across the institutional spectrum from comprehensive school to upper secondary and vocational education (Education, 2022).

One of the key challenges identified concerns the area “arithmetic and algebra”, where students’ progression from primary school to secondary education and into higher education needs further attention as failures are deemed to be of lasting and detrimental nature to mathematics learning more generally (Education, 2022). This can to some extent be considered a departure from the popular, but increasingly obsolete idea that mathematics learning can be reduced to developing generic competences without explicit and continuous attention to knowledge and skills in particular key domains.

The first solution recommended by the group of experts was to elaborate and implement an "arithmetic- and algebra strategy" for the next 8 – 10 years, to establish a broad understanding of the seriousness of the existing challenges in arithmetic and algebra across educational institutions, and to strengthen research in the didactics of arithmetic and algebra at all levels (Education, 2022).

It is my intention to contribute to the last part of this recommendation by providing a more comprehensive diagnosis of the challenges encountered in the transition from arithmetic to algebra in lower secondary school. A multi-faceted diagnosis could conceivably be used as the basis of targeted interventions like the choice and use of quality instructional materials relating to the transition from arithmetic to algebra, as considered in this thesis. As a teacher educator, my interest in the quality of instructional materials is in part related to a concern about the role that mathematics teachers could play in updating and reinforcing the teaching of algebra at the lower secondary level.

1.2 The didactic transposition as a theoretical framework for the guidance of the research.

Teacher educators (as the author of this thesis) are well situated to experience that educational problems must be considered from a wider systemic perspective and cannot be reduced to single factors like the formal preservice training, the official curriculum, textbooks etc. – they all have their place and role but neither can be expected to be independent of the other. The systemic perspective implies larger units of analysis than classical classroom research, psychometric testing etc.: we need to address the way in which mathematical knowledge (in a broad sense) is disseminated in and between educational institution as well as in society at large, or in short: didactical phenomena. I will now explain how didactic transposition provides the necessary systemic perspective for my study.

Chevallard (1985) published a foundational first elaboration of a theory on how knowledge "travels" from its origin and forms in scholarly institutions, to be subsequently selected and declared as knowledge to be taught in school, to become finally taught and learnt knowledge in didactic institutions (cf. also Chevallard & Bosch, 2020). Through case studies (not least related to the radical reforms of secondary school in the 1970s) this initial work demonstrated that the knowledge taught at school is derived from other institutions, shaped by concrete practice and organized and reorganized as “knowledge objects” which may appear quite different from their origin.

The didactic transposition in mathematics education has its origins in what Chevallard describes as *scholarly knowledge*, which concerns both academic and societal forms of mathematical practice and theory (Chevallard & Bosch, 2014). This knowledge is then transposed (selected, adapted, reformulated…) to *knowledge to be taught*; the authors or producers of this transposition are policy makers, curriculum designers, and textbook authors. This group of positions in relation to the didactical transposition is called the *noosphere*, as the persons filling these position “think” about teaching in schools but are positioned in a “sphere” around but outside of the school institution. The main role of the noosphere is to negotiate and deal with the demands that society places on the educational system, while selecting, declaring and reorganizing scholarly knowledge to form the knowledge to be taught (Chevallard, 2019). The noosphere has considerable impact on the extent to which specific objects, connections and foundations from the scholarly knowledge are preserved, or conversely abandoned in order to accommodate new and pressing needs and demands from society at large. Especially when crises arise with respect to the learnt knowledge, as was alluded to in the previous section, it can be important for locating causes to consider this earlier part of the transposition, like whether crucial elements have been omitted (Bosch & Cascón, 2006); any intervention in the school institution must take the knowledge to be taught into account. The transposition process as a whole is commonly represented as in Figure 1.

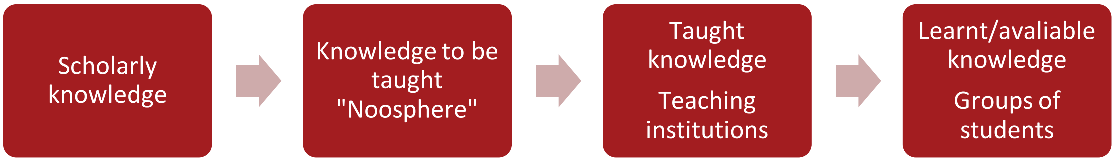


Figure 1: diagram of the process of didactic transposition, modified from (Chevallard & Bosch, 2014)

While the process appears in this rough illustration as one directional, there is of course also elements of response or retroaction in the opposite direction. If the learnt knowledge differs markedly from the taught knowledge, it can evidently impact on later versions of the taught knowledge; interaction between teachers and students can of course also make this retroaction almost immediate. If the taught knowledge undergoes changes, it may influence the noosphere and consequently the knowledge to be taught. Other opposite impacts, directly from measurements of learnt knowledge to the noosphere, were alluded to in the previous section.

Similarly poor exam results after primary school create a demand for educational institutions to improve in more or less unspecified manners. In the context of Danish strategy for arithmetic and algebra, the noosphere was entrusted with the responsibility of formulating recommendations for school institutions and (at least in theory) also for its own activity of specifying knowledge to be taught.

The transposition process within the school, from knowledge to be taught through taught knowledge to learnt knowledge is called *internal didactic transposition*, while the transposition from scholarly knowledge to knowledge to be taught, carried out by the noosphere outside of the school institution, is called *external didactic transposition* (Bosch & Gascón, 2006; Chevallard 2019). Almost all "mainstream" research on mathematics education is concerned with mechanisms and interventions in the internal didactic transposition. To make our position on the external didactic transposition clear, we elaborate on it in the next two sections.

1.3 External didactic transposition

This process is influenced by various institutions. For instance, as illustrated in section 1.1, multiple institutions hold different views on the state of algebra education in Danish schools, thereby influencing the broader educational discourse. By examining historical changes in how scholarly mathematical knowledge is transposed into curriculum resources, we may also gain some knowledge about how the knowledge to be taught in (and with it, indirectly, the teaching of) school algebra has evolved over time (Strømskag & Chevallard, 2022).

Research has shown that teachers rely massively on textbooks and other resources for teaching (e.g. Arsac et al., cited in Chevallard & Bosch, 2014). Mathematics textbook materials therefore have an important role as mediator between official guidelines and teachers work (Tesfamicael & Lundeby, 2019). It is rare that teachers who use curriculum resources to plan their lessons question the choices and the selections made in the external didactic transposition process, to the extent that it may seem almost transparent or invisible to them. It is mostly in periods of massive change that it becomes visible while in daily practice, it is perceived as a form of "school nature". It is one of the fundamental ideas of modern didactics (e.g. Brousseau 1997, p. 21) to insist on the importance of epistemological vigilance not only within the school institution, but also and particularly in relation to the external transposition.

We now briefly turn to the case of the current external didactic transposition for Danish lower secondary school (DLS), as illustration of the above abstract ideas, and as part of the context for this thesis.

1.4 The Danish external didactic transposition

The official description of mathematics in Danish primary- and lower secondary school (students aged 6 to 16) has three overall paragraphs descriptions of the purpose of mathematics in schools (Education, 2019, p.3 – translated by the author).

- §1. In the subject of mathematics, students shall develop mathematical competences and acquire skills and knowledge to enable them to behave appropriately in mathematics-related situations in their current and future daily, leisure, educational, working and social lives.
- §2. Pupils' learning must be based on their being able to experience, independently and through dialogue and collaboration with others, that mathematics requires and promotes creative activity, and that mathematics provides tools for problem solving, argumentation and communication.
- §3. The subject of mathematics must help students experience and recognize the role of mathematics in a historical, cultural and societal context, and students must be able to evaluate the use of mathematics in order to take responsibility and exert influence in a democratic community.

The more specific parts of the national program for mathematics in grades 1-9 (called “Common Objectives” and here abbreviated to (CO) are organized according to four so-called competence areas: Mathematical competences, numbers and algebra, geometry and measurement, and statistics and probability. For each of these four areas, CO specifies so-called “competence aims” to be achieved after grade 3, 6 and 9. For example, the competence aim for numbers and algebra after grade 9 is that “the students must be able to use real numbers and algebraic expressions in mathematical investigations” (Education, 2019 p.7). Algebra as a domain in CO is divided into the sectors: equations, formulae and algebraic expressions, functions. Each of these sectors contains three pairs of so-called “guiding skills and knowledge objectives “that are meant to serve as inspiration for teachers, rather than as binding aims (Quality, 2022). For example, the final skills and knowledge objectives for the “Formulas and algebraic expression” sector after grade 9 are that “The students can compare algebraic expressions” and that “the student has knowledge of rules for calculations with real numbers” (Education, 2019).

These very broadly described competence goals, which are mostly optional, can then be considered the first part of the Danish external transposition. Other noospherians subsequently produce textbooks based on this first part; the authors are typically “prominent” teachers or teacher educators, and in fact sometimes involve also authors of the CO.

Most mathematics textbooks for Danish primary and lower secondary school refer explicitly to CO. But unlike countries like Japan and Singapore (Yoshikawa, 2008; Soh, 2008), there is no infrastructure or tradition for systematic empirical testing and evaluation of textbook materials, beyond the fact that the publishing companies may exert some influence through their disciplinary editors and the requirements they can pose for authors. Danish textbooks are thus largely based on their authors’ personal didactic ideas and experiences, and sometimes ideas taken from a broader international literature on mathematics education (like publications and designs from the realistic mathematics education community). They are basically commercial products, approved and marketed by independent publishers. Any such product may be adopted by teachers or schools, with no centralized authorization or approval being in play.

1.5 Can curricula be imported?

Singapore and Japan have gained a reputation for excellence in mathematics education, mainly because of their results in international comparative surveys. These convincing and consistent outcomes are ascribed, by the research literature, to the quality of their teaching methods (roughly speaking, ambitions problem solving by students) and also the quality of the mathematics curriculum, including textbooks (Takahashi, 2021; Ginsburg et al., 2005).

This has led some to hypothesize that the use of imported curriculum elements (primarily, textbooks) from Singapore and Japan might contribute to improve students’ results in other countries (Ginsburg et al., 2005). The assumption of textbook imports is that institutions and teachers could adapt the use of the foreign (translated) textbooks to their local constraints, or even adapt the foreign official program more or less wholesale in contexts (like the US or Denmark) where the official program contains mainly optional recommendations when it comes to the concrete knowledge to be taught. The rationale for such curriculum imports has been formulated in terms of a cost-benefit analysis: the change certainly involves initial costs (translation, adaptation of texts, teacher training) but the financial value for society that is gained from students’ lifelong benefits are estimated to far outweigh the initial costs (Reys et al., 2004).

A common and rather evident argument against international curriculum import is that the material could have severe cultural biases that would be an obstacle to adaptation. (Fan et al., 2013). It is also conceivable, for a given national context, that some specific and crucial areas of mathematics are not adequately covered in the available textbooks, while international materials could be used to fill these gaps, even if the material come from different curriculum structures and cultures. Thus, rather than wholesale import, the import could be local and partial. There seems to be much less empirical (let alone research) on this hypothesis for “partial, area-specific” curriculum import. We notice here that “import” could conceivably take place between institutions in the same country, while above and in the sequel, we consider only import from one national school system to another. This, in fact, leads to some of the initially formulated questions that have driven this project: What could determine the success or failure of importing and adapting research-based mathematics textbook material for teaching specific areas of school mathematics? What are the implications for the internal didactic transposition? And our more specific interest: To what extent can the challenges for school algebra in DLS be addressed through the use of structured, research-based text material – what adaptations are needed or desirable?

We illustrate the general mechanism of curriculum import in Figure 2. The upper part of the figure shows the didactic transposition in the national context from where the curriculum is imported. The lower part of the model shows the didactic transposition in the receiving institutions.

Curriculum is often used as a collective term for both official aims and objectives, syllabuses, teaching guidelines, textbooks material and other resources that form the knowledge to be taught. In the model, we need to be able to separate the official programs form the textbook in order to examine the transposition between these two curricular resources.

EDT_1 is the external didactic transposition from Scholarly Knowledge (SK) to Official Program (OP) as part of the curriculum and knowledge to be taught, as described above. EDT_2 is the transposition from OP to Textbooks (TB), an inner-curricular transposition to be found in knowledge to be taught. The internal didactic transposition thus goes from TB in one institution to Taught Knowledge (TK) in another institution, illustrated by the arrow from TB to TK.

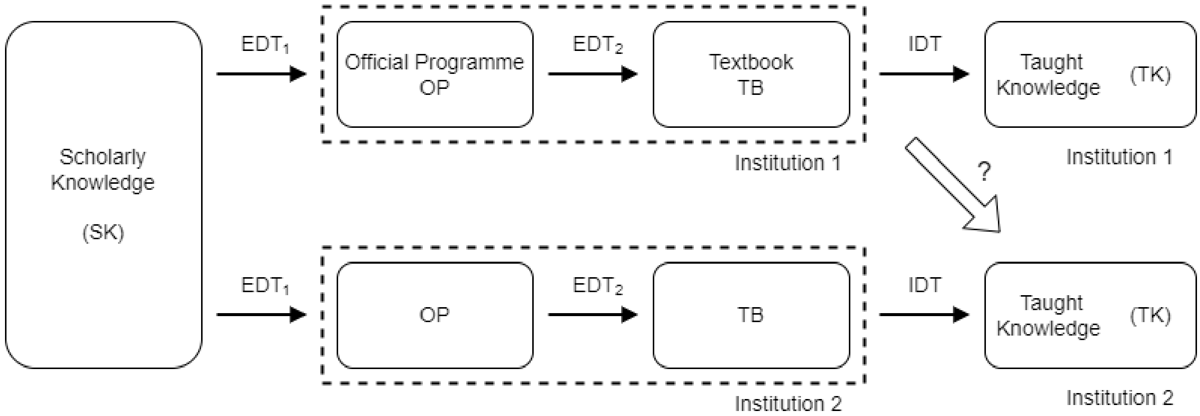


Figure 2: The transposition process of adapting foreign textbook into DLS

In order to study the didactic transposition process described in figure 2 we need theoretical and methodological tools that allow us to explain the phenomena that arise or appear when using a foreign curriculum (wholly, or in part).

More generally, we are interested in investigating the conditions and constraints for learning school algebra and describe what create and regulate this learning. We are also interested in how school algebra could be modified in a certain direction through partial textbook import, and what the consequences of such a modification would be. In the terms of Gascón (2024), these analyses are concerned with the ecology of the current or possible modalities of study in an institution. The Anthropological Theory of Didactic (ATD), in its contemporary form, provides both a theoretical foundation and methodological tools for studying ecological problems in more detail.

2 The Anthropological Theory of the Didactic

The anthropological theory of the didactic or ATD is characterized by addressing questions about teaching and learning (more generally, about didactic phenomena) from a wide variety of granularities, from the "micro-didactic" (analysis of short teaching situations with very small didactic stakes) to the "macro-didactic" (analysis of conditions and constraints on didactic phenomena that are shared by a whole civilization). The theory aims to do so in a coherent way, so that analyses at the different levels are done in a coherent way. The elaborate figure 3 summarizes two dimensions in the theory, which are related to this characterization: the levels of didactic co-determination (cf. also Chevallard, 2002, p.10 or Chevallard & Bosch, 2020) used, and the corresponding the levels of a praxeology (section 2.2). Here, the levels of co-determination concern the origin and scope of conditions and constraints on the didactic, while the praxeological levels relate to the different levels of the human activity (like mathematics) which also serve to organize and teach it.

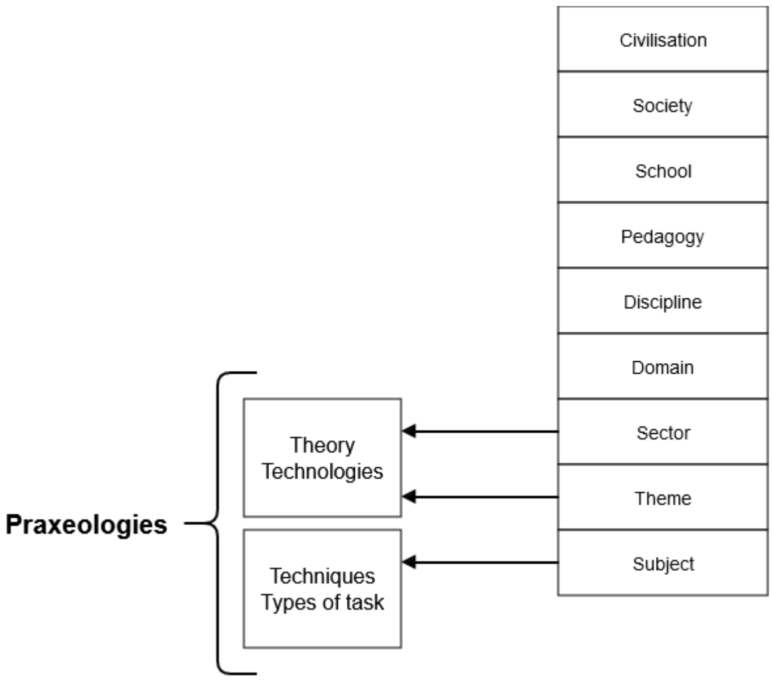


Figure 3: Model of mathematical praxeologies realized in school and the relation to levels of didactic codetermination, modified from (Artigue & Winsløw, 2010, Fig. 1).

In this thesis we focus on a specific school institution (DLS) and the domain of algebra within the discipline of mathematics, both as they are described and practiced within this school.

To investigate how knowledge is transposed within and between institutions, the model of didactic co-determinacy levels can provide a structured overview of the institutional conditions for didactical transposition (Artigue & Winsløw, 2010; Chevallard, 2019). For instance, when we investigate the use of a Japanese textbook chapter in a Danish classroom, the possible hindrance coming from cultural difference amounts to a possible constraint coming from the level of Danish society or of Western civilization that would make the use difficult or impossible. The model, of course, does not say that this is the case, but we can situate the hypothesis within the model. Similarly, if the textbook would rely on a technique from arithmetic which is not taught in the Danish school, we would have a constraint coming from the subject level. In other words, the ecology for a given or imagined didactic practice (or form of internal didactic transposition) can be represented in a structured way where the level indicates the origin of observed or imagined conditions and constraints.

2.1 Institution in ATD

In ATD, all modelling of didactic phenomena is based on the notion of person, institution, and institutional positions (Chevallard, 2019). A person is any human being. An institution is any kind of created reality of which human beings can be members. Examples of institutions could be a family, a school class, or a football team. In each institution there are a number of positions (Bosch & Chevallard, 2020). In a family, there are, for instance, the positions of father, mother and children. In a school class, there are at least the positions of teacher and student. A football team will often include the positions of goalkeeper and midfielder etc.

Institutional positions are filled by persons who become subjects in the institution (Strømskag & Chevallard, 2022). In the noosphere there are positions such as committee members, ministerial curriculum developer, textbook author, or "prominent math teacher". Notice that a person usually occupies a position for some finite time, so often the position remains, the person changes. Nevertheless, the institutional positions a person has held will often shape the person. Conversely, as subject in the institution, you can also influence and change the position(s) you occupy. Finally, an institution is simply a configuration of positions, and institutions generally have a considerable stability in spite of changing positions and persons filling them. This means that person, institutions and institutional positions are closely linked. They are also essentially linked to the next notion we will discuss, that of praxeology, since the positions within an institution are very often defined by their relationship to praxeologies for which the institutions serve as habitats and in which they can be created, disseminated and developed.

2.2 The Praxeologies

Every human activity, and their outputs, can be described in terms of praxeologies (Chevallard, 1985; Gascon & Nicolás, 2024). This statement is the core of the theory of human activity proposed by the ATD. In ATD, mathematical activity and the study of mathematics is firmly placed within the spectrum of human activity (Chevallard, 1999). A praxeology consists of a praxis block and a logos block. Praxis described what is done and how: (solving) a type of task T through the use of a corresponding technique τ which serves to solve tasks of the type T. Logos is discourse (including inner discourse) about praxis and comes in two forms: technology θ (discourse about techniques) and theory Θ (discourse for and about technology, to define and justify objects beyond the individual technique… abstraction from concrete objects is perhaps one of the most fundamental particularities of human capacity).
A praxeological organization consist of a practical-technical block $[T/\tau]$ and a technological-theoretical block $[\theta/\Theta]$. A punctual praxeology is noted $[T/\tau/\theta/\Theta]$ and consist of the four Ts type of task $T,$ technique τ, technology θ and theory Θ (Chevallard, 1999).

As an example, we will first describe a common human activity, washing clothes, in terms of praxeologies. A laundry task could be washing white towels. This task can be solved by loading the towels into the washing machine, adding the right amount of detergent and starting the cotton wash program at 60 degrees. This task and the associated technique constitute the praxis block. The logos block consists of discourse about cotton towels which should be washed at least at 60 degrees and with the amount of detergent appropriate to the type and amount of textiles and their dirtiness and not to forget that white textiles should be washed separately from colored textiles. This discourse can be explained by the theory that bacteria are destroyed at 60 degrees and therefore towels should be washed at a temperature that destroys bacteria. The type and amount of detergent must match the type and amount of textile, so the clothes are washed clean but without detergent residue remaining in the textiles after washing. Similarly, the discourse about not mixing white and colored textiles can be explained by the theory that excess dye in textiles can be transferred from dark to light textiles via the water in which they are washed. Similarly, you can describe practices for washing silk clothes or synthetic sportswear, which require different washing techniques but draw on the same technology and theory about detergents and temperature. This allows us to create laundry models where different type of tasks and their corresponding techniques have the same level of technology and theory.

The example of laundry makes it easier to see that task types are not given from nature, but are artefacts produced by, existing in, and varying between institutions.
Many learn to wash clothes though knowledge about practice e.g. by imitating experienced launderers, which means using effective techniques with limited relations to the technology and theory. In this situation the logos exist but is not fully visible to the learner, and the dominant model for laundry will include a fragmented or invisible level of theory. Similarly, you may find that the technique of not mixing colored and white clothes has an impact on the quality of the wash, which can be explained by technology and theory. Industrial developments and the use of machines influence many human activities, including laundry, emphasizing that praxeologies are not static but can also change over time. Although there may seem to be a long way from washing clothes to doing mathematics, both can be described in terms of praxeologies and the institutions in which they live (Chevallard, 1999).

In mathematics, a concrete task could be "Rewrite $2(3+x)$ ", which is of the task type T: Rewrite $a(b+x)$ where $a, b \in \mathbb{R}$, and can be solved using the technique τ: $a(b+x) = ab + ax = ax + ab$. The first step can be justified by the theory of the distributive law, and the second step, to rewrite the expression, can be justified by the convention that terms with x are written first. This means that Logos $[\theta/\Theta]$ consists of the explanation of the above technique and (essentially) the distributive law. Together, this forms a punctual praxeology.
In the example above, the type of task T can be solved by a simple technique τ, which is evident when both T and τ are written in a general form. An exercise in a math textbook may contain different questions, not all of which can be answered by one or several techniques. This is in contrast to a problem that can always be solved using one or more techniques. This means that a question can be identified with the type of task that can be solved with a technique (Winsløw et al., 2013). The term" technique" is used in a broad sense to refer to what is done to deal with the concrete task. The anthropological approach assumes that any task, or the resolution of any problem, requires the existence of a technique, even though the technique is hidden or difficult to describe (Barbé et a., 2005).

It is convenient to understand a mathematical praxeology as a type of mathematical organization (MO), where a punctual MO consists of a type of task T_i and the corresponding technique τ_i (Bosch & Gascón, 2006). When a set of punctual MOs is explained by using the same technological discourse, they construct a local MO (LMO) characterized by its technology. Likewise, LMOs with the same theoretical discourse can give rise to regional MOs (RMO). In this organization, it is important to be aware the punctual MOs can be integrated into different LMOs, and similar LMOs can be integrated into different RMOs (Barbe et al., 2005).

2.3 Praxeological change

In the early years of school, students are trained to follow the rules of arithmetic by calculating one arithmetic expression to get another that is equal to it, given the result in canonical form. The expression $3 + 5 = 8$ can be read, if you do the calculation $3 + 5$ correctly – then you will get 8. In other words, = is read from left to right as "makes". The concept of equivalence is symmetric and means "same value", and in algebra it is used about both isolated expression and entire equations. For instance, $3 + x$ and $x + 3$ are equivalent (or equal) expressions, because they are equal for any value of the variable x. Similarly, $3 + x = 8$ and $x = 5$ are equivalent because they are true for exactly the same values of the variable x. In these cases, the equals sign does not mean "makes" or "something needs to be done." In the transition from arithmetic to algebra and particular when introducing algebraic expressions and working with equations, it is necessary to understand the equal sign as a symbol for equivalence, and the notion of equivalent equations (Kieran, 1981). Thus, the learning of algebra requires the learner not only to learn new knowledge, but also to unlearn or reorganize existing knowledge (Brousseau, 1990).

The change from understanding the equals sign as an operator symbol to a symbol of equivalence requires a change in practice and theory. The notion of praxeological change can be used in order to describe such transitions in terms of praxeologies (Putra, 2019): it can mean that existing praxeologies need to be modified or rejected in favor of new ones. The use of praxeologies allows us to describe what is sometimes referred to as "conceptual change" with a very high degree of precision. A praxeological change must be expected to be harder to effectuate within a group of learners than the simple addition of new praxeological elements.

We now turn to outline the notions of ATD which are used to describe the essential moments in teaching-learning processes, with explicit reference to the parts of the taught and learnt praxeologies.

2.4 Didactic moments

Just like we defined mathematical organizations, a didactic organization consist of task types, techniques, technology and theory, where the tasks relate to teaching a given mathematical praxeology (Chevallard & Bosch, 2020). More precisely the didactic tasks relate to organize particular *moments* at which *specific parts* of the mathematical praxeology can be developed by the learner. There are six such moments and they are called *didactic moments* or *moments of study* (Barbe et al., 2005). The first moment of study is the moment of the *first encounter* with a type of task T. The second moment concerns the *exploration* of the type of task T with the emergence of a first technique τ used to solve T. In the third moment the *construction (or identification) of the technological and theoretical block* $[\theta/\Theta]$ used to explain and justify τ, and is closely interrelated to each of the other moments (Chevallard & Bosch, 2020). The fourth moment concerns the *technical work* and is the moment to work on routinizing and refining the technique(s). *Institutionalization* of the entire praxeology produced $[T, \tau, \theta, \Theta]$ is the aim of the fifth moment. The sixth and final moment is the moment to *evaluate* the praxeology and is linked to the moment of institutionalization (Chevallard, 1999; Barbe et al., 2005). A "complete" realization of the six moments of the didactic process will create a mathematical organization (MO) that goes beyond the simple solution of a single mathematical task.

2.5 Epistemological reference model for elementary algebra

The construction of epistemological reference models (REM) is a central tool in ATD, and these models are formulated in terms of praxeologies (García et al., 2006). A REM is constructed by theoretical and empirical data from scholarly knowledge, knowledge to be taught and knowledge taught and learnt, even erroneous parts of these, depending on the purpose of the model (Ruiz-Munzón et al., 2013). The construction of an REM for elementary algebra provides an opportunity to reinterpret what is considered as algebra in schools and to examine the role of elementary algebra in relation to other domains (Munzón et al. 2015). In that way the REM has the function of a working hypothesis for the researcher and provides opportunities to compare and include different praxeologies related to school algebra (Bosch, 2015).

When we describe the *dominant way* of teaching or otherwise approaching a mathematical praxeology within an institution, we do so using the categories of REM to show how it includes some elements of the praxeology and its uses and exclude others. This more or less reduced mathematical praxeology is called the dominant epistemological model (DEM) (Lucas et al., 2019). To establish the precise elements of the DEM, we base our analysis on empirical data such as official objectives, textbook material and written exams. Overall, the DEM is a two-edged sword: it can create cohesion within the school institution, but also contribute to overly reduce the praxeology to be taught. It lies in the qualification "dominant" that the DEM has a large degree of stability and is rarely or never questioned from within the institution. For that reason, it is essential for the researcher not simply to adopt the DEM as REM.

3 Literature background

To investigate the tentative hypothesis that the transition from arithmetic to algebra, which is a well-known challenge for teachers and students in Danish comprehensive schools (cf. section 1.1), can be facilitated at least to some extent by the use of textbook resources from highly successful school institutions (in the sense of ATD), we need to draw on several different parts of the research literature. This involves, besides *related research within ATD*, also research in other paradigms concerning *curriculum resources and textbook materials,* as well as *the teaching and learning of algebra in schools*. To navigate this intersection of broad research areas that draw on different research traditions the literature review will focus on the three *intersections*: between ATD and school algebra, ATD and textbooks as curriculum resources, and school algebra and textbooks, as illustrated in Figure 4.

The joint intersection between the three areas of ATD, school algebra and textbooks as curriculum resources is naturally of particular interest.

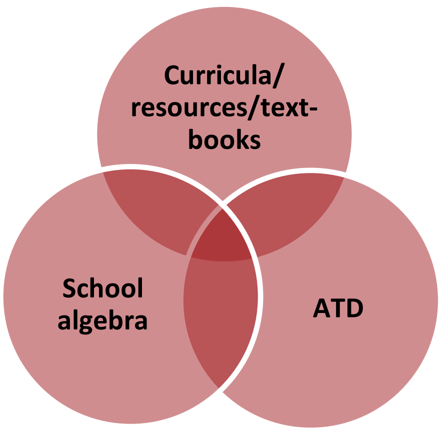


Figure 4: Model of the intersection of the research areas

3.1 Methodology

To collect data for the literature review, I chose to use Google Scholar as this search engine provides access to a wide range of sources in didactics, without including a lot of teaching materials. Unlike databases such as Education Resources Information Center (ERIC), Google Scholar indexes scholarly articles, books and dissertations. Google scholar includes not only peer-reviewed articles, but also what might be called grey literature, such as reports on development projects, ministerial committee work etc. This search engine therefore covers a broader range of texts on a given topic, which may be relevant to an institutional analysis. Other advantages include the fact that Google Scholar is freely available (potentially increasing transparency) and that you can use advanced search strings for instance to find item that cite a given source. The broad search results one gets with Google scholar also represents a challenge in using the database, as it is not as specialized as ERIC. Another advantage of ERIC is the transparency resulting from the explicit description of keywords used for cataloguing, and what journals and peer review resources are used for the search.

For pragmatic reasons, only English keywords will be used, so that primarily English language articles will be found and read. When articles are written in a language other than English, I have applied a machine translation using Deepl, being aware that nuances and technical terms may be lost or degraded in the translation. The use of translated articles has been made because parts of the first and important research in ATD and on school algebra has not been published in English; one important example is the paper *Le passage de l'arithmétique à l'algèbrique dans l'enseignement des mathématiques au collège* (Chevallard, 1989), as we shall see.

3.2 Method 1: search by keyword

To carry out a systematic search the use of keywords is essential. The search string and associated result will be documented in a table, followed by a description of exclusion and inclusion criteria. Following a review of the title and abstract, the articles will be grouped according to their thematic content.

Systematic literature searches can provide an overview of a research field; however, any search will its limitations. For instance, key articles may not be identified in the search results if the search terms do not appear in the article. In fact, almost any search method risks bypassing relevant results. The example of the paper by Chevallard (1989) mentioned above illustrates this in a striking way: it provides a central contribution to research on the transition from arithmetic to algebra, carried out in the context of French collège (middle school, students aged 10 to 15). The paper outlines an important development of algebra in that institution and contributes theoretical ideas that are relevant beyond this context. However, the article was published in *Petit x*, a practitioner oriented journal which aims to promote the dissemination of research and development in mathematics and mathematics education (Petit, 2024) and is an example of an article that will not appear when searching on "ATD and algebra" in ERIC, or for that matter in the Google Scholar search with the search string "+ATD +algebra +school +"lower secondary"". We captured this paper by the use of another method, targeted citation search (see section 3.3).

3.2.1 The first keyword search for ATD and school algebra.

The first keyword search is based on the intersection between ATD and school algebra. ATD and algebra as domain are closely related research fields, as evidenced by more than 8600 results at the first Google Scholar search, table 1. Narrowing the institutional conditions by adding "lower secondary" and excluding "high school" provides a body of literature emerges that form the basis for further analysis.

Table 1. Overview of keyword used and results for the ATD and school algebra search

	Search string in Google Scholar	Results
1	+ATD + algebra	8690
2	+ATD + algebra +school	3510
3	+ATD +algebra +school +"lower secondary"	240
4	+ATD +algebra +school +"lower secondary" –"high school"	128

Due to the specific goals of this thesis, we chose to further reduce this initial body of 128 items by manually removing texts about teacher training, teaching at university level, papers focusing on geometry and digital resources, and papers on study a research path (SRP) design. The remaining texts can be placed within the following four categories: *Dialogue between theories*, *Institutional transitions*, *Praxeological and epistemological analysis of school algebra*, *Teacher knowledge and practice including teaching experiments*, all related to lower secondary algebra. The category "*Dialogue between theories*" is excluded here, as this study uses ATD and does not focus on abstract relations between research paradigms. In the category of institutional transitions, we furthermore excluded articles whose primary focus was on the transition from primary to secondary or from lower secondary to upper secondary school. The reason for this last elimination is that we are primarily interested in the teaching within lower secondary school (although with the adaptation of textbook material from another, foreign, institution at the same level, but this is very different from what is treated in the excluded papers).

3.2.2 The second keyword search for textbooks and algebra

The second search clearly shows how much literature there is in the field of "algebra and textbooks", table 2. By adding "lower secondary" and excluding "high school" and "upper secondary" the institutional setting is narrowed.

Table 2. Overview of keyword used and results for the textbook and school algebra search

	Search string in Google Scholar	Results
1	+textbook +algebra	253000
2	+textbook +algebra +school	137000
3	+textbook +algebra +school +"lower secondary"	2940
4	+textbook +algebra +school +"lower secondary" –"high school" –"upper secondary"	826
5	+textbook +algebra +school +"lower secondary" –"high school –"Upper secondary" -electronic -digital	393
6	+textbook +algebra +school +"lower secondary" –"high school –"Upper secondary" -electronic -digital -geometry	101

Exclusion by the keywords "electronic" and "digital" removes articles that discuss the use of digital textbooks and the use of Computer Algebra Systems (CAS). As we are particularly interested in the transition from arithmetic to algebra, articles with geometry as a keyword are excluded. This exclusion also includes articles with a primary focus on interaction between geometry and algebra.

The remaining articles from the second keyword search leads to defining additional categories:

- *Students' positions and backgrounds*, including articles on ethnic and cultural backgrounds, high and low achievers, gender research, and beliefs about algebra as a domain and mathematics as a discipline.
- *Algebra as a modelling tool in other disciplines,* with a primary focus on teaching other disciplines, particularly science.
- *Generic teaching methods* such as cooperative learning and flipped classroom.
- *The subject- and theme-specific papers,* which include studies on fractions, linear equations, inequalities, quadratic equations and logarithms in the transition from arithmetic to algebra.
- *Theoretical dimensions of algebraic thinking* in schools, including algebraic reasoning, functional reasoning, analytical and structural reasoning.

- *The role of mathematics textbooks*, including how teachers and students use textbooks and comparative studies of textbooks.

Texts in the first three categories were excluded as they appeared less or not relevant to our project.

3.2.3 The third keyword search for ATD and textbooks

The third keywords search aims to find research carried out within ATD, and that examine the knowledge to be taught based on the textbook and how the knowledge to be taught is transposed to taught knowledge based on the textbook.

Table 3. Overview of keyword used and results for the ATD and textbook search

	Search string in Google Scholar	Results
1	+ATD +textbook	11100
2	+ATD +textbook +mathematics	2380
3	+ATD +textbook +mathematics +school	2030
4	+ATD +textbook +mathematics +school +"lower secondary"	230
5	+ATD +textbook +mathematics +school +"lower secondary" -"high school"	103

This third search also excludes articles in geometry and statistics and the use of digital resources. The remaining articles can be grouped into the following categories: *Lesson comparison and lesson studies, problem solving and inquiry-based learning, SRP and questioning the world, praxeological analysis*. Only the last category was retained. It includes comparative studies focusing on both mathematical praxeology and didactic praxeology.

3.3 Method 2: Targeted citation search

In order to address the limitations of a keyword-based literature search, a targeted citation search was employed to supplement the search results by keyword. Citation tracking is an umbrella term for several different methods, all of which collect related articles from “seed-references”. Using citation relationships, it is possible to find additional qualified studies by either “backward” of “forward” tracking (Hirt et al., 2023). In this way, citation tracking enables the monitoring of the evolution of a research domain by focusing on highly cited and therefore influential articles. In this citation search we will use forward tracking with selected “seed-references” in ATD and paradigms on curriculum resources and textbook materials, and teaching and learning algebra in schools.

Within each of the three intersections, ATD and school algebra, school algebra and curriculum resources, ATD and curriculum resources (figure 4) three “seed-references” were selected for “forward” tracking. The criteria for “seed-references” were that they must be more than 20 years old, have been cited more than 300 times. Selecting articles that are more than 20 years old as “seed-references” can help to monitor subsequent research and how fundamental ideas and concepts have evolved over time. Having been cited more than 300 times indicate, in a rough way, that the text is considered useful and valuable. The high number of citations suggests that the research has influenced subsequent studies and still relevant.

From the more than 300 articles citing each “seed-reference”, the most cited articles and new articles related to the basic themes of this thesis were selected for the literature review. The articles identified through the targeted citation search will contribute to the existing thematic categorization identified through the keywork search.

3.3.1 First seed-reference in ATD and school algebra.

The article “Le passage de l’arithmétique à l’algèbrique dans l’enseignement des mathématiques au collège” by Chevallard (1989) was identified as “seed-reference” for the targeted citation search in the field of ATD. The article, which has been cited more than 340 times according to Google scholar, was published in 1989 and can be placed in the early works in ATD and school algebra. The author is the founder of the Anthropological Theory of the Didactic (ATD) as a research framework and has later written extensively on school algebra.

3.3.2 Second seed-reference in school algebra and curriculum resources

I the field of algebra in schools, several researchers have contributed to the development of theory over the last 20 years (cf. section 5.4). Carolyn Kieran has been engaging in research on school algebra since 1980s, during which time she has conducted comprehensive literature reviews (Kieran, 2007; 2022). These reviews represent a significant contribution to the field, providing a detailed overview of the evolution of school algebra. As “seed-reference” in the intersection of algebra in schools and curriculum resources, we will use Kieran’s article “Algebraic thinking in the Early Grades: What Is It?” published in 2004 and cited over 800 times according to Google scholar. Kieran (2004) examines how different curricula defines algebra and algebraic thinking. By comparing curricula from China, Singapore, Korea and the United States, she outlines commonalities such as focus on generalization, problem solving and modelling.

These commonalities form the basis of Kieran's model of school algebra in terms of activities, which consist of generational-, transformational- and global meta-level activities (Kieran, 1996). The three-part model of algebraic activities includes implicitly the letter-symbolic representation. In global meta-level mathematical activities, algebra is used as a modelling tool to model more general mathematical processes and activity (Kieran 2004). In this "seed-reference" paper, Kieran offers a definition of algebraic thinking in the early years, based on the global meta-level activity of algebra, and thus form the starting point for the "forward" citation search.

3.3.3 Third seed-reference in curriculum resources and ATD

In the article "Twenty-Five Years of the Didactic Transposition" Bosch and Gascón describe the development of ATD from the first summer school in "Didactic of Mathematics" in 1980 to 2005, where the historical view of ATD is presented. As described in (cf. section 1.2), the notion of didactic transposition formed the basis for the later development of the theory. A development that in the early years was mainly driven by the French-speaking research community, and very soon also by the Spanish-speaking community. Two of the key researchers who have contributed to the development of ATD through research contributions in English are Marianna Bosch and Josep Gascón. The above-mentioned article has been cited more the 300 times and fulfils the citation criterion for "seed-references". In their early research, both Bosch and Gascón worked with elementary algebra, where algebra is viewed not only as a mathematical organization, like arithmetic and geometry, but as a process that affect either an entire mathematical organization (Bolea et al., 2001).

3.4 Literature review related to School algebra

The literature identifies different principal ways to introduce algebra in school: as a theoretical generalization of the study of arithmetic and numbers, as a new sector motivated by needs for the study of functions, and as a tool for modelling. The approaches and their qualities are still a theme of many current papers.

In 2012, Bosch and Chevallard express that it is unclear what is meant by "Elementary algebra" in mathematics education and schools, and more broadly in society, and it is difficult to find work in educational research that examines what is taught under the heading of "elementary algebra" (Munzón et al., 2015). In response to the lack of overview of empirical studies on school algebra, Eriksson (2022) presents a comprehensive literature review. Teaching students aged five to twelve can be broadly organized within three traditions: arithmetic thinking traditions developing arithmetic thinking first, developing arithmetic and algebra at the same time or algebraic thinking tradition developing algebraic thinking first (Eriksson, 2022). That view of the field of research on early and elementary algebra is reinforced by a further literature review, in which Kieran (2022) describes theoretical dimensions of early algebraic thinking and excerpts from empirical findings.

3.4.1 Theoretical dimensions of algebraic thinking

Generalization is the common thread running through Kieran's three overarching types of algebraic thinking, namely analytic thinking, structural thinking and functional thinking (Kieran, 2022).

Analytical thinking is what distinguishes algebraic thinking from arithmetic thinking according to Radford (2014). When indeterminate quantities such as unknowns and variables are treated analytically by considering them as if they were known quantities, and thus performing calculations as one would with known quantities, this can be defined as algebraic thinking according to Radford (2018). For Radford, the use of explicit algebraic notation is not necessary for algebraic thinking, since the description and labelling of indeterminate quantities can be done using natural language, gesture and unconventional signs (Kieran, 2022). Radford uses the concept of analyticity as part of the analytic dimension of early algebraic thinking, which relates in particular to the activity of patterning (Radford, 2018).

Structural thinking focuses on relationship and properties (Kieran, 2022). In the tradition that Eriksson describes as algebraic thinking first, students are introduced in the early years of schooling to relationships between sets that also contain letter symbols (Eriksson, 2022). Kaput (2008) has described another aspect of structural thinking that relates to the generalized arithmetic view of algebra. In this type of structural thinking there is a focus on numerical aspects, with reasoning about general structures, as the generalization of arithmetic operations and their properties.

According to Kaput (2008), algebraic reasoning includes functional thinking, where functional thinking is used to generalize relationships between co-varying quantities. Functional thinking enables you to analyze functional expressions through different representations, to describe deviations from a given pattern or rule (Kieran, 2022).

The above only provides a very coarse theoretical overview of the many dimensions of early algebraic thinking identified by the references, without including examples of empirical studies linked to analytic, structural and functional thinking. While the literature review by

Kieran (2022) outlines theoretical and empirical dimensions of algebraic thinking, Kaas (2019) attempts through a literature review to categorize the approaches to early algebra teaching (algebra in primary school) that have been proposed in the research literature. This categorization is divided into activities and topics, where generalizing relationships and properties and reasoning with unknown quantities are the activity type. Here, arithmetic and numbers, arithmetic and quantities and functional relationships are considered different approaches to the subject area (Kaas, 2019).

The above theoretical approaches to school algebra provide a picture of the diversity of early algebra, which Kieran describes as a multidimensional field including three main dimensions of algebraic thinking, analytic thinking, structural thinking and functional thinking (Kieran, 2022). Despite these many studies of algebraic thinking, Hodgen et al. (2018) still considers that there is missing research in many of the fundamental questions about teaching and learning elementary algebra. The authors provide both a description of algebraic thinking and a critical look at the research field, as well as perspectives on new directions (Hodgen et al., 2018). The authors further argue that the theoretical frameworks used in algebraic thinking research can be categorized into three groups (1) Conceptual frameworks, namely skeletal structures of justification, rather than structures of explanation based in a formal theory, (2) General theories of teaching and learning, used to study school algebra, (3) Holistic theories are frameworks that encompass a methodology for instructional design (Hodgen et al., 2018 p. 35). The first category thus contains the theoretical approaches described above, where Kaas (2019) finds structures in the content approach and Kieran (2022) in algebraic thinking as a research field. As examples of general theories belonging to the second category, Hodgen et al. presents, among others, the semiotic theory and the cognitive theory of instrument use (Hodgen et al., 2018). ATD as a theoretical framework for the study of elementary algebra can be placed in the third category, as will be described in more detail below.

Based on the theoretical review, Hodgen et al. ask two key questions: How do the theoretical frameworks relate to research problems in algebra teaching and learning, and to what extent are the theoretical approaches complementary or contradictory?

On one hand, there is historically different approaches to early algebra (Eriksson, 2022) and a diverse theoretical description of algebra in school (Kieran, 2007; 2022). On the other hand, and despite these theoretical variations, empirical data describes a more uniform current practice where school algebra in the Western world appears to be dominated by a narrow focus on training isolated techniques related to notation and formulas, with a widespread neglect of present algebra as a modelling tool (Herscovics & Linchevski, 1994; Strømskag & Chevallard, 2022). Algebra as a modelling tool has been central to the work of Chevallard (1989) and modelling problems with algebra, and recognizing and evaluating algebraic models is also more broadly considered fundamental to algebra learning (Jupri & Drijvers, 2016).

3.5 Literature review related to ATD and school algebra

ATD as a research program has evolved since the presentation of didactic transposition and the introduction of the notion of praxeologies by Chevallard (1999). In the "seed-reference" by Chevallard (1989), the theory of didactic transposition is used as theoretical foundation to describe the transition from arithmetic to algebra in college mathematics education in France. An analysis of the manipulation of algebraic expressions in college reveals that it has the sole purpose of training certain algebraic tricks and has no mathematical goal beyond this training. The "rules" of this manipulation are unmotivated, learnt and expressed by instructions, which also have a standardized form (Chevallard, 1989). Munzón et al. (2015) present a global synthesis of the contributions made by ATD to the problem of teaching elementary algebra. As described above, there are many theoretical approaches to elementary algebra. Despite this diverse and widespread research on elementary algebra, Munzón et al. (2015) call for work in educational research that examines what is understood by elementary algebra and actually taught as elementary algebra in mathematics education. What praxis and logos constitute the taught knowledge of elementary algebra and not least, what knowledge and activities are not taught in schools (Munzón et al., 2015). In Spanish schools, the same situation described by Chevallard (1989) can be found in the teaching and learning of elementary algebra, where students learn to write, factorize and simplify expressions as an end itself and not as a tool for problem solving (Cosan, in press; Munzón et al., 2015).

3.5.1 Praxeological and epistemological analysis of school algebra

The teaching of mathematics in schools, and in this context school algebra, can be put in further perspective through the study of older mathematic textbooks. The epistemological study of the concept of formula and its development by Strømskag and Chevallard (2022) is based on a historical analysis of curriculum and textbook development, where they conclude that the curriculum is both evidence and cause of the disappearance in schools of algebra as a modelling tool (Strømskag & Chevallard, 2022).

In ATD, the focus in schools on algebra as a modelling tool was previously advocated by Chevallard (1998), Bolea et al. (1998; 2001; 2004) and Ruiz-Munzón et al. (2013; 2015; 2020). Algebra as a modelling tool includes models of intra-mathematical systems like calculations patterns, and of extra-mathematical systems, like the study of quantitative relations in other disciplines, e.g. economy, physics and biology (Bolea et al, 2001). In this way, algebra does not appear as a mathematical organization at the same level as the other organizations (e.g. arithmetic, geometry), but as a modelling tool for all mathematical organizations in the school (Bolea et al, 2001; Bosch, 2015).

3.5.2 Levels of the algebraization process

Algebraic modelling occurs through an algebraization-process that begins in primary school and continues through secondary education to university level. In ATD, the characterization of "algebraized" mathematical activity and the "degree of algebraization" are of particular interest (Ruiz-Munzón et al., 2013). To detect and analyze general levels in the school algebra to be taught and locate which aspects of the algebraization process are weak or difficult to introduce in schools, a three-stage model of the algebraization process can be used (Bosch, 2015). In the three-stage model of algebraization defined by Ruiz-Munzón et al. (2013), arithmetic can be identified as the domain of calculation programs (CP). The first stage of algebraization occurs as learners consider the CP as a whole and not only as a process. In the second stage, letters are introduced as parameters and unknowns, to model the relationship between CPs. The third and last stage of the algebraization process appears when the number of variables of the CP is not limited to one, and the distinction between unknowns and parameters is eliminated (Ruiz-Munzón et al., 2013). Munzón et al. (2015) uses the three-stage model of algebraization as a reference system to analyze how elementary algebra is presented in the Spanish lower secondary algebra.

3.6 Literature review related to Curriculum resources

Curriculum theorists use categories to analyze different types of curricula and to clarify their meaning. In a very general description, the "formal curriculum" refers to the objectives and activities outlined in school policies or described in textbooks. The "Intended curriculum" represents teachers' goals, while the "enacted" of "experienced" curriculum is what actually happens in the classroom (Remillard, 2005). Researchers mostly focus on the enacted curriculum, emphasizing the active role of teachers in its design – in brief, the internal didacticc transposition. Understanding the relationship between the written and the "lived" curriculum involves exploring how teachers construct the lived curriculum and the role of resources such as textbook material in this process (Remillard, 2005)

Curriculum resources include traditional curriculum materials such as textbooks and official curriculum guidelines, but also include teaching guides, students' notebooks, electronic tools and online materials. This diversity of resources appears to be increasingly influencing curriculum materials, curriculum development, teacher training and everyday classroom practice. In line with Pepin and Guedet (2018), Remillard (2018) and Rezat et al. (2021), we will occasionally use the term "curriculum resources" to include textbooks as well as the variety of printed or digital materials that were designed to support teachers', all elements that comprise the body of the "knowledge to be taught" in the transposition process. The term "curriculum" is used when a resource pays particular attention to the sequence of learning topics at the grade or age level, or to the content associated with a particular domain in school education.

Through a literature review, Rezat et al. (2021) identified three objectives for change through curriculum resources. First, curriculum resources can be used as instrument to change the goals of mathematics education and thus the mathematical content taught. Secondly, curriculum resources can be designed to implement change in mathematics teaching, and finally, curriculum resources define mathematics as a school subject, so renewal of the first imply renewal of the profile of the subject. Thus, curriculum resources are a tool to improve student's knowledge, but they can also help to change students' beliefs and attitudes towards mathematics (Rezat et al., 2021).

Although the perception of materials for teaching algebra has changed over time with the inclusion of new types of resources such as CAS (Computer Algebra Systems) and dynamic geometry programs, the textbook is still considered the most important resource for teachers (Straesser, 2011). The use of the term "curriculum resources" rather than "curriculum material" is an attempt to broaden the perspective on the elements available for the teachers' work to include also other resources that define the knowledge to be taught.

There has been a significant theoretical development to describe the use of curriculum resources in mathematics education. Researchers have reached a consensus that the utilization of a curriculum resource is not a direct implementation but involves an interaction between the teacher and the resource (Remillard, 2012). Using mathematics curriculum text designed to guide teachers in planning daily lessons, Remillard (2012) explores and theorizes the relationships teachers develop with curriculum resources.

3.6.1 The textbook as a resource

A literature review by Fan et al. (2013) highlights the widely recognized importance of textbooks in teaching and learning. It is noted that textbooks are powerful resources for mathematical education because they can introduce readers to unfamiliar concepts, provide a structure for the teaching and learning of mathematics and provide access to mathematical knowledge (Fan et al., 2013). To classify the literature in mathematics textbook research, Fan et al. (2013) used the four categories: textbook use, analysis and comparison of textbooks, the role of textbooks, and other areas of research (Fan et al., 2013). The use of mathematics textbooks in the United States has until the publication of the NCTM Standards in 1989, the primary focus of mathematics textbooks in the United States was on student exercises and practice problems. However, because the standards address both the types of mathematical problems students should solve and the way in which these problems should be taught, more attention was paid to pedagogical guidance for the teacher (Remillard, 2012). Research showed that teachers find it challenging to use the so-called standards-based teaching materials (which were new at the time of the study) and that many teachers use the materials in ways other than those intended by the designers (Remillard, 2012).

Studies of teachers' use of textbooks in different countries show that mathematics textbooks are used by teachers in two dominant ways. One is as guide to mathematical content and as a source for mathematical tasks. The other is as a guide to approaches to teaching and how content should be presented (Rezat, 2012).

A study by Rezat (2012) examines four teachers' explicit references to the textbook. The ways in which the teachers refer explicitly to the textbook can be characterized along three dimensions; directly or indirectly, specific and general, voluntary or obligatory (Rezat, 2012). When the teacher talks about the textbook, the students' attention is drawn directly to the book. However, students can also be influenced indirectly by the mere use of the teacher of a textbook. The teacher can refer to specific sections, examples or tasks in the book, or more generally to the textbook as a place where students can find answers to their questions. Teachers may also explicitly refer to students' optional use of the book if they require assistance, and therefore does not require them to use the book if they do not need such assistance. Conversely, students may be required to utilize the book due to the necessity of working with the assignments contained therein (Rezat, 2012). Whether it is a directly or indirectly use of the textbook Howson (2013) argues, that good textbooks are more important for good math results in schools, rather than other factors such as new IT equipment or nice classrooms.

Despite the consensus that textbook are of importance for the teaching of mathematics, many teachers and textbook writers are unaware of the cognitive difficulties in the learning of mathematics and especially school algebra. This can result in students not having the time to build on the pre-algebraic foundation they have developed in primary school, not understanding the meaning of the new symbolism and carrying out meaningless operations and reduction of symbol strings they do not associate with any meaning (Herscovics & Linchevski, 1994). Although textbooks and curriculum resources have a significant impact, they cannot transform teaching and learning practice on their own, and more knowledge is needed about the characteristics of curriculum resources that support the implementation of change (Rezat et al., 2021).

3.6.2 Curriculum structure

The objectives, aims and content of a curriculum are central to the knowledge to be taught, but the overall structure and organization of the curriculum content is also important. One way of organizing the curriculum is through a tiered structure, where content areas build on previous areas with a clear progression. This is a traditional curriculum structure, where subject areas have their own goals and elements of assessment. A structured approach to curriculum design, where content is organized in a step-by-step, sequential progression. I this linear structure, mathematics subject areas are introduced in a structured sequence, that builds on prior knowledge and has a clear emphasis on progression. In contrast to the linear curriculum structure, which employs a step-by-step approach to knowledge acquisition, the spiral curriculum is based on a more iterative process.

A spiral curriculum is based on the overall objectives or goal of the curriculum, where the content is an iterative review of topics, subjects or themes throughout the program. The features of a spiral curriculum are that topics are revisited, there are increasing levels of difficulty, new learning is related to previous learning, and students' competences increase with each visit until the final overall objectives are achieved (Harder & Stamper, 1999). The value of a spiral curriculum lies in the following six key aspects according to (Harder & Stamper, 1999): *Reinforcement*, where topics are revisited, which helps students retain and deepen their understanding over time and solves the common problem of forgetting previously learnt knowledge. *Progressions* from simple to complex, where the students are introduced to concepts at a manageable level and gradually build on previous knowledge, increasing understanding. *Integration*, where the spiral approach breaks down traditional boundaries between MO. *Logical* *sequence*, where the spiral curriculum provides a structured flow that helps students navigate the complexity of mathematics as subject. *Higher-level objectives*, where the emphasis is on applying knowledge and skills rather than simply memorizing facts. *Flexibility*, where the curriculum allows for flexibility so that students can progress to more advanced stages if they have mastered previous content. The official Danish curriculum “Common Objectives” (CO) is a competence-based curriculum with a spiral structure. On the other hand, the Japanese curriculum has a more linear and step-by-step structure with a steady but slow progression (Paper II).

Despite the very different curriculum structure, the overall objectives of mathematics in Danish and Japanese schools are quite similar. The Japanese School Education law state in paragraph two that “… school education should be committed to enhancing its instruction to enable pupils, to solidly acquire basic and fundamental knowledge and skills, to foster the ability to think, to make decisions, to express themselves in ways that are necessary to solve problems by using acquired knowledge and skills, and to cultivate an attitude of proactive learning (Nakayasy, 2016). This is similar to the general descriptions of the purpose of mathematics in Danish schools (cf. section 1.4). Thus, the overall aims of the mathematical education in the Danish and Japanese schools align, while the detailed program propose different ways to reach the aim. Although the overall educational goals are aligned, analyses of the specific subject areas (e.g. the transition from arithmetic to algebra), reveal different approaches in terms of textbook content and didactic principles (Paper II).

3.6.3 Subject- and themes related to the transition from arithmetic to algebra.

There is a long tradition of analyzing and comparing textbooks within and between education systems (Sayers, et al., 2019). A study by Ding (2016) aims to create better conditions for students to learn inverse relations and argues for the potential of using Chinese textbooks in the United States. In a comparison of Chinese and American textbooks, they examine how items of the task type $T: a + x = b$, where $a, b \in N$ with the corresponding technique τ: filling in the missing number using inverse relations, e.g. solving $8 + (\,) = 14$ by thinking $14 - 8$ is represented by the parenthesis (Ding, 2016). The main difference between the Chinese and American textbook system from Grade 1 to 6 was that in Chinese textbooks, learning is structured over time with an emphasis on systematic structural relations, including the inverse quantities relationships (Ding, 2016).

The context and progression of the distributive property in Chinese and American mathematics textbooks is the subject of another comparative study (Ding & Li, 2010). The results based in coding the tasks from textbooks in grade 2 to 6 show that the American textbooks are dominated by calculations and less by word-problems. The Chinese textbook begin to develop students' intuitive understanding of the distributive property in Grade 2, while the American textbooks do so in Grade3. The crucial difference is that the Chinese text extensively discusses the distributive property when it is first formally introduced, whereas only one of the American textbooks has such a focus. This is evidenced by the fact that the Chinese 4th grade textbook has a specific chapter called "Operational Properties" on distributive property only which contains, three lessons with exercises the focus on the distributive property. After the formal introduction of the distributive property, there is a progression from students using the property with repeated variables and problem solving to extensive use of distributive property in grade six, solving equations and using equations to solve word problems. In contrast, both US textbooks used distributive property primarily for computation (Ding & Li, 2010).

3.6.4 Institutional transitions

Although there is a substantial body of research examining the analysis and comparison of mathematics textbooks between countries, there is little evidence on how the use of an imported textbook from one cultural context to another affects the didactic traditions of a system (Sayers et al., 2019). A Swedish study analyses the opportunities for learning fundamental number sense in three first-year textbooks in Sweden, where one of the textbooks is from Swedish and two are imported from Finland and Singapore respectively, which are generally recognized as successful countries in international studies such as PISA and TIMSS (Sayers et al., 2019). The analyses show both similarities and differences between the three textbooks depending in the analytic approach chosen. It is therefore emphasized by the authors that it is unwise to uncritically import textbook materials from the so-called successful countries if the import "attempts to fix something that is not broken" (Sayers et al. 2019, p.521). The article does not address the converse idea, namely that it may be appropriate to import textbook material in mathematical areas need "to be fixed". The following studies by Krammer (1985) and Ginsburg et al. (2005) investigates IDT from TB to TK (figure 2).

Krammer (1985) conceptualized the textbook as a classroom context variable and compared the teaching practices of teachers using three different mathematics textbooks in grade 8 classed in the Netherlands. The study, based on classroom observations, tests, and questionnaires, found significant differences in teaching practices. These included variations in the frequency of higher-order questions, seatwork, academic conversation, and students' perceptions of remedial help, all closely linked to the features of the textbook they used. Krammer (1985) questioned whether the observed consistency between textbook and teaching practice was due to teachers following textbook blindly or because the teachers' selection of textbooks followed from their teaching styles.

In the exploratory study by Ginsburg et al. (2005) they compare the mathematics education systems in Singapore and the US, focusing on primary school, where basic mathematics skills are built. The study examines important differences in the institutional frameworks, including textbooks, assessments, and teacher qualifications, in the two countries.
It shows that Singapore's success is due to its coherent system, which includes a coherent national framework for education, problem-based textbooks, rigorous assessments and highly qualified, mastery-oriented teachers. In contrast, the US system lacks a coherent content focus, relies on traditional textbooks that emphasize rote learning, and has fewer challenging assessments. In addition, US teachers often lack adequate training in mathematics, and at-risk students receive inadequate support. The study also presents pilot results from US schools that implemented Singapore's textbooks without reproducing all features of the full educational system. Both strengths and weaknesses were observed in piloting the implementation of the Singapore mathematics curriculum, in comparison to the traditional mathematics curriculum. The strengths were greater depth and less breadth in the curriculum, so that fewer topics were covered each year, with an emphasis on mastery level. Mathematics topics were revisited throughout the curriculum with higher levels of mathematical content and higher levels of mathematical thinking. The textbook material presents concepts clearly using pictures, numbers and words and the challenging word problems allowed students to become critical thinkers. Finally, algebraic ideas and problems were introduced at an earlier age than in US books. The main challenges with the use of Singapore textbook resources were that some of the strategies used were new to both students and teachers. There were also language differences, including different metric units. Written communication skills were not emphasized and there were not enough practical activities in grade 7 and 8. Finally, the Singapore resources did not match the state framework, e.g. the probability strand was not covered (Ginsburg et al., 2005). The implementation of the Singapore mathematics textbooks produced mixed results across the pilot sites. In some cases, particularly in smaller sites with stable enrolments, or in sites with gifted students, students showed significant progress. However, the variability in results and the concerns expressed by teachers suggest that careful introduction of entire textbooks is necessary for their effective use. Unlike US textbooks, the Singapore's curriculum does not repeat content

as much, creating challenges for students transferring from schools with different curricula and for students who struggle with mathematics (Ginsburg et al., 2005).

Many studies of curriculum materials start with the text and look at how closely teachers follow or change it. These studies often take a positivist view, assuming that, under ideal conditions, a close match between written and enacted curriculum is possible. As a result, researchers focus on how curriculum writers can provide clearer guidance to teachers. This focus is not surprising, given the widespread use of textbooks and the tendency to see the materials as a potential vehicle for change (Remillard, 2005).

3.7 Literature background summary

The literature search based on keywords and seed references yielded a large field of relevant literature within each of the thematic areas. The above literature review is not exhaustive but includes a broad range of articles relevant to the project. However, at the intersection of ATD, school algebra and textbooks, there are primarily praxeological and epistemological studies and no studies of experimental use of foreign textbooks. Studies comparing mathematics textbooks from so-called high-achieving countries such as Singapore, China and Japan with American and European textbooks, show that Asian textbooks are characterized by greater depth and less breadth, with fewer topics covered each year and an emphasis on mastery and clear progression.

Mathematics textbooks from Asian countries are only used with limited success in countries with different curriculum structures, according to the sparse and non-systematic evidence available. We note that a common feature across the few studies of such use that they concern the adoption of entire textbooks rather than selected parts.

This background literature lays the foundation for testing the hypothesis that foreign, research-based textbook materials can support students and teachers in the transition from arithmetic to algebra in DLS, through the use of selected chapters. By doing so, the project will contribute to the intersection of the research fields Figure 4, by exploring the conditions and constraints for the selective import of textbook material between institutions.

4 Research Questions

In the original application to the Council of Educational Research, the project title was “The Abstraction Gap – transition from arithmetic to algebra”. After having explored the definition of school algebra as a modelling tool and literature background on the corresponding research areas and in particular the challenges of the transition from arithmetic to algebra, I replaced the idea of an abstraction gap by this latter transition. It became clear to me that the transition from arithmetic to algebra is difficult but not abrupt like a gap, and that school algebra as a domain is intertwined with all other mathematical domains. Instead of the image of a gap to be bridged, it is more fruitful to think of school algebra as a modelling tool that is used in all domains of mathematics (at least after primary school), and in many domains of disciplines outside mathematics, such as physics, biology and economics.

I order to base my diagnosis and intervention on an adequate picture of what algebraic praxeologies are currently taught in DLS and to what extent theoretical elements from arithmetic are used or modelled, I began by addressing

> **RQ1: What algebraic praxeologies are currently taught in the Danish lower secondary school, and to what extent does the teaching offer a systematic progression at the level of theory?**

My original idea of a “gap” reflects a practitioner’s experience of a sometimes-sharp divide between students who appear to have “cracked the code” of algebra and those who have not. Indeed, with a detailed model of the algebraic praxeologies at stake, one could begin to investigate this in a more precise way, based on empirical data of various origins:

> **RQ2: What algebraic techniques are particularly problematic for Danish students at early lower secondary level and what theoretical gaps does these imply?**

The third research question presents a discussion of the findings from research question one and two. This is not a separate research question in any of the papers and, in principle, also more of a “discussion issue” related to the previous two questions. As it is often the case in journal papers, the papers included in this thesis did not have space for a systematic and comprehensive literature review. However, such a review was included as a separate section in this introduction (Chap.3), and we took advantage of this to include also a more thorough discussion of how the findings from the Danish context relate to the broader range of previous international research (Chap. 7).

RQ3: How does previous international research relate to answers found for RQ1 and RQ2?

From the outset of the project, a main goal was to experiment the use of certain textbook materials from Singapore or Japan that are based on thorough and broad practice research (cf. Miyakawa & Winsløw, 2019; MEXT, 2023). The aim was to investigate the extent to which the targeted use of research-based textbook materials can help overcome the current challenges in the transition from arithmetic to algebra in DLS:

RQ4: What effects can be seen in Danish lower secondary school when implementing adapted research-based resources from other countries to overcome the challenges identified (RQ1, RQ2)?

Some elements of this question could, in retrospect, be nuanced or at least elaborated. It is not immediately clear what implementing resources means, in particular, how much support and direction is given to the teachers' appropriation and use of the resource. Any "effects" should be understood in terms of differences, which is complicated for two reasons: even when focusing on differences in students' praxeological equipment, any didactic process results in such differences, and isolating what is somehow caused by the resource is difficult especially since one cannot know what the use of the habitual resource would have produced. In ATD, we would rather consider the effects in terms of new constraints and conditions for the didactic process as a whole, involving also impact on the teachers' practice.

The following chapter outlines the way the research questions posed were addressed further on.

5 Methodology

To respond to the four research questions (Chap. 4) in ATD, a variety of methods and models will be used. A praxeological analysis of knowledge to be taught and knowledge taught in DLS will be used to answer RQ1: What algebraic praxeologies are currently taught in the DLS, and to what extent does the teaching offer a systematic progression at the level of theory.

5.1 Praxeological analysis to model algebraic praxeologies taught in DLS

The algebraic knowledge learnt in school is only a part of the internal didactic transposition, and it is therefore essential to consider the other elements of the internal didactic transposition of school algebra in order to gain a more comprehensive understanding of school algebra in Denmark. The construction of an epistemological reference model (REM) for elementary algebra in Danish lower secondary school provides an opportunity to investigate and interpret the status of school algebra in Danish lower secondary school (Paper I).

To answer the research question "What algebraic praxeologies are currently taught in the DLS, and to what extent does the teaching offer a systematic progression at the level of theory?" We will look at the knowledge to be taught by analyzing curriculum resources, common objectives (Education, 2019), national written exam after lower secondary school year 2019 (Education, 2024) and the textbook material KonteXt+ (Thomsen et al., 2015), as illustrated in figure 5.

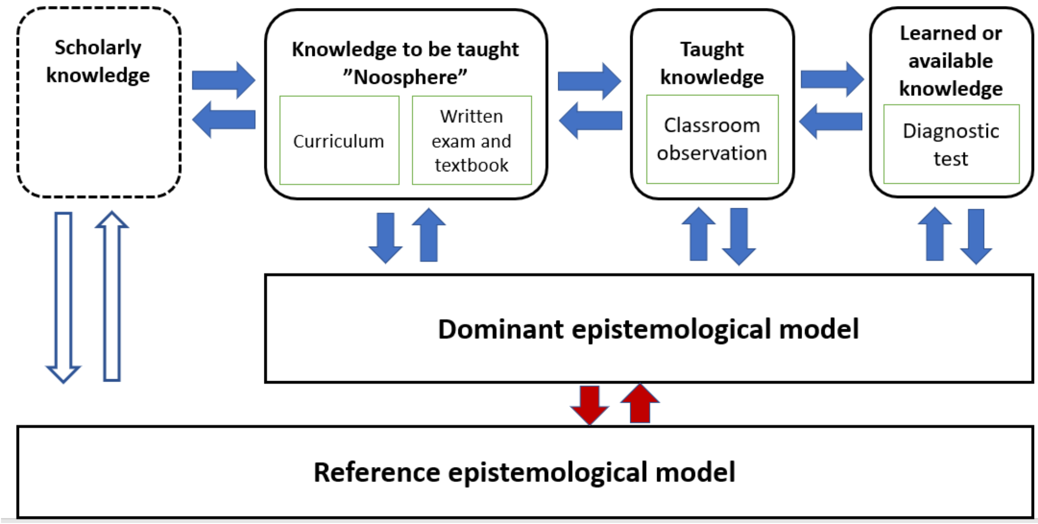


Figure 5. The empirical data that models the DEM and REM (Paper I)

5.1.1 Identifying the Dominant Epistemological Model (DEM)

The first step in modelling the DEM is to detect the exercises in the curriculum material associated that forms the current algebra knowledge to be taught in lower secondary school. The second step is to divide the exercises into type of tasks T_i and the corresponding technique τ_i and describe the current related knowledge block $[\theta/\Theta]$. As we are particularly interested in elementary algebra, instrumented techniques such as using CAS tools and spreadsheets to construct and solve equations will not be considered in detail. The empirical data for this first step is the national curriculum for DLS "Common core" (CO), written exam after DLS and the textbook material KonteXt+ (Johnsen et al. 2005; 2009; Thomsen et al., 2016). The next step in modelling the DEM is to obtain empirical data from the knowledge taught (figure 5).

In this project, we carried out observations of the teaching of elementary algebra in three different schools, for a total of 12 lessons in the January 2022. Here, "the teaching of elementary algebra" reflects the teachers' selection of lessons where algebra was taught according to them; there is no guarantee that the lessons were somehow representative, and some of the lessons and little or no algebraic content. We used participant observation (Emerson et al., 2001) and field notes were recorded during the teaching and combined with transcriptions of the classroom dialogue (Paper I). To describe and analyze how a study process construct and reconstructs a mathematical organization, we will use the six moments of didactic process as an analytical tool to analyze the selected observations (Barbe et al., 2005) (Cf. Chap. 3.4).

The purpose of these observations was to complement the analysis of textbooks with episodes from actual teaching, in view of identifying the DEM (algebraic praxeologies currently taught in DLS). The construction of a dominant praxeological model (DEM) is appropriate to describe the algebra prevalent in educational institutions. While algebra as a domain poses challenges that can be difficult to see and understand when algebraic techniques and theory is internalized and institutionalized to such an extent that it can be difficult to see beyond. The dominant model constructed through praxeological analysis of current knowledge in institutions, is referred to by Gascón (2024) as the current epistemological model (CEM).

The analysis will present the syntheses of the DEM in thematic chunks associated with the transition from arithmetic to algebra. The aim is to gain insight into the knowledge to be taught and the knowledge taught, in order to answer RQ1 and as a baseline for our subsequent experiment.

To inform the DEM with current knowledge learnt in DLS, a diagnostic paper and pencil test was developed and conducted (cf. section 5.2.4). The outcome of the diagnostic test also contributes to answer RQ2: What algebraic techniques are particularly problematic for Danish students at early lower secondary level and what theoretical gaps does these imply, we will develop a diagnostic test tool.

5.2 Diagnostic test to diagnose problematic algebraic techniques I DLS

It is a long-standing and widespread problem that large groups of students seem to get stuck in the transition from arithmetic to algebra (Herscovics & Linchevski, 1994). A significant part of the didactic research in basic algebra focuses on the difficulties that students encounter when they first engage with algebra in lower secondary school (Munzón et al, 2015). To get insight into what algebraic techniques are particularly problematic for Danish students in the transition from arithmetic to algebra in lower secondary school and what theoretical gaps these imply, we developed a diagnostic test inspired by a project on middle school arithmetic (Cosan, 2021).

5.2.1 Construction of the diagnostic test tool

The diagnostic test tool is based on the REM. Items designed to test a simple technique are also simple to construct (Paper IV). As an example, the item $4 + 5 \cdot 2 =$ is the type of tasks T: calculation of arithmetic expressions with multiplication of integers before addition and subtraction. The item is designed to display the technique used. There are also variations of type of tasks that can be solved using the same technique for a more nuanced picture of the scope of the technique used. Example "Show and explain how you calculate $3 - 2 + 5 \cdot 2 =$ ", where the calculations are designed to diagnose the techniques used and by requiring the student to "explain", there is an opportunity to gain access to the logos part of the praxeology.

As described in the section on ATD and School algebra (cf. Chap. 3.5), algebraic modelling occurs through a process of algebraization that begins in primary school and continues through secondary education to university level. In order to diagnose the general level of school algebra taught and to identify which aspects of the transition from arithmetic to algebra appear to be weak, the diagnostic test will include items on the first stage of the three-stage model of the algebraizations process. The first stage of algebraizations occurs when students consider CP as a whole ad not just as a process (Ruiz-Munzón et al., 2013; Bosch 2015). "Show how to calculate $17 + 29 + 132 - 52 + 52 - 29 =$ " and the item $8 + 4 = \ \ + 5$ will provide insight into the techniques used and to what extent the arithmetic foundations for the algebraization process is present. The equations $2x = 16 + 2$, and $7x - 7 = 13 - 3x$ provide insight into the algebraic techniques used and the extent of the first level of algebraization (cf. section 3.5.2).

5.2.2 Algebraic modelling in the diagnostic test

As previously described (cf. section 5.4) the use of algebra as a modelling tool is a central issue in the discussions about the role of school algebra (Chevallard & Strømskag, 2022). The diagnostic test includes items that includes algebraic modelling in geometry with the task type T: determine the perimeter of the polygon, and the task type T: determine the area of a polygon with all sides being either parallel or orthogonal.

In addition to gain knowledge of students' praxis blocks according to arithmetic and algebra, the diagnostic test also includes items designed to gauge knowledge of their logos blocks.

5.2.3 The level of theory in the diagnostic test

The distributive law is part of the theoretical level of both arithmetic and algebra and is the crucial link between addition and multiplication (and field axiom). To diagnose the state of the distributive property there are different variations of items with techniques based on the distributive law. Examples $(2+3)\cdot 5-1=$, $32+3\cdot(7-5)=$, $k(5+3)=$, "Explain how to calculate $(a+b)(a+b)=$ ".

In ATD these theory elements are empirical object and contributes to state the level of algebraic techniques and theory in Danish lower secondary school.

5.2.4 Conducting the diagnostic test in DLS

The diagnostic test was first piloted by 25 grade 7 students (13-14-year-old). Based on the pilot test, items that all students answered correctly and items that none of the students answered correctly were removed. Some of the items that used techniques for addition and subtracting fractions were removed, because the results did not contribute to further knowledge in relation to Cosan (2021). The revised diagnostic test was carried out into two different schools, four grade 7 classes in February 2022 and one grade 8 class in Marts 2022. The tests were conducted after the Covid-19 pandemic and not all students were present. The total number of grade 7 students (12-13-year-old) is 69 and the total number of grade 8 students (13-14-year-old) is 22 students. The students had 45 minutes to answer the 67-item paper-and-pencil diagnostic test (Paper I).

The thorough diagnosis of the current state of school algebra forms the basis for the experiment of using foreign textbook in DLS.

5.3 The experiment

We are interested in how school algebra can be changes in a certain direction to address some of the challenges stated through the diagnosis. But how and what would be the consequences of such a change? To investigate this, we ask RQ4: What effects can be seen in Danish lower secondary school when implementing adapted research-based resources from other countries to overcome the challenges identified (RQ1, RQ2). In order to conduct such an experiment, it is essential to consider a number of methodological and ethical issues.

In the process of translating the knowledge to be taught into the knowledge taught, the textbook has a role as a mediator, where the form and content of the textbook has consequences for the teaching and the knowledge learnt. To explore the extent to which the transition from arithmetic to algebra can be strengthened by using textbook resources from another country we use the Japanese textbook *Junior High School Mathematics:1* (Isoda & Tall, 2019). The basis for the selection is that the textbook material comes from Japan, which is one of the top five performing countries in Trends in International Mathematics and Science Studies (TIMSS) 2019 (Mullis et al., 2020). The material is based on systematic empirical research and translated into English. The Japanese textbook has an explicit and theory-based approach and includes an explicit description of the use of algebraic notation form (Paper II). In order to use the Japanese textbook material in DLS, it requires at least a translation to Danish.

5.3.1 The translation of the textbook material

There are three levels of considerations when using a foreign textbook, acceptance, adaption and appraisal (Howson, 2013). You can choose to accept the textbook material in their full form, with the “real” examples included, even though they may be of less relevance or importance to the students who will be using the material. Adaption of content, e.g. by changing currencies and changing place names or other superficial details. The last level involves an appraisal of the materials. At this level, curriculum resources from another country are examined to identify potential ideas or strategies that can be incorporated into home-produced texts (Howson, 2013).

I this project, we want the translation of the textbook chapter to be as close to full acceptance as possible. To ensure that only the most essential elements of the material were changes and adapted, the chapter was first taught to one student in its original form. Observations from this collaboration meant that the names of the artificial persons in the material were changes from the Japanese names Yui and Takumi to Aya and Toke, to move the students focus from pronunciation to content. Another small subset of the material was also changed.

The sentence “In algebraic expressions, you can remove the multiplication sign ×” which can be found in the text box with “How to express product” was modified to “In algebraic expressions, you can remove the multiplication sign ×. The multiplication sign can also be written as a dot ·”. Because in Danish textbooks, the dot is mainly used as a multiplication sign. However, the multiplication sign × is not unfamiliar to students as it appears on calculators and in older textbook material. To make the use of signs explicit, which is one of the hallmarks of the material (Paper II), this information on the relationship between the cross and the dot has been added.

The chapter “Algebraic Expression” is in Japan used in the first part of junior high school, when students are 12 years old. To determine when the first encounter with algebraic expression should take place in DLS, the CO states that students must “find solutions to simple equations using informal methods” and “use simple algebraic expressions for calculations” in grade 6 to 9 (Education, 2019). As the CO do not specify this further, the timing of the teaching experiment was based in conversations with mathematics teachers at the school where the experiment was to be conducted. The first two month of grade 8 were chosen as the time to use the Japanese textbook chapter “Algebraic Expression” in Junior High School Mathematics:1 (Isoda & Tall, 2019).

To investigate what effects can be seen in Danish lower secondary school when using Japanese textbook materials we will use the concept of praxeological change (Putra, 2019) (cf. section 2.3).

5.4 Praxeological change based on observations and teacher interviews

There are three areas in particular where the Japanese textbook chapter excels. It is the use of a broad opening problem that is referred throughout the chapter. The explicit use of algebraic notation form and the explicit technical descriptions with theoretical justifications for the techniques used (Paper II). Based on these key areas, classroom observations with notes, photographs and video recordings are used to select teaching episodes for further analysis. The selected teaching episodes will form the basis of the subsequent teacher interviews, which will follow the photo elicitation interview method, where transcripts and photographs of the episode are used to generate verbal discussion (Nissen et al., 2016). A pre- and post-diagnostic test (cf. section 5.2) is used to gain insight into the knowledge learnt.

There are ethical issues that need to be considered when conducting experiments in general, and especially in school institutions where teachers and students are involved.

5.5 Ethical considerations

In the project, the mathematics teachers voluntarily signed up for the experiment based on an information letter sent to the mathematics supervisors in the schools. This letter described the aim of the project, how teachers and students were expected to contribute, and what kind of data would be collected during the experiment. In this letter the aim of the project was described and how teachers and students should contribute and what kind of data I will collect during the project. The intention of the information letter was to ensure transparency and openness about interest, research plan, methods and results, not only in reporting but also in relation to the participants in the project (Jensen et al., 2020). To ensure that the participants knew the purpose before signing the cooperation agreement, I held a meeting before the experiment started to inform the teachers about the purpose of the intervention. I made it clear that the intervention was an experiment with only suspected benefits. By participating in the experiment, schools are contributing to educational research. If the intervention is found to have a positive impact on how school algebra is taught and learnt in school, then the experiment will have a positive impact on the school as an institution. Although the benefits may not be immediately apparent, the experiment could form part of a longer-term strategy to enhance the knowledge of mathematics teachers and students in the field of school algebra, with the potential for further improvement over time. While there is a possibility that the intervention may have a negative impact, this likely to be limited given that the experiment utilizes validated and empirically tested textbook material.

We base our study in ATD, with a focus on the institutional conditions and constraints when teaching algebraic expression using a Japanese textbook chapter. The study aims to describe praxeological changes (Putra, 2019), when using a foreign textbook chapter. It should be noted that this research does not focus on individuals, but rather on the didactic system as a whole.

5.6 Responsible conduct of research in the project

The project integrates quantitative and qualitative methods in line with the ATD methodological tradition to classify mathematical praxeologies and in relation to modelling praxeologies (Garcia et al., 2006). The project will comply with the Danish Code of Conduct for Research Integrity (Science, 2019) and the General Data Protection Regulation as described in the Official Journal of the European Union Regulation 2016/679 (Regulation (EU) 679, 2016). Because “The purpose of the GDPR is to impose a uniform level of privacy protection in all Member States when data is processed in the Member State or transferred to other Member States within the EU/EEA” (Jensen et al., 2020, p.119), and “Any information which can directly or indirectly be linked to an identifiable person counts as personal” (Jensen et al., 2020, p.120), the teachers participating in the experiment was informed that all empirical data collected for the research purpose, including the diagnostic test results, observations, and interviews, would be anonymized in any subsequent publications. This signifies that 'the individuals are not or are no longer identifiable and could not be identified by further processing of the data' (Jensen et al., 2020). During the analysis of the diagnostic test results, the students is anonymized for both ethical and practical reasons. As stated by (Jensen et al., 2020, p. 58) “When personal data is fully anonymized, it can be managed in the same way as other non-sensitive data”. This approach allows for the discussion of the analysis and results with colleagues without the risk of identifying the schools, teachers or students.

6 Results

In this section we present the main results of the thesis understood as answers to the four research questions (section 4). Notice that subsections 6.1-6.2 reflect the analysis of algebraic knowledge to be taught, actually taught and learnt (or not learnt) in DLS.

6.1 Algebraic praxeologies in Danish lower secondary school

We have organized the analysis of the algebraic knowledge to be taught in DLS according to the various stages and documents that were identified in Section 3: the official program (CO), the national exam, and textbooks (figure 5). In order to go beyond analyzing the aimed-at (final) praxeologies, we also look at the progression suggested by the CO and especially the textbooks, and with particular attention to the progression with respect to the theory level and with respect to the levels of algebraization (cf. section 3.5.2).

6.1.1 The progression of school algebra in the common objectives

In Denmark, the CO for mathematics serve as official guidelines for primary and lower secondary schools (Education, 2019). These goals form a competence-based curriculum with mathematical organization that combines mathematical topics in number and algebra, geometry and measurement, and statistics and probability. The six competencies' associates with each topic are: problem solving, modelling, reasoning and thinking, representations and symbol processing, communication, and assistive technology. The aim of the competencies' is for "the student to act with judgement in complex situations with mathematics" (Education, 2019, p. 9). The overall aim for students after lover secondary school (grade 9.) according to the MO number and algebra is, that the students can use real numbers and algebraic expressions in mathematical investigations. IN order to gain an overview of the manner in which the overarching objective is delineated in the CO, we have constructed a table 4 of the MO in CO.

In the CO algebra is divided into three RMOs: 'equations', 'formulas and algebraic expressions' and 'functions'. Each RMO is further subdivided into learning objectives (LMOs), which are subdivided into techniques and the knowledge required for these techniques table 4.

Table 4. MO in the CO of the algebraic domain in DLS (Tonnesen, RDM, Table 1)

RMO_1 Equations		**RMO_2 Formulas and algebraic expressions**		**RMO_3 Functions**	
$LMO_{1,1}$		**$LMO_{2,1}$**		**$LMO_{3,1}$**	
The student can develop methods for solving equations.	The student has knowledge of strategies for solving equations.	The student can describe relationships between simple algebraic expressions and geometric representations.	The student has knowledge of geometric representations of algebraic expressions.	The student can use linear functions to describe relationships and changes.	The student has knowledge of representations of linear functions.
$LMO_{1,2}$		**$LMO_{2,2}$**		**$LMO_{3,2}$**	
The student can build and solve equations and simple inequalities.	The student has knowledge of equation solving with and without digital tools.	The student can rewrite and calculate with variables.	The student has knowledge of methods for rewriting and calculations with variables, including with digital tools.	The student can use non-linear functions to describe relationships and changes.	The student has knowledge of representations of non-linear functions.
$LMO_{1,3}$		**$LMO_{2,3}$**			
The student can build and solve simple systems of equations.	The student has knowledge of graphical solutions of simple systems of equations.	The student can compare algebraic expressions.	The student has knowledge of the rules for calculating with real numbers.		

On the one hand $LMO_{2,1}$ and $LMO_{2,3}$ refers to algebra as an intra-mathematical modelling tool e.g. $LMO_{2,1}$ could include modelling geometric relationships as the perimeter and areas of a polygon and $LMO_{2,3}$ could include associativity for addition and multiplication. On the other hand, $LMO_{2,1}$ can refer to “geometric algebra”, a method where geometric models are used to justify mathematical properties such as the distributive property (Paper IV). Based on these to interpretations, it can be argued that $LMO_{2,1}$ and $LMO_{2,3}$ permits the utilization of algebra as an intra- and extra mathematical modelling tool, but is conditional on $LMO_{2,2}$. This is a MO where the spiral structure allows for integration and flexibility between the RMOs and LMOs and it is possible to achieve higher-level objectives by applying knowledge and skills rather than just memorizing facts (cf. Chap. 5.6.3). The process of developing a progression from simple to more complex problems, in which fundamental algebraic concepts are introduced (e.g. the use of variables) in a structured and incremental manner, is through the EDT_2 (figure 2.). In order to narrow the EDT_2, it is useful to analyze that types of tasks are included in the mathematical national exam after DLS.

6.1.2 Algebraic praxeologies and levels of algebraization in the national written exam

A complete praxeological analysis of the types of tasks appearing in the national exam (with and without aids) was elaborated by Poulsen (2015). A main outcome is that most tasks are of the type “solve a first order equation”, and that the only really necessary technique for solving the equations is the technique of substitution (guessing a solution by trial-and-error). In particular, solutions are always small positive integers. There are also a few examples of algebraic modelling, often with the model being presented by the exercise and merely to be used by the examinees. In this section presents an analysis of three illustrative examples, with a particular emphasis on the level of algebraization.

In the Danish paper and pencil unaided written exam after grade 9, tasks like $40 + \_\_\_ = 50 + 15$ appear frequently. The items can be solved “mentally”, without any algebraic reasoning, and is a very basic example of the first stage of the algebraization process where the students consider the CP “40 plus something” as a whole. In the written exam there are also equations in the more common algebraic notation form:

$$3x + 1 = 10, \qquad x =$$

$$5x - 3 = 2x + 18, \qquad x =$$

$$\frac{2(x+4)}{x} = 6, \qquad x =$$

In fact all three equations can be solved by substitutions of small positive integers $x \in \{1,2,3, \dots ,7\}$, which, a priori, is expected to be the dominant technique used. In the third equation, the substitution with two integers is successful and the student can be satisfied with finding one solution, as they have never encountered equations with any other number of solutions than one. By using substitution as a technique for solving equations, the algebraization process remains at the first stage (Paper I).

The last example is a mathematical problem from the national written exam (2017).

In an item, students are asked to explore a series of squares that can be modelled by a number pattern, in which K_n designated the number of squares in a certain number pattern. In the last question, students are asked the following "You must show that the formula $K_n = r(r - 1 + n)$ can be rewritten as $K_n = r(n - 1) + r^2$" (Education, 2024). In this example, the result is given and all that is required of the students is that they should provide "correct transformations with at least one intermediate result" such as $r^2 + r(-1 + n)$, according to the exam instructions. In REM, this rewriting requires explicit reference to theory, with simple use of the distributive property, the meaning of exponents and the commutative property of addition (Paper I). We also note that there is no questioning the importance of rewriting when using algebra as a modelling tool; the above rewriting has no visible function at all and appears as an unmotivated trick that the students should explain. In the last example, the level of algebraization also remains at the first level.

In order to gain insight into the progression from CP to the initial stage of the first level of algebraization, essential praxis and logos blocks in arithmetic will serve as the foundation for a praxeological analysis of the progression in the textbook material for DLS.

6.1.3 Praxeological analysis of progression in the textbooks

In the Danish Textbook KonteXt+, the chapter titles refer to the mathematical content, but the sections are divided according to different types of tasks and activities (Paper II, Table 3). This structure reflects the spiral and integrated approach of the CO, in which mathematical competences and praxeologies are developed and revisited over several years.

Case: Textbook progression in fraction praxeologies through grade 5-9

An example is the problem of ordering two given fractions, where you have to decide which one is the largest. In this type of task, the two fractions either can both be simple fractions, fractions with like denominators and different numerators or fractions with like numerators and different denominators.

Table 5. Theme in the DEM according to fractions in the textbook material (Paper IV, Table 1)

Type of task	Techniques	Kon-text+ 5	Kon-text+ 6	Kon-text+ 7	Kon-text+ 8	Kon-text+ 9
T_{19}: Given two simple fractions $\frac{1}{a}$ and $\frac{1}{b}$, which is largest	τ_{19}: The fraction with the lowest denominator is largest i.e., $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$					
T_{20}: Examine which fraction with like denominator and different numerators, $\frac{a}{c}$ and $\frac{b}{c}$ is largest.	τ_{20}: The fraction with the highest numerator is largest i.e., $if\ a < b\ \Rightarrow \frac{a}{c} < \frac{b}{c}$					
T_{21}: Examine which fraction with like numeration and different denominators fraction $\frac{a}{b}$ and $\frac{a}{c}$ is largest.	τ_{21}: The fraction with lowest denominator is largest i.e., $if\ b < c\ \Rightarrow \frac{a}{b} > \frac{a}{c}$					

These types of tasks are all located in the textbooks KonteXt+5 to KonteXt+9 (Grade 5 to 9) table 5. In the textbooks, it is a group of practices that are taught together and unified by a common discourse in techniques that includes characteristics of the particular cases and descriptions of the techniques (Paper IV). Therefore, the potential for advancement that could be achieved by treating the various tasks in a consecutive manner is not realized. This results in the presence of all three types of tasks within the textbooks over the course of five years. The serves to illustrate how task types and techniques constitute an integral part of the arithmetic foundation upon which the transition to algebra is based.

In order to gain knowledge of the theoretical elements of arithmetic that are built upon in school algebra, we will analyze the level of theory in DLS textbooks, using the distributive property as an illustrative example.

Case: The level of theory in DLS textbooks - the distributive property.

The distributive property is the crucial link between addition and multiplication (and field axiom) that forms part of the theoretical level of both arithmetic and algebra in REM (Paper V). The progression at the level of theory can be illustrated by the introduction to the distributive property in the Danish textbook material, which appears several times in different grades and in different forms. The distributive property is introduced in different contexts and gradually integrated into both arithmetic and algebra, through special cases and tasks of increasing difficulty (Paper II). As an example, the textbook emphasizes that "You can use geometric figures to model arithmetic rules with letters" (Lindhardt et al. 2021, p. 95). To illustrate, an example is given where a rectangle is divided vertically into two smaller rectangles; the textbook claims it proves the distributive property. In fact, the area of the large rectangle is $c \cdot (a + b)$ and can be written as the sum of the two parts of the rectangle $a \cdot c + c \cdot b$. The generality and variations of the distributive property remain implicit: it is not pointed out to the students that the example is special (assume that $a, b > 0$) (Paper V) and so does not prove the general property. "Distributivity" as an assumption or axiom in algebra is generally not introduced. The description of the distributive property in algebra in the textbook is not explicit linked to knowledge of the distributive property in arithmetic. Consequently, the theoretical levels associated with arithmetic are not used as a foundation for the introduction of theoretical elements in algebra, and the coherence and connection of MOs that CO allows for is not created. To ascertain whether this is merely an isolated case, we will look at the level of algebraization in the textbooks, with a view to determine the implicit or explicit theoretical level of the books in question.

6.1.4 The level of algebraization in DLS textbooks

In the KonteXt+ series of textbooks, algebra is described as the "language of mathematics" and readers learn that "many of the arithmetic rules and notations that apply to numbers, also apply to letters. If you are not sure how to calculate with letters, you can often try with numbers" (Hansen et al., 2016b). The description of algebra as the language of mathematics therefore competes with simple tricks to solve algebraic exercises arithmetical techniques (Paper I). The term "rules" is used to refer to both mathematical properties (e.g., the commutative law) and notational conventions (e.g., writing $a \cdot x$ as ax), in particular, with no distinction between theory and technology (Paper II).

In the Danish textbook, the algebraization process occurs mainly at the first level (arithmetic), with CP represented symbolically. More generally, in DEM, key algebraic conventions and principles emerge through repeated exposure rather than formal introduction, leading to algebraization in gradual and implicit ways. The Danish curriculum resources use common sense terms like "calculation rules" rather than the precise mathematical language about mathematical properties.

The extent to which this “common sense approach” is also reflected in the knowledge taught will be accessed by analyzing algebraic praxeologies current taught in DLS.

6.1.5 Taught algebraic praxeologies in DLS

In this project, we have only carried out limited observations of “actually taught” algebra in DLS, prior to the experimental phase (cf. section 5.1.1). The observations revealed a general structure where didactic processes are limited to moments of first encounter, moments of exploration and moments of technical work, but without the constitution of a shared technologic-theoretical block and in particular without justification of the techniques worked on. The focus on exploration and technical work reflects priorities in the textbook material. This means that the third moment of the study process (section 2.4) is limited and the foundation for the fifth moment (institutionalization of the entire praxeology) is missing.

It will be easier to grasp these overall observations through a concrete episode.

In a grade 8 lesson on solving equations, the teacher seeks to establish a "recipe" for solving equations of the form $ax + b = c$. The teacher does not use such symbolism but proceeds with examples were a, b and c are given. The first example he proposes is $3(x - 4) = 6$. The students begin to discuss the difference between "reduce" and "solve." After a while, the teacher writes $3(x - 4) = 6$ on the whiteboard and asks the students to explain how to solve the equation.

A student explains that she solves the equation mentally, by thinking what inside the parentheses would give two. The teacher labels this method as "trial and error”.

Another student perceives $(x - 4)$ as a unit and at the same time as an expression with a variable and rewrite the equation to $(x - 4) = \frac{6}{3}$ and says, “Then we get x minus four” (Paper I). The teacher acknowledges this but cuts off the exploration to explain a very different method.

The teacher writes $3(x - 4) = 6$ on the whiteboard again and draws curved arrows from 3 to x and from 3 to 4. After this implicit ostension of the distributive law, the teacher writes $3x - 12 = 6$ and says “I have just multiplied into the parentheses. Now I want to move -12 to the other side”. This technique of "moving" numbers is frequent in teachers’ explanations of operations on algebraic equations. Then the teacher writes $3x$ =18 on the whiteboard and says “Then we must find out what to multiply 3 by, to get 18. We should only divide by 3” and writes $x = 6$ on the whiteboard. A student asks: “…. how do we know that we could not subtract 12 from $3x$?” (referring to the step $3x - 12 = 6$). The teacher exclaims, "You cannot subtract numbers from x's," but does not prove the justification asked for by the student. This, indeed, would require a technological-theoretical environment involving the meaning of equivalent expressions and equations. The textbook provides no support for creating such an environment, which could associate more general principles to the given example. Instead, the teacher merely demonstrates an alternative technique which is not more efficient for solving the given task than the spontaneous techniques developed by the students. The technology he must use remains idiosyncratic and highly informal (curly arrows to “show” how to “multiply into” a parenthesis, “moving” numbers etc.).

We note here that although this episode may seem rather particular, the textbook’s lack of support of the third (and therefore also the fifth) moment is quite general (see also section 6.3), and we consider this the principal cause of their absence in the above and many other observed episodes. This reinforces the interest of experimenting alternative text material in view of enabling teachers to organize more complete didactic processes.

6.1.6 Summarizing main points of algebraic praxeologies currently taught in DLS.

The preceding results of the praxeological analyses of what algebraic praxeologies are currently taught in the Danish lower secondary school, and to what extent does the teaching offer a systematic progression at the level of theory, can be summarized in the following headings:

- **Focus on arithmetic and algebraic techniques**.
  The analysis of the textbook material and the written exam reveals a predominant focus on arithmetic an algebraic techniques. Additionally, the textbook place significant emphasis on repetition, whereby the specific types of tasks and associated techniques are reiterated across multiple grade levels.

- **Limitations in the transition from arithmetic to Algebra**
  Arithmetic praxeologies form the basis of the initial introduction to school algebra, but the algebra presented is often a set of rules or techniques rather than a fully developed system.

The theoretical elements of arithmetic (e.g. distributive property) are not linked to algebra, and the possible coherence and connections between MOs of arithmetic and MOs of algebra are not established.

- **Implicit level of theory**
  The teaching of basic algebra tends to focus on techniques with little emphasis on justifying these techniques or developing theoretical concepts (e.g., distributive property). Key theoretical elements such as commutative and distributive properties are rarely introduced explicitly in mathematical terms. While arithmetic praxeologies serve as the foundation for the initial introduction to algebra, the theoretical progression is often explicit, with algebra typically presented as a set of "rules" or techniques rather than as fully theorized system.

- **Implications of the CO structure and approach**
  The CO employs a gradual, spiraling approach to algebraic praxeologies, with an emphasis on providing a description of competencies that students should attain. For instance, the CO for grade 9 outlines that students should be able to "set up and solve equations and simple inequalities". However, while the CO is comprehensive, it does not explicitly address theoretical element such as the commutative and distributive properties in mathematical terms.

It is therefore anticipated that the structure and content of CO, textbooks and final exams will facilitate a certain degree of technical proficiency, although there may be limited scope for systematic theoretical progression. The extent to which this phenomenon is present in the knowledge learnt will be discussed in the following section 6.4 .

6.2 Problematic algebraic techniques in DLS

The following chapter explores what algebraic techniques are particularly problematic for students in DLS and what theoretical gaps these challenges reveal. This is achieved by initially examining the types of tasks related to algebra that are not answered correctly by a significant proportion of students at the national exam after DLS. To gain further insight into the techniques students use to solve these problematic types of tasks, we will analyze a selection of types of task from the diagnostic test (cf. section 5.2).

6.2.1 Problematic algebraic techniques in the national written exam after DLS

After grade 9, students sit a national and compulsory written exam in mathematics for the first time in their school life. Around 60.000 students take this exam every year. In this section, we provide some data from the exams held in 2018 and 2019. The test has two parts, one part without aids, where the students only indicate solutions, and another part where students also write explanations of their results and can use digital tools. It is not possible to gain access to actual student responses from the second part; therefore, the data as a whole provide no insight into the actual techniques used by students. From the accessible data on the exam, we can only see the percentages of students who have correct or partially correct (sometimes at 2 levels) answers, and therefore we can only use these data to identify types of task that are particularly problematic. In the next section, we report on a diagnostic test that we carried out to gain further insight into the techniques students use (or do not use) for these problematic types of task. Table 6 and 7 provide a few examples of items from the first part of the exam, and the share of students who provided correct answers.

Table 6. Correct student answers of items in the written exam without aids 2018 (Paper I)

Item Year 2018	Task Solve the equations	Type of task	Correct answer
7.1	$6x + 4 = 28$ $x =$	T: Solve $Ax + B = C$	88%
7.2	$\frac{x}{2} + 1 = 5$ $x =$	T: Solve first degree equation wich include fraction	66%
7.3	$4 \cdot (x - 2) = 2x + 6$, $x =$	T: Solve first degree equation with brackets and unknown on both sides	44%

Table 7. Correct student answers of items in the written exam without aids 2019 (Paper I)

Item Year 2019	Task Solve the equations	Type of task	Correct answer
11.1	$3x + 1 = 10$ $x =$	T: Solve $Ax + B = C$	89%
11.2	$5x - 3 = 2x + 18$, $x =$	T: Solve $Ax + B = Cx + D$	61%
11.3	$\frac{2(x + 4)}{x} = 6$ $x =$	T: Solve first degree equation which include brackets and a fraction	46%

The results from table 7 and 8 show that the presence of the unknown on both sides of the equation, of brackets and of fractions, all lead to types of tasks which are more difficult than the simple form $\mathrm{Ax} + \mathrm{B} = \mathrm{C}$ (where A, B and C are natural numbers), even more so if they are combined (the last row). A priori analysis suggests that all items can be solved by the technique τ: trial-and-error by substitution of one-digit natural numbers, and for other reasons it is very plausible that most students apply this technique consistently. If so, it is still considerably more difficult to execute the technique in the tasks that involve parentheses, fraction, unknown on both sides, or (worse) a combination of these. The other items in table 7 and 8 can also be solved using the τ: trial-and-error by substitution of integers, but this technique requires more CPs. Of course all the equations can also be solved by algebraic rewriting techniques like τ:addition, subtraction, multiplication or division in both sides of the equal sign, but we don't know to what extent the students may attempt to do that.

Another area of difficulty for students in the written exam is the use of algebra as a modelling tool. In the national written exam in June 2019, 70% of the students were able to correctly determine the perimeter of a polygon with sides given in terms of an unknown (so a side length could be for instance $2x$). Only 26% of the students could determine the area of the same polygon correctly, even if all sides were orthogonal or parallel to one another.

6.2.2 Problematic techniques according to the diagnostic test

In order to gain further insight into the algebraic techniques used, the algebra tasks in the national exam were replicated in the diagnostic test.

Table 8. Correct and incorrect student answers of items in the diagnostic test (Paper I)

Item	Item Nr.	Correct Grade 7	Correct Grade 8	Incorrect Grade 7	Incorrect Grade 8	No Answer Grade 7	No answer Grade 8
$2x = 10$ $x = \_$	2.5	44 64%	18 82%	10 15%	0 0%	15 21%	4 18%
$2x = 16 + 2$ $x = \_\_\_$	3.6	29 42%	12 55%	19 28%	3 14%	21 30%	7 32%
$7x - 7 = 13 - 3x$ $x = \_\_\_$	4.7	15 22%	3 14%	20 29%	4 18%	34 49%	15 68%
$x \cdot \frac{3}{4} = \frac{15}{20}$ $x = \_\_$	7.6	17 25%	8 36%	7 10%	1 5%	45 65%	13 59%

Table 8 illustrates that approximately 50% of the students in grade 7 and 8 answered item 3.6 correctly.

By analyzing the percentage of correct answers for each item, we can get a picture of which type of tasks are challenging for students in DLS. But we can also gain insight into the institutional differences where item 4.7 of the task type T: Solve the first-degree equation of the form $\mathrm{Ax} + \mathrm{B} = \mathrm{Cx} + \mathrm{D}$, seems to cause more problems for students in grade 8 than in grade 7. According to our a priori analysis (Paper I), we expect that the presence of the unknown on both sides of the equations will make it difficult to use substitution, and that students will therefore solve item 4.7 by collect same type of terms and the reduce the coefficient of x. This is not the case in the examples figure 6 and 7 (Paper I).

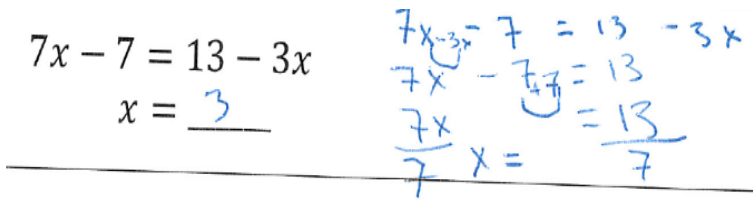


Figure 6. Student answer to item 4.7 in the diagnostic test (Paper I)

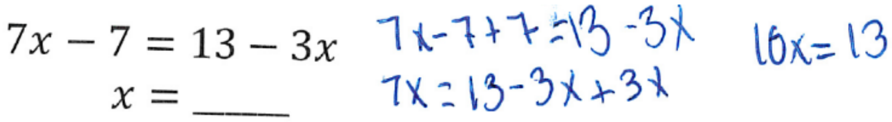


Figure 7. Student answer to item 4.7 in the diagnostic test (Paper I)

The responses of the students in figures 6 and 7 illustrate two examples of the techniques used. It can be seen that the attempts to use algebraic methods were not successful. The addition of opposite numbers and the distributive property were unsuccessful, and students attempted to use arithmetic operations with limited success. These τ^* in figures 6 and 7 are reiterated in different forms in the incorrect answers to item 4.7 (table 8).

A synthesis of the data pertaining to the number of students who solved the algebraic items in national and diagnostic tests, respectively, reveals a prevalence of problematic algebraic techniques among students in grade 7, 8 and 9. Specifically, approximately half of the students' exhibited deficiencies in their ability to utilize algebraic techniques. This phenomenon may be attributes to an overemphasis on arithmetic and algebraic techniques, coupled with an implicit theoretical level (cf. section 6.3) that hinder a praxeological change.

6.2.3 The level of algebraization according to the diagnostic test

According to the DEM, the algebraization process occurs in a gradual and implicit manner, whereby key algebraic conventions and principles emerge through repeated exposure rather than formal introduction. The level of algebraization is mainly observed at the first level, with CP represented symbolically (cf. section 6.1.4). Therefore, it is crucial to consider the arithmetic foundation upon which the algebraization process is based. In the following section, we will present examples of the fragile arithmetic foundation and its implications.

The diagnostic test (cf. section 5.2) includes items from both arithmetic and school algebra, the former in view of investigating students' prerequisites for the transition from arithmetic to algebra we are studying. In item 5.1 in the diagnostic test, students are asked to show how they carry out a given, lengthy calculation (see figure 8). The item provides an opportunity to assess to what extent students see the CP as a whole or as consecutive series of independent processes, and thus determine their readiness for the first level of algebraization (cf. section 3.5.2).

Table 9. Correct and incorrect student answers of item 5.1 in the diagnostic test

Item	Item Nr.	Correct Grade 7	Correct Grade 8	Incorrect Grade 7	Incorrect Grade 8	No Answer Grade 7	No answer Grade 8
15+38+121-42+42-39=	5.1	33 48%	7 32%	21 15%	7 32%	15 21%	8 36%

Table 9 illustrates the outcomes of item 5.1 in the diagnostic test, which revealed that less than half of the students provided the correct response. This suggests that lengthy calculation is not a straightforward task type for the students in grade 7 and 8. In order to gain insight into the techniques used, we will examine three responses to the task, each of which demonstrate a variation og the technique τ: stepwise addition and subtraction.

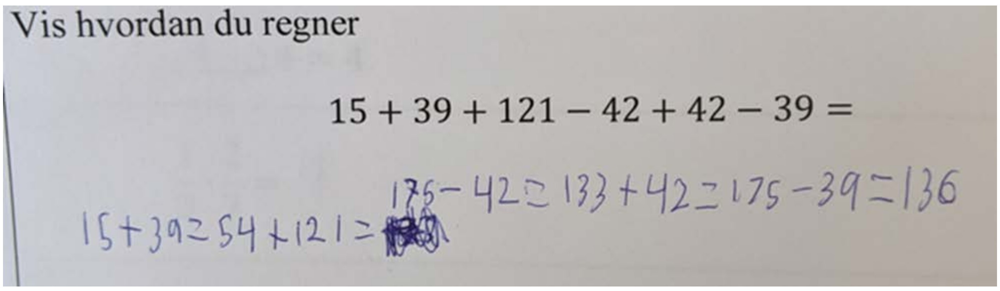


Figure 8. CP with stepwise addition and subtraction

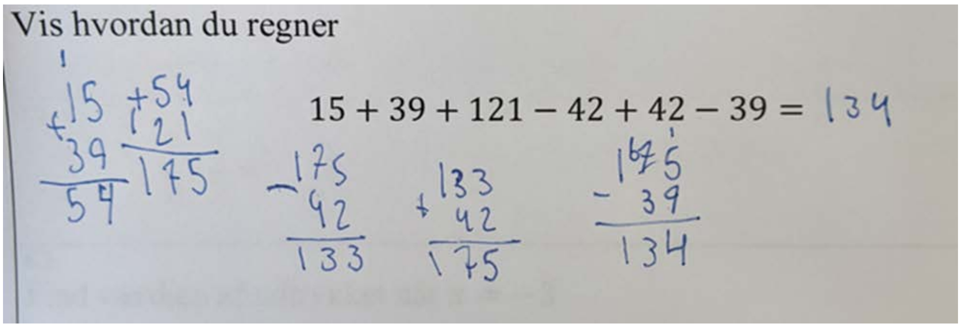


Figure 9. CP with stepwise addition and subtraction in a vertical schema

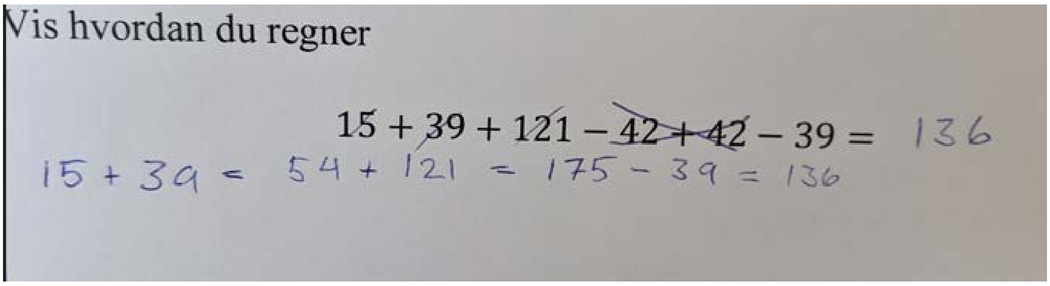


Figure 10. CP with the addition of the opposite and stepwise calculation

The response in figure 8 employs the horizontal scheme of the CP, whereas figure 9 utilizes the vertical scheme of the CP. Despite the differing schemes, the techniques use in figures 8 and 9 are the same. Figure 10 illustrates the application of two fundamental arithmetical techniques τ: addition of the opposite, is used first and then τ: calculation the sum by repeated addition or subtraction of two integers at a time. The responses also offer insight into technological-theoretical element, as the technique τ: addition of the opposite, is only employed when the integers are written in immediate succession, as in the case of $-42 + 42$ (figure 10). However, it is not utilized in instance where the integers are not written next to each other, such as in the case of + 39 and -39. This serves to illustrate that the theoretical elements, such as the commutative property, are not part of the students' topos.

In the transition from arithmetic to algebra, several praxeological changes (cf. section 2.3) are required, including the representation of relations (Kieran, 2004). In arithmetic, an expression such as $8 + 4$ is understood as the CP, where the objective is to find the sum. However, it the expression $8 + 4 = \_\_ + 5$ is viewed as a CP in arithmetic but not in algebra, it can result in an incorrect value of 12, despite the correct answer being 7. This was the case for approximately 50% of the responses to the diagnostic test.

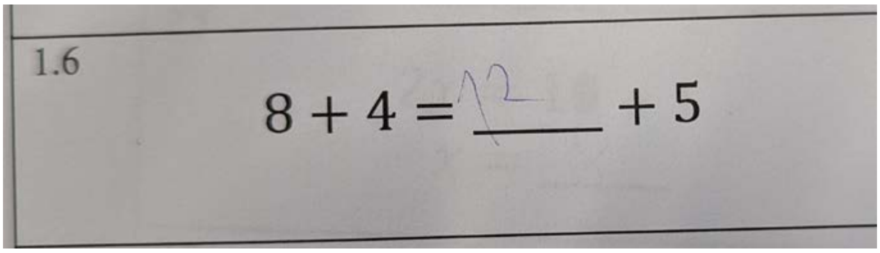


Figure 11. Response where the CP is not perceived as a unified entity.

In item 1.6, figure 11, the CP is not perceived as a unified entity, but rather as a CP with addition "from left to right", which results in an erroneous response. The technology of the equal sign is constrained and presents an obstacle to achieving the first level of algebraization.

6.2.4 Summarizing main points on problematic algebraic techniques in DLS.

The results of the praxeological analyzes of the national written exam and the diagnostic test have provided insights into problematic algebraic techniques which can be summarized under the following heading:

- **Problematic algebraic techniques identified**
  In the national exam and diagnostic test, half of the students struggles with basic algebraic techniques, including trial-and-error substitution, manipulation of equations, and solving first-degree equations with variables on both side of the equal sign.
  Only a quarter of the students could solve simple algebraic modelling items related to the area of polygons (Paper V).

- **The transition from arithmetic to algebra**
  The use of arithmetic operations to solve equations is only partially successful at best. Some of the errors that arise in attempts to use algebraic rewriting techniques can be explained by students' doing operations only on one side of the equation, mimicking calculations learnt in arithmetic. Other difficulties, like parenthesis or fractions adding difficulty for using trial-and-error substitution, are caused by non-mastery of parentheses and division in the context of integers. This emphasizes the importance of a solid foundation in arithmetic to ensure a successful transition from arithmetic to algebra.

6.3 Japanese textbook use in DLS: conditions, constraints and outcomes

In order to describe and analyze the outcome of our experiment – teachers using a Japanese textbook chapter in DLS – we will look at praxeological changes in *the algebraic knowledge to be taught*, and in *the algebraic knowledge actually taught and learnt*. The textbook chapter was chosen because it introduces algebraic expressions based on mathematical investigation and arithmetic expressions with a clear technological and theoretical justification and a carefully chosen example of algebra as a modelling tool (Paper II). Based on the analysis of the internal transposition observed, we aim to identify the main conditions and constraints, which the use of the foreign textbook meets in the experiment, as hypotheses for what may occur more generally for this case of curriculum import.

This section presents the main findings from the actual use carried out by the teachers and students. The findings are presented in accordance with the didactic transposition (see figure 2), which is presented in inverse order (from learnt knowledge, to knowledge to be taught). The process commences with an examination of the results from the diagnostic test. This is followed by an investigation of the results in accordance with taught knowledge based on observation and interviews, and textbook analysis to determine the knowledge to be taught.

6.3.1 Praxeological change based on the diagnostic pre- and post-test

The diagnostic test was designed with two distinct purposes in mind. Primarily, it was intended to contribute to the thorough diagnosis of knowledge learnt by DLS students at large. Secondly, it was designed to serve as an indicator of potential praxeological change when using Japanese textbook chapter in DLS, by comparing students' responses (techniques, technology and to some extent theory) before and after the experiment.

The students completed the diagnostic test immediately prior to and following the intervention period. To avoid memorization of the test results, superficial numerical changes in the tasks were employed in the post-test (cf. section 5.2.1). Table 10 presents the percentage of correct responses to the items (same types of task) in the pre- and post-test, where 56 students answered the pre-test, and 60 students answered the post-test. Only pre- and post-test results for items assessing the same type of task and associated technique are used for this study. Consequently, the total score will not be used as there is no direct comparability between alle items in the pre- and post-tests. Furthermore, the relatively modest size of the test group must be considered.

Table 10. Percentage of correct answers in pre- and post-tests according to item number.

Percentage of correct answers in pre- and post-test. 56 students completed the pre-test, and 60 students completed the post-test both groups from grade 8 at the same school. Items with a significant positive difference in percentage between pre- and post-test are highlighted using boldface numbers and grey background. Items with a significant negative difference are highlighted using only boldface numbers. Items in italic assess technological or theoretical element but these are not directly comparable between pre- and post-test because of significant variations in the content.

Item	Pre	Post	Item	Pre	Post	Item	Pre	Post	Item	Pre	Post
1.1	89	82	2.1	82	83	3.1	87	92	4.1	91	93
1.2	46	52	2.2	95	95	**3.2**	**36**	**77**	4.2	62	67
1.3	82	70	2.3	63	67	3.3	58	62	4.3	35	40
1.4	**82**	**28**	2.4	83	92	**3.4**	**44**	**65**	4.4	85	43
1.5	**35**	**47**	2.5	78	80	3.5	35	*75*	**4.5**	**7**	**52**
1.6	58	65	**2.6**	**63**	**75**	3.6	67	*48*	4.6	7	*45*
1.7	**80**	**90**	**2.7**	**60**	**70**	3.7.	64	*28*	**4.7**	**27**	**43**
1.8	91	*13*	**2.8**	**33**	**62**	3.8.	0	*21*	**4.8**	**13**	**42**
1.9	**18**	**33**	2.9	67	73				4.9	44	

Item	Pre	Post	Item	Pre	Post	Item	Pre	Post	Item	Pre	Post
5.1	**47**	**67**	**6.1**	**56**	**75**	7.1	95	87	8.1	89	87
5.2	40	47	6.2	38	35	**7.2**	**87**	**37**	8.2	16	25
5.3	85	92	6.3	35	53	7.3	38	33	8.3	29	42
5.4	35	30	**6.4**	**31**	**57**	**7.4**	**38**	**50**	**8.4**	**51**	**27**
5.5	49	50	6.5	71	68	7.5	2	42	8.5	24	*15*
5.6	4	*65*	6.6	27	35	**7.6**	**35**	**65**	8.6.a	18	*12*
5.7	5	*33*	6.7	62	47	7.7.a	4	50	8.6.b	5	
5.8	20	25	**6.8**	**25**	**42**	7.7.b	2	40			

The data presented in table 10 demonstrates that the majority of items are answered with a similar percentage of correct answers.

Table 12. Exceptional items with significant decline from the pre- to the post-tests

Item no.	Pre-test item	Post-test item
1.4	$6 + (-5) =$	$-(23 - 7) =$
8.4	Write an equation to fit the statement: I am twice as old as my son	Write an equation to fit the statement: I am twice as old as my son

The lack of correct responses to item 1.4 can be attributed to the fact that the variation was done incorrectly. The pre-and post-items do not test the same technique, due to the different uses of the additive inverse in the pre-test and post-test (table 12). The lower success rate of item 8.4 in the post-test is related to time pressure in the post-test, due to the addition of further items with theoretical content.

The items that had significantly positive change can be grouped into three main categories: types of task that employ arithmetic techniques (table 12), types of task that employ algebraic techniques (table 13) and types of task that involve algebra as a modelling tool (table 14). The following sections will describe conditions and constraints for praxeological change, with a focus on these three groups of types of task.

6.3.2 Praxeological change according to arithmetic

A praxeological change is essential to address the problematic arithmetic and algebraic techniques outlined in section 6.3. The extent to which the textbook chapter can contribute to a praxeological change will be discussed below by examining the arithmetic items in the diagnostic test that have demonstrated the most significant positive change.

Table 12. Significant positive items all of which utilize arithmetic techniques

Item no.	Item	Type of task	Techniques
5.1	Show how you calculate $17 + 29 + 132 - 52 + 52 - 29 =$	Long calculation with addition and subtraction	Addition and subtraction
2.7	$-18 : 3 =$	Division of negative integer	Division
6.1	$\_\_\_\_ : 4 = 4$	Division of an unknown	Opposite operation or Trial-and-error-substitution
7.4	$16 - 9 - 2 \cdot 3 =$	Calculation with multiplication and subtraction	Multiplication and subtraction
1.5	$7 - (-9) =$	Subtraction of negative integer	$a - (-b) = a + b$
2.6	$5 \cdot (-7) =$	Multiplication of negative integer	$a \cdot (-b) = -ab$
3.2	$k(5 + 3) =$	Calculation with bracket	$k(a + b) = ka + kb$
3.4	$(-12) + (-9) =$	Addition of two negative integers	$(-a) + (-b) = -(a + b)$
2.8	Put $a = 3$ and $b = 4$ and calculate $2a + 3b =$	Substitution of integers in an algebraic expression	Calculation by substitution

With the exception of item 2.8, all items in table 11 consist of typical arithmetic tasks that one would expect eighth-grade students to solve easily. As an example, item 5.1 has been discussed. (see section 6.3.3). It seems reasonable to posit that the significantly correct answers to items involving negative terms and "bracket rules" can be attributed to the textbook's explicit use of notation (see episode 2/2, Paper III) and theory about "bracket rules" (see episode 2/1, Paper III).

The explicit description of the use of algebraic notation is a key element in the textbook material (Paper II). Given that the textbook chapter used in the experiment pertain to algebraic expressions, it is also of interest to ascertain whether this is also reflected in the diagnostic post-test.

6.3.3 Praxeological change from arithmetic to algebra in the Japanese textbook chapter

The transition from arithmetic to algebra requires a praxeological change. The textbook chapter "algebraic expression" used for the teaching experiment starts with an opening problem where the students have to determine how many sticks are needed to make chains of squares of different lengths (se front page of this thesis). The devolution of the opening problem where the teacher hands over the responsibility of the enquiry process to the students is straightforward, because the students can draw on already established techniques. The opening problem acts as a catalyst, introducing the variable x to represent the number of squares in the chain. This sets the stage for an important shift, moving from arithmetic expressions to algebraic modelling of the patterns. The episode has a profound effect on the learnt algebra knowledge: as the students work through the problem, they realize that two algebraic models to determine the number of sticks needed for a chain of given length, give the same result. This leads the students to assume that the two models are equivalent.

Although the textbook emphasizes that algebraic expressions can both represent and solve the problem (Isoda & Tall, 2019, p.63), this idea is not explicitly emphasized be the teacher (Paper III). This is an example of a point which is not taught, but possibly still experienced by the students.

This first encounter with algebraic expressions and the further exploration of the initial problem gives rise to the emergence of techniques to solve the task, which are subsequently validated throughout the chapter through the construction of a technologico-theoretical environment. To what extent the technical moment that follows (after the above two) contributes to improvement of algebraic techniques can be answered to some extent by the diagnostic test.

6.3.4 Praxeological change in relation to algebra

The diagnostic test contains a number of items of the type "reduction with one or two variables" "Solve a first-degree equation". These types of task can also be found in the textbook chapter. Table 13 shows that the use of the algebraic techniques is significantly improved in the post-test (see Table 10 for the exact success rates).

Table 13. Items with significant positive change - all of which utilize algebraic techniques

Item no	Item	Type of task	Techniques
1.7	Reduce $7a - 3a =$	Reduction with one variable	Subtraction of terms
4.8	Reduce $15a + 8b - 2 \cdot 4b =$	Reduction with two variables	Simplify by collecting and reduce same kind of terms
6.4	Reduce $7 + 3x + 32 - 2y + 2y - 3x =$	Reduction with two variables	Simplify by collecting and reduce same kind of terms
4.7	$7x - 7 = 13 - 3x$ $x =$ _____	Solve a first-degree equation	Addition, subtraction, multiplication and division on both side of the equal sign
7.6	$x \cdot \frac{3}{4} = \frac{15}{4}$ $x =$ ____	Solve a first-degree equation	Addition or subtraction on both sides of the equal sign or Trial-and-error-substitution
4.5	Rewrite with use of powers $a \cdot a \cdot a \cdot b \cdot b =$	Multiplication of variables	Collect same variables and write the answer by using the rule of power

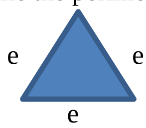
It is evident that the students have developed their algebraic techniques for reducing algebraic expressions and solving first-degree equations. This corresponds to content emphasized in the Japanese textbook.

Item 4.5, which addresses the use of exponents, merits particular attention as it is related with a specific technology-oriented activity of the book (see Isoda & Tall, 2019 p. 71). Paper III describes in more detail how this activity is deployed in one class (episode 3, Paper III). The activity involved first realizing that the exponent of a^n increases by 1 when the expression is multiplied by a. And two students are given the task of finding out what a^{-1} and a^{-2} would then reasonably be. The students seem to think that decreasing the exponent by one is equivalent to subtracting a. Instead of addressing this incorrect reasoning, the teacher lets the students use a calculator to explore the problem numerically when $a = 2$. The students infer that $2^{-2} = \frac{1}{2^2}$, where this assumption is based on the authority of the calculator and not on the algebraic reasoning modelled by the textbook (Paper III).

Notwithstanding the observation that the theoretical elements of the textbook chapter are not fully utilized into the teaching, the evidence presented in item 4.5 suggests that it continues to exert a beneficial influence at the technical and technological level.

The last group of items that were rated as significantly positive is type of task that highlight algebra as a modelling tool (table 1).

Table 14. Items with significant positive change to algebra as a modelling tool.

Item no.	Item	Type of task	Technique
1.9	Draw or write a story that fits the equation: $\mathbf{2x = 10}$	Describe a relationship that fits a linear equation	Model the relationship between the terms ax and b
6.8	Determine the perimeter of the triangle 	Determine the perimeter of a triangle where all side lengths are given	Addition of the three side lengths

Items 1.9 and 6.8 (see figure 14) exemplify the capacity to model simple intra-mathematical and extra-mathematical relationships through algebra, as well as the ability to translate algebraic equations into practical situations. Despite the relatively simple expressions, it is encouraging if the Japanese textbook chapter can contribute to the utilization of algebra as a modelling tool (cf. section 3.4) especially when you consider the national test results (cf. section 6.3.1).

The above test results and case examples indicate that there is a potential for praxeological change at student level. As we shall now see the material also contain potential for praxeological change at teacher level.

6.3.5 Mathematics teachers praxeological change

In the chapter and the textbook in general, there is an explicit distinction between laws (such as the commutative property) and conventions (such as the order of terms in expressions).

This explicit discussion of notation and techniques is not fully utilized in the observed episodes (Paper III). This may be because the teachers are not completely certain about the meaning of explicit rewriting rules (like exponentiation rules), and when a change in notation is simply due to conventions (like $2 \cdot x = 2x$). Teachers are not used to teach such explanation as they are not part of the Danish textbook (Paper II). It may also be that the teachers are, more generally, not fully aware of the importance of making technical and theoretical elements explicit. However, in the interview with the teachers, there are signs that this potential is beginning to be realized for them. The mathematics tutor says that the grade 8 mathematics teaching team has become aware of the importance of the explicit notation, and through the use of the Japanese textbook chapter they have explored topics that they may have previously overlooked, which is a sign of a praxeological change for the teachers (Paper III).

7 Discussion

The above results (section 6) are based on a comprehensive diagnosis of algebraic praxeologies currently taught in DLS and problematic algebraic techniques in particular. The discussion of these finding will be further informed by the previous research in ATD, school algebra and curriculum resources as well as relevant international studies identified through the literature review (cf. section 3). In both the diagnostic and the experiment school algebra was considered as a modelling tool, which also forms the foundation for a discussion of the particular approach to elementary algebra.

7.1 Algebra as a modelling tool

Algebra as a modelling tool to model intra- and extra-mathematical system can be grouped into three common categories: constructing the algebraic model, rewriting the algebraic model and applying the algebraic model (Chevallard, 1989). The same overall structure is currently applied to research on Danish school algebra, where a current praxeological reference model for school algebra at the transition from lower to upper secondary level consist of three local algebraic organizations: set up an algebraic model, substituting in an algebraic model and rewrite (operate on) an algebraic model (Cosan, in press). In the dominant model with lower secondary school, only substitution in an algebraic model is given substantial attention – whether in subjects concerned with algebraic models in extra-algebraic settings (like the computation of circle areas by substituting in the model $A = \pi r^2$) or in activities with equations. Students do not authonomously set up algebraic models, and they hardly work on rewriting them (Paper I). Thus, for the students there is rarely any visible connection between the algebraic expression and a system to be modelled, which is emphasized as essential to school algebra as a modelling tool by Strømskag & Chevallard (2022).

In the DEM for lower secondary schools, applying the algebra models consists mainly in substitutions of one or more variables by integers. In general, the logos blocks do not include "formal mathematical definitions and axioms" (Paper I). If we look at CO, one could easily get impression that algebra as an intra-mathematical and especially extra-mathematical modelling tool would be required or at least encourages in DLS mathematics (cf. Section 6.1.1). But the teachers do not have concrete support to realize such an interpretation, in the form of resources for their teaching and occasions to learn how to use them.

According to the research and innovation program by Strømskag & Chevallard (2022) students should start by looking at a system and a related question with mathematical elements. They should then create a model of the system tailored to the questions, using elementary algebra and including relevant parameters. As they work through the model, students should focus on finding an answer to the question. The key is that during this process students naturally encounter and explore algebraic concepts, refine their understanding and learn to use these tools efficient (Strømskag & Chevallard, 2022). This structure of a tailored opening problem, which leads to a study process the goes beyond the simple solution of a single mathematical problem and ends with a mathematical organization, is precisely what is contained in the Japanese textbook chapter (Paper II). This textbook structure where algebra is used as a modelling tool, is not generally used.

In the study of the most widely used textbooks in Spanish secondary schools, the researchers found that the content related of algebra is mainly focused on solving equations, with an "introduction to algebraic language" (Munzón et al. 2015). Algebraic techniques are learnt implicitly through reference to arithmetic and number sense, although algebraic arithmetic is governed by different syntactic rules than arithmetic. Whereas in arithmetic you often simplify each operation you do before moving to the next, in algebra you can benefit from "complexifying" the calculations and manipulations you do (Munzón, et al. 2015)

In Spanish schools the teaching and learning of elementary algebra focuses on students learn to write, factorize and simplify expressions as an end itself not as a tool for problem solving (Munzón et al. 2015). The same situation was described by Chevallard (1989) where analysis of the manipulation of algebraic expressions in France college (LSS) revealed that there was no mathematical aim beyond the training of algebraic skills and the rules of the algebraic manipulation were unmotivated and used as procedures for the sake of the use (Chevallard, 1989).

The current teaching of school algebra in DLS maintain the same limitations previously described for algebra in French and Spanish schools (Munzón et al., 2015).
Formal learning cannot replicate all the manipulations students need when using algebra as a modelling tool. This results in a vocabulary focused on specific operations (e.g., calculate, simplify, develop, factorize), which not only fails to teach fundaments algebraic skills but also prevents students from learning how to choose the most appropriate transformation for a given problem (Munzón et al. 2015).

On the one hand, the competence-based and spiral curriculum give opportunity for praxeological change over years in study process where the moment of the first encounter with T, the moment of exploration of and emergence of a technique τ can be part of one grade level and the theory block can be part of later grade level. It requires that the study process with the moment to build the technological and theoretical block are based on the previous study process and praxeologies. In the selected algebra lessons, students discuss the difference between the words "reduce" and "solve" (cf. section 6.2). It reflects the connection between the type of task and the corresponding technique to solve the task, where it is crucial for the students to define the type of task to apply the corresponding technique.

7.2 The use of foreign textbook material

The article "A comparative study of didactic moments in a first chapter on algebra in Danish and Japanese middle school textbooks" (Paper II) highlights the broad opening-problem as a catalyst for creating a general algebraic model of the system tailored to the questions posed by the textbook. A process that invites the students to naturally encounter and explore algebraic concepts by including relevant theoretical elements, which contributes to refine their understanding and learn to use these tools efficient. These conditions fulfil the requirements for a study process that contributes to e more "authentic" study and use of algebra in schools, as described by Strømskag & Chevallard (2022).

If we compare the conditions and constraints of our project using Japanese textbook chapters with the experience of American Schools adopting Singaporean mathematics textbooks, there are differences to consider. While the Schools in U.S. used the entire textbook system, in this project we used only a selected chapter. While the use of the Japanese textbook chapter was intended to help in the transition from arithmetic to algebra and fill theoretical "gaps" in the Danish system, one of the challenges with the Singapore textbook system was that it did not fulfil the official goals for teaching.

By selecting specific chapters from a textbook that follows a curriculum with a clear, coherent progression, it is possible to adapt the material into a spiral curriculum structure, where topics are revisited with increasing depth and breadth. On the other hand, focusing on a single chapter in isolation misses the benefits of this progression, which involves building on prior knowledge. Or even worse, the use of the isolated chapter becomes difficult or impossible because of students insufficiently solid or extensive prior knowledge.

The experiment with the use of Japanese textbooks material took places in DLS that could be described as average in terms of number of students in the classes, students' well-being amount and the percentage of students who went on to secondary school, but "below the national average" in terms of the compulsory mathematics test (Paper I). In the experiment the use of Japanese materials did not require any special institutional conditions, in contrast to the US schools, where it was concluded that replication of the Singapore successes would require major reforms of the US school mathematical system (Ginsburg et al. 2005). Conversely, through teacher interviews, we gain insight into the teachers' discussions about the mathematical and didactic content of the material, which has been a requisite component of the teachers' preparation for teaching with the textbook chapter in the experiment. It would be beneficial to pursue these institutional conditions (and constraints) in future research projects investigating the utilization og Japanese textbook material in DLS.

9 Conclusion

The aim of this study was to explore the current algebraic praxeologies taught in Danish lower secondary schools, focusing on the extent to which the teaching provides a systematic progression at the level of theory in the transition from arithmetic to algebra. After diagnosing what algebraic techniques are particularly problematic for Danish students at lower secondary level, we have selected textbook material for a teaching experiment, to overcome these challenges and investigate the conditions and constrains when using research-based textbook material in the transition from arithmetic to algebra.

9.1 The status of school algebra in Danish lower secondary school

A fundamental viewpoint from the outset of this project has been to consider external and internal didactic transpositions of algebra as intimately connected, with the textbook being an important interface. The textbook is crafted in and by the external didactic transposition, but it is also, even materially, present in the classroom.

The teaching of school algebra in DLS is based on the premise that algebra should be presented as a seamless continuation (if not almost part of) arithmetic. In the textbooks, algebraic symbolism is introduced relatively early, also in contexts where it is not strictly necessary (cf. Paper II). Algebraic expressions appear in ready-made models ("formulae", e.g. for the circumference of a circle in terms of its diameter) and in equations where the solution is to be found through trial-and-error (substitution of small natural numbers). In terms of the levels of algebraization this corresponds to the first level where calculation programs are presented algebraically and used (but not constructed, questioned or modified). Another specificity is that the numbers used in these calculations' programs are almost invariably integers (and mostly, positive integers). These restrictions are not mandated by the official directives (CO) but are implicitly present in the equally official final written exam, where they are followed. The CO do not formulate specific requirements in terms of algebraic techniques or theoretical notions (like commutative law, inverse operation etc.), and indeed coherent and precise algebraic discourse is rare or absent both in the textbooks and in observed teaching. Technical work, for instance with solving equations or handling parentheses, then come to rely on informal language and rules. At the final exam, relative few (often less than 50%) students succeed with algebraic task of the limited type described above. In the diagnostic test, a wider area of arithmetic and algebraic tasks was proposed, also in order to gain closer insights into the techniques and logos mastered by the students. A main outcome was that some of the students cannot even access the first level of algebraization because of difficulties with arithmetical praxeologies, like viewing longer computations structurally and not as step-by-step instructions. These difficulties are by no means universal: some students can solve items in the diagnostic test which go beyond the official requirements as implicitly formulated through the final written exams. Exceptional items like the one discussed in the introduction to Paper I also show that almost nobody (3%) get to master relatively basic rewriting rules for algebraic expressions.

9.2 The method and meaning of the diagnosis

The results outlined in previous section relies on a variety of data sources: the official program, textbooks, classroom observations, and results from a test based on an elaborate REM. The connected analysis of these is made possible by the explicit praxeological reference model and by the theoretical situation of the various data in relation to the didactic transposition. The outcome is both to exhibit the (normally implicit) DEM of algebra in the institution (DLS) and its consequences in terms of the teaching and learning that is realized within the institution. The systemic viewpoint is necessary to identify relations and effects among the many elements which could otherwise be seen as independent factors or symptoms of “the algebra problem” – which, in a vague sense, is “felt” by neighboring institutions. A diagnosis of the kind realized in Paper I does not aim to confirm such vague impression but to collect, analyze and combine evidence that can provide a sharper picture than the institutional beliefs. The diagnosis can also serve as the foundation and context of interventions like our experiment with curriculum import.

9.3 The potential of curriculum import

A positive condition for our experiment was the fact that at the school where the Japanese text on initial algebra was experimented, the school had adopted a single textbook system for lower secondary mathematics; the teachers habitually combined resources (text, exercises) from different sources, including the internet. The mathematics teachers also shared such materials among them and frequently planned parallel teaching together. We do not know how common these conditions are in DLS. It certainly made it easier to have a whole team of (five) mathematics teachers volunteer to work with the Japanese material for about a month, and to do so collaboratively and simultaneously in all three grade 8 classes of the school. These positive conditions may not be generally available, and we do not know to what extent they were instrumental for the outcomes observed. It made us have more direct access to their didactical rationales and methods. On the other hand, the school was not in any way a privileged context in terms of general parameters like grades, social context etc.

In summary, the teachers used the inductive approach of the chapter, with the challenging launch question supporting many points along the way, beginning with the students' work to model what appears initially an arithmetic problem.

The teachers also appreciated and effectively used the parts of the text that discuss new notational conventions in algebra (e.g. $2 \cdot x = 2x$ but NOT $x \cdot 2 = x2$) explicitly, rather than leaving them to be assimilated tacitly.

There are other elements which were realized less successfully. Generally speaking, they relate to theoretical aspects of algebra which are not habitually taught in DLS, like

- situations where students should discover or derive a theoretical point from a pattern of examples, like when students work on the definition $x^{-n} = 1/x^n$ for positive integers n;
- establishing clear connections between abstract theoretical laws (like the distributive) and examples as well with previous knowledge from arithmetic.
- make full use of the structure of the chapter so that the cumulative progression of the material becomes explicit in the teaching.

As one could expect we identified both potentials and obstacles in the experiment. On the side of the teachers, the main potential was probably that they explicitly say they learnt new ideas about teaching and algebra from using the material. On the student side, the diagnostic test suggest that they advanced even in areas (like positive integer exponents) where the teaching did not follow Japanese didactic principles for managing students' problem solving. We cannot, however, in any way claim that their outcome from the experimental teaching was *better* than it would have been otherwise; so, results are more about the phenomena, conditions and constraints to expect.

10 References

Artigue, M., & Winsløw, C. (2010). International comparative studies on mathematics education: A viewpoint from the anthropological theory of didactics. *Recherches en didactique des mathématiques*, 30(1), 47-82.

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice: The case of limits of functions in Spanish high schools. *Educational Studies in Mathematics*, 59(1–3), 235–268. https://doi.org/10.1007/s10649-005-5889-z.

Bjerril, S. (2024). On in five Danish students is too bad at mathematics to become an "engaged citizen". *Folkeskolen*. Retrieved from https://www.folkeskolen.dk/finland-matematik-norge/pisa-hver-femte-danske-elev-regner-for-darligt-til-at-blive-en-engageret-borger/4751618

Bolea, P., Bosch, M., & Gascón, J. (1998). The role of algebraization in the study of a mathematical organization. In I. Schwank (Ed.), *Proceedings of the First Conference of the European Society for Research in Mathematics Education* (Vol. II, pp. 135–145). CERME.

Bolea, P., Bosch, M., & Gascón, J. (2001). La transposición didáctica de organizaciones matemáticas en proceso de algebrización: el caso de la proporcionalidad. [The didactic transposition of mathematical organisations in the process of algebraization: the case of proportionality]. *Recherches en didactique des mathématiques,* 21(3), 247-304.

Bolea, P., Bosch, M., & Gascón, J. (2004). Why is modelling not included in the teaching of algebra at secondary school. *Quaderni di Ricerca in Didattica*, 14, 125-133.

Bosch, M. (2015). Doing research within the anthropological theory of the didactic: The case of school algebra. In S. J. Cho (Ed.), *Selected regular lectures from the 12th International Congress on Mathematical Education* (pp. 51–69). Switzerland: Springer International Publishing.

Bosch, M., & Cascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI bulletin*, 58(58), 51-65.

Bosch, M., Chevallard, Y., Garcia, F. J., & Monaghan, J. (2020)(Eds.). *Working with the anthropological theory of the didactic in mathematics education. Routledge*. https://doi.org/10.4324/9780429198168

Brousseau, G. (1990). ¿ Qué pueden aportar a los enseñantes los diferentes enfoques de la didáctica de las matemáticas?(Primera parte). [What can different approaches to the didactics of mathematics bring to teachers]. *Enseñanza de las ciencias. Revista de investigación y experiencias didácticas*, 8(3), 259-267.

Chevallard, Y. (1991). La Transposition didactique. Du Savoir savant Au savoir enseigné. [Didactic transposition. From learnt knowledge to taught knowledge]. *La pensée Sauvage*.

Chevallard, Y. (1989). Le passage de l'arithmétique a l'algébrique dans l'enseignement des mathématiques au collège – Deuxième partie. *Perspectives curriculaires: la notion de modélisation* [The transition between arithmetic and algebra in the teaching of mathematics at secondary school level – Second part. Curricula approaches: The notion of modelling]. *Petitx*, 19, 45–75.

Chevallard Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en didactique des mathématiques*, 19(2), 221-226.

Chevallard, Y. (2002). Organiser l'étude : 3. Ecologie & régulation. In *Actes de la XIe école d'été de didactique des mathématiques* (pp. 41-56). La Pensée Sauvage.

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic an attempt at a principled approach. *Hiroshima journal of mathematics education, 12*, 71-114.

Chevallard, Y., & Bosch, M. (2014). Didactic transposition in mathematics education. In Lerman, S. (eds) *Encyclopedia of Mathematics Education*. Springer. https://doi.org/10.1007/978-94-007-4978-8_48

Chevallard, Y. & Bosch, M. (2020a). Anthropological Theory of the Didactic (ATD). In Lerman, S. (eds) *Encyclopedia of Mathematics Education*. Springer. https://doi.org/10.1007/978-3-030-15789-0_100034

Chevallard, Y., & Bosch, M. (2020). Introduction. A short (and somewhat subjective) glossary of the ATD. In Bosch, M., Chevallard, Y., Garcia, F. J., & Monaghan, J. (Eds), *Working with the anthropological theory of the didactic in mathematics education* (pp. 1-4). *Routledge*. https://doi.org/10.4324/9780429198168

Christensen, V. T. (2019). *PISA 2018-Danish youth in an international comparison*. VIVE, The Danish National Centre for Social Science Research and Analysis. Retrieved from https://pure.vive.dk/ws/files/3875344/211100_PISA_2018_A_summary_of_the_Danish_Results_TG.pdf

Cosan, D. D. (2021). *A diagnostic test for Danish middle school arithmetic*. [MSc dissertation, Department of Science Education, University of Copenhagen]. Retrieved from https://www.ind.ku.dk/publikationer/studenterserien/studenterserien-publikationer/a-diagnostic-test-for-danish-middle-school-arithmetics/

Cosan, D. D. (in press). Praxeological differences in institutional transition: The case of school algebra. *Annales de Didactique et des Science Cognitives*.

Ding, M. (2016). Opportunities to learn: Inverse relations in U.S. and Chinese textbooks. *Mathematical Thinking and Learning*, 18(1), 45–68. https://doi.org/10.1080/10986065.2016.1107819

Ding, M., & Li, X. (2010). A comparative analysis of the distributive property in U.S. and Chinese elementary mathematics textbooks. *Cognition and Instruction*, 28(2), 146–180. https://doi.org/10.1080/07370001003638553

Danish Ministry of Education (2019). Common objectives: Mathematics. [Fælles mål: Matematik] National Agency for Education and Quality. Retrieved from https://emu.dk/sites/default/files/2020-09/GSK_F%C3%A6llesM%C3%A5l_Matematik.pdf

Danish Ministry of Education (2022). *Joint development of Mathematics. By expert group for mathematics*. Retrieved from https://www.uvm.dk/-/media/filer/uvm/aktuelt/pdf22/okt/221011-ekspertgruppens-rapport-om-faelles-udvikling-af-matematik.pdf

Danish Ministry of Education (2024). *Test collection*. Ministry of Children and Education. Retrieved from https://www.xn--prvebanken-1cb.dk/proevematerialer/FAP/MAT/MATFP9

Emerson, R. M., Fretz, R. I., & Shaw L. L. (2001). Participant observation and fieldnotes. In P. Atkinson (Ed.), *Handbook of Ethnography* (pp. 352-368). Sage.

Eriksson, H. (2022). Teaching algebraic thinking within early algebra—A literature review. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)* (pp. 481–488). CERME.

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: Development status and directions. *ZDM-Mathematics Education*, 45(5), 633-646.

Garcia, F. J., Pérez, J. G., Higueras, L. R. & Bosch, M. C. (2006). Mathematical modelling as a tool for the connection of school mathematics. *Zentralblatt für Didaktik der Mathematik* 38, 226-246. https://doi.org/10.1007/BF02652807

Gascón, J. (2024). Contributions of the anthropological theory of the didactic to the epistemological programme of research in mathematics education. *ZDM Mathematics Education* 56, 1319-1330 https://doi.org/10.1007/s11858-024-01563-1

Gascón, J., & Nicolás, P. (2024). On the theory of praxeologies. In I. Florensa, N. Ruiz-Munzón, K. Markulin, B. Barquero, M. Bosch, & Y. Chevallard (Eds.) *Extended Abstracts 2022. CITAD 2022. Trends in mathematics* (vol. 16, pp. 193-196) Birkhâuser. https://doi.org/10.1007/978-3-031-55939-6_13

Ginsburg, A., Leinwand, S., Anstrom, T., & Pollock, E. (2005). *What the United States can learn from Singapore's world-class mathematics system (and what Singapore can learn from the United States): An exploratory study.* American Institutes for Research. Retrieved from https://files.eric.ed.gov/fulltext/ED491632.pdf

Gonczi, A. (2006). The OECD: Its role in the key competencies debate and in the promotion of lifelong learning. In P. Hager & S. Holland (Eds.), *Graduate attributes, learning and employability* (pp. 105-124). Springer.

Grønmo, L. S. (2018). The role of algebra in school mathematics. In *Invited Lectures form the 13th International Congress on Mathematical Education* (pp. 175-193). Springer.

Harden, R. M. & Stamper, N. (1999). What is a spiral curriculum? *Medical teacher*, 21(2), 141-143. https://doi.org/10.1080/01421599979752

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational studies in mathematics*, 27(1), 59-78.

Hirt, J., Nordhausen, T., Appenzeller-Herzog, C., & Ewald, H. (2023). Citation tracking for systematic literature searching: A scoping review. *Research Synthesis Methods*, 14(8), 563-579.

Hodgen, J., Oldenburg, R., & Strømskag, H. (2018). Algebraic thinking. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger, & K. Ruthven (Eds.), *Developing research in mathematics education: Twenty years of communication, cooperation and collaboration in Europe* (pp. 32–45). Routledge.

Howson, G. (2013). The development of mathematics textbooks: historical reflections from a personal perspective. *ZDM Mathematics Education 45*, 647–658. https://doi.org/10.1007/s11858-013-0511-9

Isoda, M., & Tall, D. (2019). *Junior High School Mathematics: 1*. Gakko Tosho.

Jensen, K. K., Andersen, M. M., Whiteley, L., & Sandøe, P. (Eds.) (2020). *RCR – A Danish textbook for courses in Responsible Conduct of Research.* (4th ed.) Department of Food and Resource Economics, University of Copenhagen.

Johnsen, L. B., Hansen, N. J., Thomasen, H. & Lindhardt B. (2015). *KonteXt* 7. Alinea.

Johnsen, L. B. Christensen, M. S. Hansen, N. H. & Lindhardt, B. (2009). *KonteXt+ 9*. Alinea.

Jupri, A., & Drijvers, P. (2016). Student difficulties in mathematizing word problems in algebra. *Eurasia Journal of Mathematics, Science and Technology Education*, *12*(9), 2481-2502.

Kaas, T. (2019). Approaches to early algebra. [Tilgange til tidlig algebra]. *Nordic Studies in Mathematics Education*, *24*(3-4), 15-41.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J.J. Kaput, D. W. Carraher & M. L. Blaton (Eds.), *Algebra in the early grades* (pp.5-19). Lawrence Erlbaum Associates.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational studies in Mathematics*, 12, 317-326.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde, & A. Pérez (Eds.), *Eighth International Congress on Mathematical Education: Selected lectures* (pp. 271–290). S.A.E.M. Thales.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator* (Singapore), 8(1), 139-151.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). NCTM.

Kieran, C. (2022). The multi-dimensionality of early algebraic thinking: background, overarching dimensions, and new directions. *ZDM-Mathematics Education, 54*(6), 1131-1150.

Kjeldsen, C.C., Kristensen, R.M., Bindslev, S.G., & Christensen, A.A. (2021). *Mathematics and science/technology in 4th grade. Summary of TIMSS survey*. Aarhus University Press. Retrieved from https://unipress.dk/media/21013/matemarik-og-natur-teknologi-i-4-klasse_timss-rapporten-2023.pdf

Krammer, H. P. (1985). The textbook as classroom context variable. Teaching and teacher Education, 1(4), 273-278. Retrieved from https://www.sciencedirect.com/science/article/abs/pii/0742051X85900150?via%3Dihub

Lucas, C., Fonseca, C., Gascón, J., & Schneider, M. (2019). The phenomenotechnical potential of reference epistemological models: The case of elementary differential calculus. In M. Bosch, Y. Chevallard, F. J. García, & J. Monaghan (Eds.), *Working with the Anthropological Theory of the Didactic in Mathematics Education* (pp. 77–98). Routledge. https://doi.org/10.4324/9780429198168

Ministry of Education Culture, Sports, Science and Technology (MEXT).(2023). *Chugakko Gakushu Shido Yoryo Kaisetsu Sansu-Hen*. Retrieved from https://www.mext.go.jp/en/index.htm

Mullis, V. S., Martin, M. O., Foy, P., Kelly, D. L. & Fishbein, B. (2019). *TIMSS 2019 International Results in Mathematics and Science.* International Association for the Evaluation of Educational Achievement (IEA). Retrieved from TIMSS 2019 International Reports – TIMSS & PIRLS International Study Center at Boston College

Munzón, N. R., Bosch, M., & Gascón, J. (2015). El problema didáctico del álgebra elemental: Un análisis macro-ecológico desde la teoría antropológica de lo didáctico. *Journal of Research in Mathematics Education, 4*(2), 106-131.

Nakayasu, C. (2016). School curriculum in Japan. The Curriculum Journal, 27(1), 134–150. https://doi.org/10.1080/09585176.2016.1144518

Niss, M., & Jensen, T. H. (2011). Competences and learning of mathematics. Ideas and inspiration for developing mathematics teaching in Denmark. Roskilde Universitet. IMFUFA-tekst : I, om og med matematik og fysik No. 485. Retrieved from https://rucforsk.ruc.dk/ws/portalfiles/portal/35932281/IMFUFA_485.pdf

Nissen, S. K., Schmidt, M. C. S., & Tonnesen, P. B. (2016). Photo elicitation interviews as a way of accessing primary school students' attitudes. *Proceedings of the 13th International Congress on Mathematical Education, University* of Hamburg, Hamburg, Germany.

Paulsen, K. P., Nielsen, S. B., Damgaard, M. A., Birkving, K., Tønnesen, M. G., Jensen, A. S., Weine, M., Bunde, H. & Mandrup, T. (2021). School leaders: Teacher shortage is one of our biggest societal problems, but we already have solutions that work. *Altinget.* Retrieved from https://www.altinget.dk/uddannelse/artikel/skoleledere-laerermangel-er-et-af-vores-stoerste-samfundsproblemer-men-vi-har-allerede-loesninger-som-virker

Pepin, B., & Gueudet, G. (2018). Curriculum resources and textbooks in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 172–176). Springer. https://doi.org/10.1007/978-3-030-15789-0_40

Petit x (2024). Institut de recherche sur l'enseignement des mathématique de Grenoble. Retrieved from https://irem.univ-grenoble-alpes.fr/revues/petit-x/petit-x-365064.kjsp

Poulsen, C. S. (2015). *Basic algebra in the transition from lower secondary school to high school.* [MSc dissertations. Department of Science Education, University of Copenhagen. Retrieved from https://www.ind.ku.dk/publikationer/studenterserien/studenterserien-publikationer/studenterserie44/

Poulsen, M., Hessing, S., Andersen, M.W., Lindhardt, B. & Dalsgaard, R.S. (2019). *KonteXt+ 6*. Alinea.

Putra, Z. H. (2019). Praxeological change and the density of rational numbers: The case of pre-service teachers in Denmark and Indonesia. *Eurasia Journal of Mathematics, Science and Technology Education, 15*(5). https://doi.org/10.29333/ejmste/105867

Agency of Education and Quality. (2022). *Subject booklet – Common objectives, curriculum and guidance.* Retrieved from https://emu.dk/sites/default/files/2020-09/GSK_F%C3%A6llesM%C3%A5l_Matematik.pdf

Radford, L. (2018). The emergence of symbolic algebraic thinking in primary school. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 3–25). Springer.

Ravn, K. (2023). Danish students have never performed worse in mathematics. *Folkeskolen*. Retrieved from https://www.folkeskolen.dk/dansk-internationale-undersogelser-laesning/pisa-aldrig-har-danske-elever-klaret-sig-darligere-i-matematik/4747194

Remillard, J. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, 75(2), 211-246.

Remillard, J. T., (2012). Modes of engagement: Understanding teachers' transactions with mathematics curriculum resources. In G. Gueudet, B. Pepin, & L. Trouche (Eds.). *From text to 'Lived' resources: Mathematics curriculum materials and teacher development* (Vol. 7). Springer Science & Business Media.

Remillard, J. T. (2018). Examining teachers' interactions with curriculum resource to uncover pedagogical design capacity. *Research on mathematics textbooks and teachers' resources: Advances and issues*, 69-88. Springer.

Reys, B. J., Reys, R. E., & Chávez (2004). Why mathematics textbooks matter. *Educational Leadership Feb. 2004*. Retrieved from https://www.researchgate.net/profile/Oscar-Chavez-3/publication/234571079_Why_Mathematics_Textbooks_Matter/links/571a3d6108ae7f552a472c80/Why-Mathematics-Textbooks-Matter.pdf

Rezat, S., (2012). Interactions of teachers and students use of mathematics textbooks. In G. Gueudet, B. Pepin, & L. Trouche, (Eds.). *From text to 'Lived' resources: Mathematics curriculum materials and teacher development* (Vol. 7). Springer Science & Business Media.

Rezat, S., Fan, L., & Pepin, B. (2021). Mathematics textbooks and curriculum resources as instruments for change. *ZDM–Mathematics Education*, *53*(6), 1189-1206.

Ritzau (2020*)*. 2000 University students fail basic mathematics. First-year students at Copenhagen Business School fail math screening one week after starting their studies. *Ritzau*. Retrieved from https://nyheder.tv2.dk/samfund/2020-01-22-2000-universitetsstuderende-dumper-i-basal-matematik

Ruiz-Munzón, N., Bosch, M., & Gascón, J. (2013). Comparing approaches through a reference epistemological model: the case of school algebra. In B. Ubuz, Ç. Haser & M. A. Mariotti (Eds), *Proceedings of the 8th Conference of the European Society for Research in Mathematics Education* (pp. 2870–2979). Middle East Technical University. Retrieved from http://cerme8.metu.edu.tr/wgpapers/WG16/WG16_Bosch.pdf

Ruiz-Munzón, N., Bosch, M., & Gascón, J. (2020). Un modèle épistémologique de référence pour la recherche sur l'algèbre élémentaire. [An epistemological reference model for research into elementary algebra] *Nouveaux cahiers de la recherche en éducation*, *22*(1), 123-144.

Sayers, J., Petersson, J., Rosenqvist, E., & Andrews, P. (2019). Opportunities to learn foundational number sense in three Swedish year one textbooks: implications for the importation of overseas-authored materials. *International Journal of Mathematical Education in Science and Technology*, 52(4), 506–526. https://doi.org/10.1080/0020739X.2019.1688406

Ministry of Higher Education and Science (2014). *Danish code of conduct for research integrity*. Ministry of Higher Education and Science. Retrieved from https://ufm.dk/en/publications/2014/the-danish-code-of-conduct-for-research-integrity

Soh, C. K. (2008). An overview of mathematics education in Singapore. In Z. Usiskin, & E. Willmore (Eds.), *Mathematics Curriculum in Pacific Rim Countries – China, Japan, Korea, and Singapore* (pp. 23-36). Information Age Publishing.

Straesser, R. (2011). Foreword. In G. Gueudet, B. Pepin, B., & L. Trouche (Eds.), *From text to 'Lived' resources: Mathematics curriculum materials and teacher development* (Vol. 7). Springer Science & Business Media.

Strømskag, H., & Chevallard, Y. (2022). Elementary algebra as a modelling tool. A plea for a new curriculum. *Recherches en Didactique des Mathématique*, 42(3), 384.

Takahashi, A. (2021). *Teaching mathematics through problem-solving. A pedagogical approach from Japan*. Routledge.

Tesfamicael, S.A., & Lundeby, Ø. A. (2019). A comparative study of Norwegian and Ethiopian textbooks: the case of relations and functions using anthropological theory of didactics (ATD). *Universal Journal of Educational Research* 7(3), 754-765. http://doi.org/10.13189/ujer.2019.070315

Tofte, L. R. (2023). *Danish 15-yearold have become "significantly" worse at reading and mathematics: Caurse for concern*. DR. Retrieved from https://www.dr.dk/nyheder/politik/danske-15-aarige-er-blevet-vaesentligt-daarligere-til-laese-og-regne

Tonnesen, P. B. (2024). A comparative study of didactic moments in a first chapter on algebra in Danish and Japanese middle school textbooks. *Hiroshima Journal of Mathematics Education 17*, 1-24.

Thomsen, H., Lindhardt, B., Hansen, N. J., Johnsen, L. B. & Christensen M. S. (2016). *KonteXt+ 8*. Alinea.

Trouche, L., Gueudet, G., & Pepin, B. (2020). Documentational approach to didactics. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 237-247). Springer International Publishing. https://doi.org/10.1007/978-3-319-77487-9_100011-1

Winsløw (2006). *Didactic elements. An introduction to the didactics of mathematics and science*. Forlaget Biofolia.

Winsløw, C., Matheron, Y., & Mercier, A. (2013). Study and research courses as an epistemological model for didactics. *Educational Studies in Mathematics*, *83*(2), 267–284. https://doi.org/10.1007/s10649-012-9453-3

Yoshikawa, S. (2008). Education ministry perspectives on mathematics curriculum in Japan. In Z. Usiskin & E. Willmore (Eds.), *Mathematics Curriculum in Pacific Rim Countries – China, Japan, Korea, and Singapore* (pp. 9-22). Information Age Publishing.

Paper I

Diagnosing the state of lower secondary algebra

DIAGNOSTIC DU STATUT DE L'ALGÉBRE AU NIVEAU DU COLLÈGE

Résumé – Étant donné l'importance de l'algèbre scolaire pour l'éducation post-secondaire, il paraît naturel de se demander comment les outils de la didactique peuvent servir à l'évaluation des pratiques et des résultats liés à l'algèbre dans une institution scolaire donnée, et des relations entre ceux-ci et les objectifs officiels? Dans cet article nous proposons une approche à effectuer un tel diagnostic à la base de la théorie anthropologique du didactique, en cernant surtout la distance qui sépare les modèles dominants et épistémologiques de l'algèbre scolaire. Pour démontrer cette approche nous examinons le cas de l'algèbre au niveau du collège au Danemark. Nos résultats sont lié au phénomène du « dis-algébrisation » des mathématiques scolaires.

Mots-Clés : en français sans majuscules séparés par des virgules.

DIAGNOSING THE STATE OF LOWER SECONDARY ALGEBRA

Abstract – Given the importance of school algebra for later education, it is natural to ask what tools didactics offers to evaluate the practices and results related to algebra in a given lower secondary school institution, and the relations between the official goals and praxis. In this paper, we outline an approach to set up such a diagnosis, based on the anthropological theory of the didactic, and focusing on the distance between dominant and epistemological models of school algebra. To demonstrate this approach, we use the Danish lower secondary school as a case. The key findings are related to the phenomenon of "dis-algebraization" of school mathematics.

Key words: School algebra, curriculum, the anthropological theory of didactics, didactic transposition, reference epistemological model.

Introduction

It is a longstanding theme of research into algebra teaching in Western school systems that current practice involves a narrow focus on the training of isolated techniques related to notation and formulas, while neglecting to present algebra as a modelling tool (Herscovics & Linchevski, 1994). Yet, modelling problems with algebra and evaluating and recognising algebraic models is fundamental to algebra learning (Jupri & Drijvers, 2016). School algebra and algebra as a modelling tool has been central to the work of Chevallard (1985, 1989). Subsequently Bolea et al. (2001, 2004) and Ruiz-Munzón et al. (2013, 2020) describes school algebra and algebra as a modelling tool in different school contexts, based in the anthropological theory of didactic. Kieran (2007) provides a historical overview of the evolution of school algebra. And recently, Strømskag and Chevallard based on historical examples emerge with "A plea for a new curriculum" (Strømskag & Chevallard, 2022), where algebra is a modelling tool.

Analysing school algebraic practices and curricula is a complex endeavour. Grugeon (1997) addresses this complexity by offering a multidimensional model to analyse algebraic technique and theory in the transition between vocational and general high schools, in France. By operationalising the model, Grugeon analyses the institutional relationship of students to algebra and how the discrepancies between the institutions affect the students. It is known that lower secondary school algebra can be considered a main interface between primary school arithmetic and higher mathematics (in particular, calculus) as encountered in upper secondary school and higher education (Loveless, 2013).

This is also the case in Denmark where students' failure with basic algebra as taught in lower secondary school is identified as a central cause for widespread and increasing failures in mathematics at the high stakes exams in high school (Grønbæk et al., 2017); however, as we shall see, the discrepancies are more acute than in the classical cases, as our analysis will show that large parts of basic algebra are hardly taught. This motivates our use of Danish lower secondary school as a case. In line with Grugeon (1997), the intention of this article is to provide an analysis model with instruments to achieve a coherent analysis of the status of algebra in a given contemporary institution, where the goals related to algebra are very modest. In this way, the article contributes to diagnose the state of lower secondary algebra by applying the same theoretical foundations as Chevallard et al., but in a context which is different from that of the 1990s.

Denmark as case

Due to the focus on technical skills, the problem with basic algebra is often diagnosed in terms of students' shortcomings with computations. However, these also have a theoretical dimension, as the following example illustrates. We consider a question from the 2017 national written exam after Grade 9 (15–16-year-old students). The question is the last in a series of questions, in which K_n designates the number of squares in a certain number pattern, which most of the exercise is concerned with exploration of number patterns. However, the last question does not refer to this context and states: "You must show that the formula $K_n = r(r - 1 + n)$ can be rewritten as $K_n = r(n - 1) + r^2$".

Here, the result is given – what is required is to deliver a justification. Only 3% of the 55,260 students sitting this national exam got full marks for this question – and that only required students to provide 'correct transformations with at least one intermediate result', such as $r^2 + r(-1 + n)$. In this case, what is really required is a simple use of the distributive law, the meaning of exponents, and the commutative law for addition, but full points did not require explicit reference to such theoretical ideas. While (Chevallard, 1985, 1989), thirty years ago, described a practice where the rules for manipulating algebraic expressions are unmotivated and rewriting becomes an end in itself rather than a means to solve the problem, the requirements are certainly more modest. Since then, Bolea et al. (2001, 2004), Bosch (2015), Grugeon (1997), Ruiz-Munzón et al. (2013, 2020), Strømskag and Chevallard (2022) have continued research on the didactic problem of school algebra, all based on the anthropological theory of didactics (ATD), focusing on ideas to give it meaning.

In this case, we also consider school algebra as a modelling tool (Strømskag & Chevallard, 2022) – both as a tool that models intra-mathematical praxeologies such as calculation patterns, and as a tool to study of systems in other disciplines, such as physics and biology (Bolea et al., 2001).

The example above indicates that the didactic challenges with school algebra that Chevallard addressed in the eighties are even more acute today, at least in the case of Danish lower secondary school. However, how can we go beyond isolated examples to provide a comprehensive diagnosis of the state of students' algebraic competences, based on newer tools of ATD, such as praxeological organisations? This is the focus of the present paper.

Theoretical framework

The anthropological theory of the didactic (ATD) has emerged as a theory of mathematics education (Chevallard, 2019). Within research based on ATD, school algebra has been the object of many studies, leading to significant new insights into the algebra problem (Ruiz-Munzón et al., 2013). A central feature is the use of praxeologies to model school mathematical activity, described in more detail below. To describe how knowledge is disseminated and developed in different institutions, ATD continues to use the more classical notion of didactic transposition (Chevallard & Bosch, 2014).

The didactic transposition

The process of didactic transposition is, in the present form of the theory, a way of describing the transformation of praxeologies (described below) from being produced as scholarly praxeologies in society (often historically), to being selected and designed to be taught by persons from the noosphere, 'the sphere of those who "think" about teaching' (Chevallard & Bosch, 2014). The 'knowledge to be taught' is subsequently transformed into taught praxeologies at schools (teaching institutions) and finally becomes learned praxeologies. This transposition works with the deconstruction and reconstruction of knowledge, influencing each other and the institutions they take place in.

The Danish school curriculum is formulated and published by the Danish Ministry of Education. The praxeologies undergo a transposition from one institutional setting to another, for example, the transposition from curriculum to the national written examination after lower secondary school. Textbook authors adapt both into text and exercises to be worked on by teachers and students. In the transposition process, the praxeologies will be adopted and changed according to various needs and values of various agents (Chaachoua et al., 2019). This part of the transposition is called external didactic transposition (Bosch et al., 2021).

The internal didactic transposition is the transposition of knowledge to be taught to taught knowledge and the transposition of taught knowledge to learned knowledge (Chevallard & Bosch, 2014). As a crucial step of internal didactic transposition, we can mention teachers' use of textbook material in actual teaching, with related conditions and constraints. Another step is from students' participation in a lesson to the learning results that can be observed in students' actions in other contexts, such as an exam or test, with the learned knowledge being to some extent detectable from their performance at the final exam.

The study of the external and internal didactic transposition processes of Danish school algebra is used for modelling the praxeologies in an epistemological reference model.

Praxeologies

In ATD, human knowledge and practice are modelled by praxeologies (Chevallard, 2019). A praxeology consists of types of tasks, techniques, technology, and theory (Bosch, 2015). The 'practical block' or *praxis* is formed by the *type of tasks*, indicated as T, and the corresponding *technique*, indicated as τ used to solve T (Barbé et al., 2005).

The 'theoretical block' or *logos* consists of *technology*, indicated as θ, (discourse on the techniques, such as how they work and what tasks they can solve), and *theory*, indicated as Θ (general discourse that unifies and justifies technologies, both formally and informally). This means the techniques for performing tasks are explained and justified by a 'discourse on the technique' called technology; taking this discourse to a more abstract level yields mathematical theory, to validate the technological discourse and to connect entire praxeologies (Bosch, 2015). The anthropological approach assumes that any task, or the resolution of any problem, relies on the use of techniques, even though the techniques are hidden or difficult to describe (Barbé et al., 2005).

It is convenient to understand a mathematical praxeology as a type of mathematical organisation (MO), where a point MO contains only one type of task T and a corresponding technique τ (Bosch & Gascón, 2006). When a set of punctual MOs is explained by using the same technological discourse, they form a local mathematical organisation (LMO), characterised by this technology. Likewise, LMOs sharing the same theoretical discourse form regional mathematical organisations (RMOs).

Punctual MOs can be integrated in different LMOs, and similarly, LMOs can be integrated in different RMOs (Barbé et al., 2005).

Epistemological Reference Model (REM)

In ATD, the reference epistemological models (REMs) are formulated in terms of praxeologies (García et al., 2006). In the present paper, the REM serves to present the 'researcher's model of school algebra'. The model is based on theoretical and empirical data from analyses of mathematical teaching and learning processes, textbooks and other didactical phenomena, in a wide range of institutions. The model explicitly includes the concrete questions, materials and processes used in mathematical activities as a raison d'être of the mathematical content (Ruiz-Munzón et al., 2013).

The REM has the function of a working hypothesis for the researcher and provides opportunities to include and compare different praxeologies related to school algebra (Bosch, 2015).

The Dominant Epistemological Model (DEM)

The dominant epistemological model (DEM), by contrast, shows the dominant way of describing a phenomenon within an institution, including its official raison d'être (Lucas et al., 2019). The DEM of school algebra includes the domain of algebra knowledge to be taught, and the type of algebra activities that are emphasized in lower secondary schools. The DEM of knowledge to be taught can be informed by the curricula, textbook material, and written national exams, to exhibit the dominant praxeologies.

It is also important to consider why a particular DEM exists. In that way, the REM is detached from the DEM and provides the opportunity to exhibit phenomena and differences between praxeologies that are far from evident. According to Ruiz-Munzón et al., (2013), it is an important requirement for a REM not to adopt any of the institution's prevalent viewpoints uncritically, as it must serve to question them. Bosch (2015) argues that the construction of explicit reference models provides opportunities to ask research questions that go beyond the assumptions held by the school institution itself. As a central feature, the empirical data used to construct the DEM and the REM should come from the different institutions which are involved in the didactic transposition of knowledge (Ruiz-Munzón et al., 2013).

Didactic moments

To analyse internal didactic transposition, ATD uses the notion of didactic moments (Bosch et al., 2020): the first encounter with the type of task T, the exploration of T, with emergence of a first technique τ used to solve T, the moment of constructing the technological and theoretical block [ϴ/Θ], the moment of refining the technique(s), and the moment of the institutionalisation of the entire praxeology that has been constructed (Bosch et al., 2020).

Level of algebraization

Algebra serves as a modelling tool to model intra- and extra- mathematical systems through an algebraization process (Bolea et al., 2001, 2004). The process of algebraization is a mathematical-didactic phenomenon, that starts in primary school and continues through secondary school and university. There are, thus, different levels of algebraization, corresponding to still more advanced algebraic praxeologies. Bolea et al., (2001) and Ruiz-Munzón et al., (2013, 2020) work with algebraic modelling and levels of algebraization of a mathematical organisation in their respective REM. In the three-stage model of algebraization by Ruiz-Munzón et al., (2013, 2020) arithmetic can be identified as the domain of calculation programmes (CP). The first stage of algebraization occurs as learners consider the CP as an object and not only as a process. The second stage is introducing letters as parameters and unknowns, to model the relationship between CPs. The third and last stage of the algebraization process appears when the number of arguments of the CP is not limited, and the distinction between unknowns and parameters is eliminated (Ruiz-Munzón et al., 2020).

Research questions

The didactic transposition, the notion of praxeologies and didactic moments, and the three-stage model of algebraization provide the theoretical foundation for our approach to diagnose the state of algebra in lower secondary schools. Our main contribution is to combine these elements to achieve a more coherent analysis of the status of algebra in a given contemporary institution.

The aim of this paper is to demonstrate how the theoretical tools introduced above can be combined and deployed to produce a diagnosis of the state of school algebra within a given school institution, with Danish lower secondary school (DLS) as a case.

The case itself may not be of great importance to reader, but the methodology – involving many data sources to model the DEM and REM – is generalizable, as we will argue after presenting the case.

To investigate our case, the following research questions have been formulated:

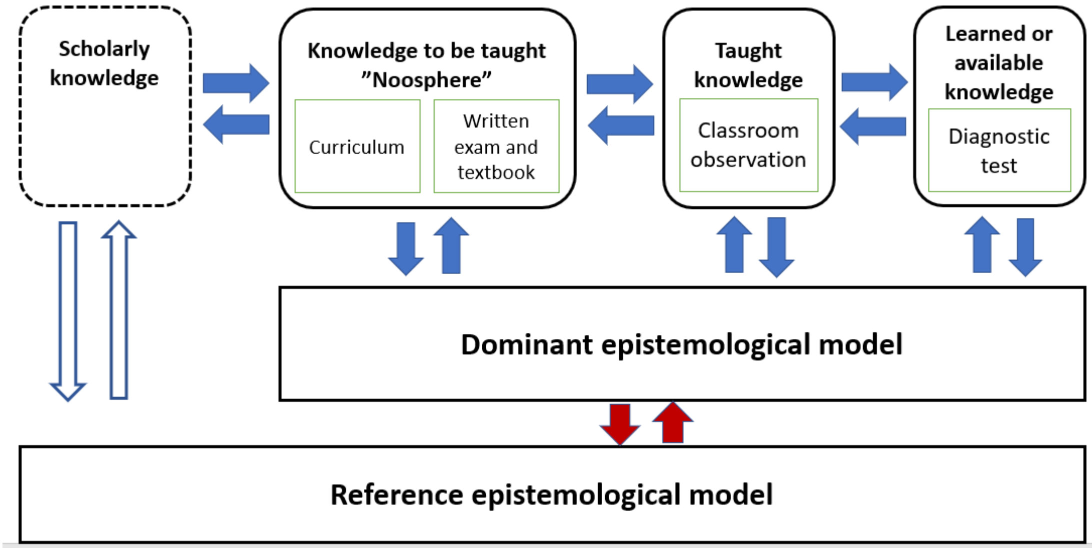
- What is the status of algebra in DLS, in terms of knowledge to be taught, taught knowledge, and learnt knowledge?
- How is algebra related to other domains such as arithmetic and geometry? What ruptures and inconsistencies are observed? To what extent is algebra used as a modelling tool?
- How can students' difficulties (such as the example of the national final exam, cf. introduction) be explained by the answers to the previous questions? What links can be made with previous studies of the algebra problem at lower secondary level?

The main advantage of the methodology proposed here is the connected analysis of all these questions, supported and enabled by the REM and DEM, as we proceed to explain in the next section.

Methodology

Reference and Dominant Epistemological Models

To answer the research questions, we will use the notions introduced above as theoretical framework to build the dominant epistemological reference models within DLS. The DEM will be compared to the REM to extract the phenomena that differ, and result in a diagnosis of school algebra in DLS.



Empirical evidence used to model the external didactic transposition

The description of algebra in the current national programme of DLS, the so-called 'common goals' (Education, 2019b), will be analysed to model algebra knowledge to be taught in DLS. The first step in the development of the DEM was to determine the topics in the national programme of DLS, which include elementary algebra and relate to algebra as a modelling tool and the three levels of algebraization (Ruiz-Munzón et al., 2013). The national programme of DLS, the written examination after lower secondary school for June 2019 and June 2021 (Education, 2019a) and textbook material by Hansen et al., (2015, 2016) were analysed. The description of the praxeologies will form the DEM of the knowledge to be taught.

The examination after DLS from 2019 was chosen because we have detailed data from the national results. The 2021 examination was selected to include a wider range of tasks. The textbook material by Hansen et al., (2015, 2016) was selected for analysis because it is from one of the most common Danish textbooks and is used in the schools where the lessons were observed.

Internal didactic transposition

We used classroom observations to gain insight into the school algebra actually taught. Two different teachers in three different classes were observed for 12 lessons in the second and fourth weeks of January 2022. We used participant observation (Emerson et al., 2001). Field notes were recorded during the teaching and combined with transcriptions of the classroom dialogue. The number of students who participated in the classes varied from 16 to 26 students, due to the COVID-19 pandemic.

We focused on a specific and characteristic episode where Grade 8 students were asked to explain how to solve an equation of the form $\mathrm{Ax} + \mathrm{B} = \mathrm{Cx} + \mathrm{D}$. The analysis was based on the ATD notion of didactic moments (Bosch et al., 2020).

Diagnostic test

To gain insight into the knowledge actually learned by students in DLS, we developed a diagnostic test tool to examine students' technical and theoretical knowledge of elementary algebra (Author, 2022). The test is based on the DEM, developed from analysis of textbook material from grade 7 and 8 (Hansen et al., 2015, 2016), the written examination after DLS and draws on a master's thesis on middle school arithmetic (Cosan, 2021). The items to test techniques are based on punctual praxeologies, consisting of simple tasks of the type which the techniques solve. For example, the type of task 'Solve first-degree equation of the form $\mathrm{Ax} = \mathrm{B}$ ' (by hand, i.e., using arithmetic operators) can be solved by the technique 'divide by A to obtain $\mathrm{x} = \frac{\mathrm{B}}{\mathrm{A}}$'. To test students' level of technology and the use of algebra as a modelling tool, open-answer questions such as 'Draw or write a story that fits the equation $2\mathrm{x} = 10$' are also included. To gain insight into the student's algebraic theory, more theoretical questions, such as 'Explain why $\mathrm{a} - (-\mathrm{a}) = 2\mathrm{a}$ ' are included in the test. The test involves systematic variations of the items to control for the significance of such variations.

The intention is to diagnose the state of DLS, grade 7-9, age 13-16. Therefore, four Grade 7 classes with a total of 69 students, and one Grade 8 class with 22 students were selected to take the 45-minute unaided paper-and-pencil diagnostic test in January and February 2022. One Grade 7 class and the Grade 8 class were used for observation. The remaining classes came from other schools in the suburbs of Copenhagen. The observation was agreed to by the teachers of the classes (no teacher refused the request, so we consider that the four classes are not a biased sample, except for their urban location).

Results

From the didactic transposition process, we extracted the phenomena in the DEM related to the praxeologies, the level of domains and the use of algebra as a modelling tool. These phenomena were analysed and linked to the REM, in view of identifying the distance between the DEM and the REM.

A Reference epistemological model for school algebra

In general, our REM states that modern algebra is a practical and theoretical tool to carry out modelling activity related to any school mathematical praxeology (Bosch, 2015). Strømskag & Chevallard (2022) have argued for a revision of the curriculum to cover the potential of elementary algebra as a modelling tool. For instance, this helps one to better understand the role of algebra in relation to arithmetic – when considering algebra as a 'tool to model arithmetic praxeologies'.

Arithmetic can be identified as the domain of calculation programmes (CPs) (Ruiz-Munzón et al., 2013). An example of a CP is the stepwise implementation of arithmetic operations that is used to solve classic arithmetic problems, such as the following item from the DLS written examination (Education, 2019a) opg. 1). Ellen wants to buy a bicycle, a helmet, and a locker for the bicycle. A drawing shows the prices from the bike shop, where a lock costs kr 249, a helmet kr 499, the blue bike kr 2750 and the red bike kr 3999. The first question is: 'How much money do the bike helmet and bike lock cost in total?'

The answer to the question is given by the CP 249+499. To calculate how much the red bike costs more than the blue bike, the CP 3999–2750 is used. The different CPs used to answer these questions are the sum and difference (of two values), which can be algebraized as an algebraic expression (of two variables): $x + y$ and $x - y$. In that way, the CP is an intermediate step of algebraization in the mathematical activity, and the first stage of algebraization occurs when the learner considers the CP as an object and not only as a process (Ruiz-Munzón et al., 2020).

Vis hvordan du regner

$$17 + 29 + 132 - 52 + 52 - 29 =$$

Figure 2. Item 5.1 in the Diagnostic Test

Figure 2 is an example of a task in which it is an advantage to rewrite down the CP before executing it, for example, $17 + 29 + 132 - 52 + 52 - 29 \rightarrow 17 + 29 - 29 + 132 \rightarrow 17 + 132 \rightarrow 149$. This practice provides an opportunity to solve the task by the technique of 'simplify[ing] the expression by applying the opposite', rather than simply deconstructing it into successive steps. The techniques used by the students will be discussed later in the article.

The second stage of algebraization is introducing letters as parameters and unknowns to model the relationship between CPs.

$$7x - 7 = 13 - 3x$$
$$x = \_\_\_\_\_$$

Figure 3. Item 4.7 in the Diagnostic Test

Figure 3 presents an example of the relation of a CP, that is, equations with one unknown and one parameter, which can be reduced to a one-variable equation. The potential algebraization of a problem into an equation is shown in Figure 3, for the problem: '7 pieces of tape minus 7 units are equal to 13 units of tape minus 3 pieces. How long is a piece of tape?' The solution is a relation between pieces and units. This kind of relation and algebraization process is the central part of the second stage of algebraization. The extent to which students can solve this type of task will also be discussed later in the article.

The third stage of the algebraization process does not distinguish between variables and parameters and there is no limit to the potential number of variables of the CP. This stage of algebraization includes the production, transformation and interpretation of formulae and it is, in general, less prevalent in lower secondary schools (Ruiz-Munzón et al., 2013).

In this case, we consider school algebra as a process of algebraization and use the three-stage model of the algebraization to set up the REM (Ruiz-Munzón et al., 2013). The three-stage model can be used as a tool to detect and analyse what kind of school algebra is taught and learnt (Bosch, 2015).

School algebra to be taught according to the curriculum

In Denmark, the so-called ‘common goals’ for mathematics (Education, 2019b), constitute the official directives for all primary and lower secondary schools. It is a competence-based curriculum, in which generic mathematical competences are combined with three mathematical topics, numbers and algebra, geometry and measurement, and statistics and probability.
The overall aim for arithmetic and algebra in ‘common goals’ for DLS, is that ‘the student can apply real numbers and algebraic expressions in mathematical investigations’ (Education, 2019b).

The mathematical organisation of the algebraic domain in the ‘common core’ is divided into three RMOs: ‘Equations’, ‘Formulas and algebraic expressions’, and ‘Functions’. Each RMO is subdivided into LMOs, which consist of two objectives, one relating to techniques and the other to knowledge about the techniques.

Table 1. Common core of the algebraic domain for DLS (Education, 2019b)

RMO_1 **Equations**		RMO_2 **Formulas and algebraic expressions**		RMO_3 **Functions**	
$LMO_{1,1}$		$LMO_{2,1}$		$LMO_{3,1}$	
The student can develop methods for solving equations.	The student has knowledge of strategies for solving equations.	The student can describe relationships between simple algebraic expressions and geometric representations.	The student has knowledge of geometric representations of algebraic expressions.	The student can use linear functions to describe relationships and changes.	The student has knowledge of representations of linear functions
$LMO_{1,2}$		$LMO_{2,2}$		$LMO_{3,2}$	
The student can build and solve equations and simple inequalities.	The student has knowledge of equation solving with and without digital tools.	The student can rewrite and calculate with variables.	The student has knowledge of methods for rewriting and calculations with variables, including with digital tools.	The student can use non-linear functions to describe relationships and changes.	The student has knowledge of representations of non-linear functions.
$LMO_{1,3}$		$LMO_{2,3}$			
The student can build and solve simple systems of equations.	The student has knowledge of graphical solution of simple systems of equations.	The student can compare algebraic expressions.	The student has knowledge of the rules for calculating with real numbers.		

At the first level of 'Formulas and algebraic expressions' (RMO_2) in Table 1 the aim is that 'The student can describe relationship between simple algebraic expressions and geometric representations' and 'The student has knowledge about geometric representations of algebraic expressions'($LMO_{2,1}$), respectively. At stage three in RMO_2 the $LMO_{2,3}$ is 'The student can compare algebraic expressions' and 'The student has knowledge about calculating rules for real numbers' (Table 1).

The $LMO_{2,1}$ and $LMO_{2,3}$ relates to algebra as a modelling tool (Bolea et al., 2001), and $LMO_{2,3}$ can be interpreted as the first stage of algebraization (Ruiz-Munzón et al., 2013, 2020). In addition to the RMOs, there is a 'focus of attention' for the lower secondary degree Grades 6 to 9, namely that 'The student can substitute numbers for variables in a simple formula' (Education, 2019b). As there are no example assignments, the dominant algebra praxeology related to the 'focus of attention' in the common goals consists of a praxis block, T: Solve a simple equation by substitution with the corresponding technique, τ: Substitution of integer. The logos block can be described with the technology.

θ: Methods to model and solve equations, and the level of theory is Θ: Knowledge of how to model and solve equations. General descriptions that do not tell teachers exactly what to teach. In practice, the knowledge to be taught is determined by textbooks and also by the centralised national exam, from which earlier items are frequently used by teachers as 'training material' for students in the final years of DLS. Both sources provide a great deal of information about the knowledge to be taught, according to the resources teachers use to organise their teaching.

Written examination praxeologies

In the paper-and-pencil written examination without aids (after grade 9), from June 2019 and June 2021, Item 6.1 (REF 2019) reads simply: '$40 + \_\_\_ = 50 + 15$'. This is an example of the first stage of algebraization in the written examination after DLS, since it can be solved by substitution. In fact, substitution is the dominant technique at the exam. The task is a variation of the algebraic type of task T: solve $A + x = B + C$. The fact that the item is not given in this standard algebraic form could reflect that the institution does not expect all pupils to master this form.

As an example of the second stage of algebraization, we considered the items in Table 2 (from the part of the exam where the students do not have access to digital tools).

Table 2. Written Examination after DLS (January 2019)

Item	Task Solve the equations	Type of task	Technique
11.1	$3x + 1 = 10$ $x =$	T: solve $Ax + B = C$ (NB: $A \neq 0$)	$\tau: x = \frac{C - B}{A}$
11.2	$5x - 3 = 2x + 18$ $x =$	T: solve $Ax + B = Cx + D$ (NB: $A \neq C$)	$\tau: x = \frac{D - B}{A - C}$, $A \neq C$
11.3	$\frac{2(x + 4)}{x} = 6$ $x =$	T: solve $\frac{P(x)}{Q(x)} = A$ (P, Q first degree polynomials)	τ: substitute $x \epsilon \{1,2,3, ..\}$

In item 11.3 (Table 2), the students are expected to solve the equation by substitution of small positive integers. The substitution by two integers succeeds, and the students may be satisfied by finding one solution, as they never encountered equations with more than one. The use of substitution as a technique to solve equations implies that the algebraization remains at the first stage. The second stage of algebraization – relying on using operators on expressions and equations – is not required.

In Table 2, the type of task T and the corresponding technique τ form a point-praxeology and a part of the $LMO_{1,2}$ 'solving equations' with the shared technological discourse 'Manipulation of variables as if they were numbers'. The LMO is a part of the RMO (Chaachoua et al., 2019). In this case, the LMO 'Solving equations' is a part of RMO_1 'Equations' and forms the DEM.

The third stage of algebraization, which includes the production, transformation and interpretation of formulae (Ruiz-Munzón et al., 2013), is rarely present in the written examination after DLS. When there are questions that require this third stage, the items are highly scaffolded, as the example mentioned in the introduction illustrates.

To sum up: the goal of 'The student can substitute numbers for variables in a simple formula'(Education, 2019b) dominates the final exams at DLS, and students are rarely presented with tasks that go beyond the first step of algebraization. Algebra as a modelling tool is implicit described in the common core and not specified through assignments.

School Algebra to be Taught according to Textbook Material

One of the most common textbook series used in DLS is *Kontext+* (Hansen et al., 2016). We chose two single textbook exercise and a theoretical section as a case for the praxeological analysis. The purpose of this is to illustrate the difference between the intended techniques and theory according to the assignment text (DEM), and the variation of techniques to solve the tasks and the level of theory according to the REM.

A typical textbook example from grade 8 is shown in Figure 4. In a. to d. the task is to 'Reduce the expression as much as possible' and 1) is a stepwise example of the process.

Trin 3	Opgave 3
1) $10 + 5x = 3x + 16$	**a.** $6x + 2 = 3x + 14$
$5x - 3x = 16 - 10$	**b.** $6x - 12 = 14x + 52$
$2x = 6$	**c.** $-5x + 33 = 5x - 17$
$x = 3$	**d.** $-3x + 8 = -2x - 13$

Figure 4. Exercise 3, p. 100, in Kontext+8 by Hansen et al. (2006).

The four different tasks in Figure 4 are of the same type, $Ax + B = Cx + D$, as they can be solved by the same techniques. The techniques can be described as addition, subtraction, multiplication and/or division on both sides of the equal sign. An example is shown in the left column as a recipe. The first step is the addition of $(-3x - 10)$ on both sides of the equal sign. The next step is reduction by subtraction and, finally, division by the coefficient of the unknown, to get $x = 3$. The shared technology discourse is 'Manipulation of variables as if they were numbers' and is in line with the dominant praxeology presented earlier, where the theoretical level is based on knowledge of arithmetic operators.

In the textbook *Kontext+8* (Hansen et al., 2016b), the authors describe algebra as the language of mathematics, and in *Kontext+7*, the section "Algebra and arithmetic" (see Figure 5) begins with the description, 'Many of the arithmetic rules and notations that apply to numbers, also apply to letters. If you are not sure how to calculate with letters, you can often try with numbers' (Hansen et al., 2016b). An example of what Chevallard (1989) describes as a key point of curriculum development, where 'numerism' and concreteness are paramount and exclude algebra as a modelling tool. The description of algebra as the language of mathematics is limited to praxeologies for arithmetic problems in the DEM and is considered as an initial praxeology for the first level of algebraization (Ruiz-Munzón et al., 2020).

Algebra og regneudtryk

Mange af de regneregler og skrivemåder, som gælder for tal, gælder også for bogstaver. Er man i tvivl, hvordan man regner med bogstaverne, kan man ofte prøve efter med tal.

Eksempel

$a + a + a = 3 \cdot a$	$2 + 2 + 2 = 3 \cdot 2$
$a + b = b + a$ og $a \cdot b = b \cdot a$	$2 + 5 = 5 + 2$ og $2 \cdot 5 = 5 \cdot 2$
$a \cdot (b + c) = ab + ac$	$2 \cdot (5 + 7) = 2 \cdot 5 + 2 \cdot 7$
$a + (b + c) = a + b + c$	$2 + (5 + 7) = 2 + 5 + 7$
$a - (b + c) = a - b - c$	$2 - (5 + 7) = 2 - 5 - 7$
$1a = a$	$1 \cdot 7 = 7$

Figure 5. Example, p.102, in Kontext+7 by Hansen et al. (2006).

This forms a textbook praxeology (Table 3) in the dominant epistemological reference model of the knowledge to be taught and contribute to the basis for the taught knowledge.

Table 3. Example of Praxeology

T	Type of task	Solve first degree equation of the form $Ax + B = Cx + D$, by using arithmetic operators.
τ	Techniques	Addition, subtraction, multiplication and/or division on both sides of the equal sign (use opposite arithmetic operators).
θ	Technology	Manipulation of variables as if they were numbers.
Θ	Theory	Rules and notation from arithmetic can be used in algebra.

In the second example the assignment text is ‘Use the guessing methods to solve the equations’ (Hansen et al., 2015), and the first four items are: 1) $x + 20 = 130$, 2) $2x - 35 = 135$, 3) $2(x + 5) = 80$ and 4) $\frac{1}{2}x = 20$. This means that the four items should be solved by the same technique, that is, τ:Trial-and-error, if we are following the guidance. The DEM considers all four tasks as the praxis: T:Solve first degree equation, with the corresponding technique τ:Trial-and-error. The REM divides the items into three different types of tasks. Item 1) is T: Solve $x + B = C$, with the corresponding technique τ: Subtract B from C to get $x = C - B$, and item 2) is T: Solve $Ax + B = C$, which can be solved by the technique τ: $\frac{C-B}{A}$. Item 3) is solved by using the distributive law $\tau\text{: } a(b + c) = ab + ac$, which entails an equation of the form $Ax + B = C$, and can be solved by the same technique as item 2). Item 4) involves calculation with fractions but is the task type T: Solve first degree equation of the form $Ax = B$, with the corresponding technique $x = \frac{B}{A}$ where $A, B \in Q$.

As illustrated in Figure 5, the book emphasises that there is a strong link between arithmetic and algebra, explained as follows: ‘Many of the arithmetic rules and notation that apply to numbers, also apply to letters. If you are not sure how to calculate with letters, you can often try with numbers’ (Hansen et al., 2016). This is consistent with the DEM, where the substitution and trial-and-error techniques are dominant in relation to equation solving.

On the other hand, there are also instructions to solve first-degree equation of the form $Ax + B = Cx + D$, using a specific CP, as illustrated in Figure 4. The techniques can be described as τ: Addition, subtraction, multiplication and/or division on both sides of the equal sign.

How this praxeology is transformed from knowledge to be taught to taught knowledge in DLS will be illustrated with an example.

The Rigidity of the Algebraic Techniques Actually Taught

The episode is from an algebra lesson in Grade 8 (13–14-year-old students) and can be placed in $LMO_{1,1}$. In the following, E is the teacher and S_n the students. The teacher starts the lesson as follows:

> E: We have talked about equations that can be solved. I would like a recipe for that.

The teacher’s intention is to institutionalize and evaluate previous work with equations of the form $x + B = C$. The teacher’s wish to establish a “recipe” can be interpreted as the explanation of T and the emergence of τ to solve the type of task T, but the students start to discuss the difference between ‘reduce’ and ‘solve’. After 20 minutes of discussion, the teacher E writes $3(x - 4) = 6$ on the whiteboard and asks the students to explain how to solve the task. As shown above, the task can be solved by first using the distributive law $\tau\text{: } a(b + c) = ab + ac$ and then the technique $\tau\text{: } \frac{C-B}{A}$, according to the REM. The moment of exploration and the emergence of τ begins.

> S_1: I do not know how to explain it, because I do it all in my head. I’m thinking of what inside the parentheses gives two.

When the student says, ‘because I do it all in my head’, it is her technique used to solve the task, described in the following sentence about what is inside the parentheses. This technique is based on the evidence that $3 \cdot 2 = 6$ (or 6:3 = 2) and that the only way to get 6 with 3 as a factor is by doing $3 \cdot 2$, the implicit technology. The teacher puts a name to this technique:

E: Yes. In a way you use trial and error, and you can do that. Well done.

The teacher names the emergent technique ‘trial and error’ (which does not correspond to what the student does) and evaluates the student’s technique by the comment ‘Well done’.

The teacher writes 6 on the whiteboard.

E: It is not wrong to guess and try.

S_2: There is a secret multiplication sign behind 3 and therefore is 6 divided by 3?

E: I do not understand.

S_2: Can I show you?

E: Yes.

The student S_2 walks to the whiteboard and writes a fraction line under 6 and writes 3 as the denominator. The student perceives $(x - 4)$ as a unit and at the same time as an expression with a variable. The comment about the secret multiplication sign indicates that the student S_2 shows insight into the written algebra discourse, but it is invisible because it remains oral.

E: Then we get x minus four equals two.

S_2: I do not know what I meant.

The moment of exploration of T and the emergence of τ end abruptly without continuing into the moment to build the theory block. The reason for this break could be the increasing small talk in the classroom. [The teacher asks the students to concentrate.]

E: You can also solve it in another way.

[The teacher writes $3(x - 4) = 6$ on the whiteboard and draws curved arrows from 3 to x and from 3 to 4. He then writes $3x - 12 = 6$ on the whiteboard on the right side of the first equation, without any oral explanation.]

E: I have just multiplied in the parentheses. Now I want to move -12 to the other side.

This is a moment of institutionalization by implementation of the preferred institutional technique. The teacher uses the distributive property of multiplication to rewrite the expression. This level of theory remains implicit for the students. The process of ‘moving’ 12 to the other side is an informal way of using and explaining the technique τ: Addition of an integer on both sides of the equal sign.

S_3: Then it's plus.

E: Then we get 3 times x is 18.

[The teacher writes $3x = 18$ on the whiteboard.]

E: Then we must find out what to multiply 3 with to get 18. We should only divide by 3.

[The teacher writes $x = 6$ on the whiteboard.]

S_1: But how do we know not to subtract 12 from 3x? How do I know if 12 should be removed or subtracted from something?

E: You cannot subtract numbers from x's.

[Small talk in the classroom.]

One student (perhaps several) can solve the equation orally with arithmetical technology and do not need the algebraic symbolic manipulation to solve the task.

The solution is not validated directly by the teacher, but the fact that she presents another technique may suggest to students that the first method is perhaps not ideal. The teacher's presentation involves informal ideas such as 'moving' objects in equations, and the student who questions this 'move' get a reply from the teacher that involves a new informal and unjustified claim ('You cannot subtract numbers from x's').

Learned knowledge according to the diagnostic test

The diagnostic test was taken by 69 Grade 7 students (12–13-year-old) and 22 Grade 8 students (13–14-year-old) from DLS. The students had 45 minutes to complete the 67-item paper-and-pencil test.

Table 4. Summary of Types of Answers in the Test

Level	Partici-pants	Test items	Correct	Incorrect	No answer	Sum of answers
Grade 7	69	67	1622 35%	1229 27%	1772 38%	4623
Grade 8	22	67	553 38%	373 25%	548 37%	1474

Table 5 contains an a priori analysis of T where Table 4 and Table 6 present the results related to T. Items 2.5 and 3.6 are of the task type T: Solve $Ax = B$, with the corresponding technique τ: Divide by A to get $x = \frac{B}{A}$. The interesting aspect is whether or not the equation can be solved by substitution.

Table 5. A priori analysis of T, related to the items from Table x and x

T: Solve Ax = B	τ: Divide by A to get $x = \frac{B}{A}$.	τ: Substitution by x∈N.
$2x = 10$ $x = \_$	$2x = 10$ $x = \frac{10}{2}$ $x = 5$	$2x = 10$ $2 \cdot 1 = 2$ $2 \cdot 2 = 4$ $2 \cdot 3 = 6$ $2 \cdot 4 = 8$ $2 \cdot 5 = 10$
$2x = 16 + 2$ $x = \_\_$	$2x = 16 + 2$ $2x = 18$ $x = \frac{18}{2}$ $x = 9$	$2x = 16 + 2$ $2x = 18$ $2 \cdot 4 = 8$ $2 \cdot 6 = 12$ $2 \cdot 8 = 16$ $2 \cdot 9 = 18$
$7x - 7$ $= 13 - 3x$ $x = \_\_$	$7x - 7$ $= 13 - 3x$ $7x + 3x$ $= 13 + 7$ $10x = 20$ $x = \frac{20}{10}$ $x = 10$	$7x - 7 = 13 - 3x$ $7 \cdot 1 - 7 = 13 - 3 \cdot 1$ $7 \cdot 2 - 7 = 13 - 3 \cdot 2$ $7 \cdot 3 - 7 = 13 - 3 \cdot 3$ ….. $7 \cdot 10 + -7 = 13 - 3 \cdot 10$
$x \cdot \frac{3}{4} = \frac{15}{20}$ $x = \_$	$x \cdot \frac{3}{4} = \frac{15}{20}$ $x = \frac{15}{20} \cdot \frac{4}{3}$ $x = 1$	$x \cdot \frac{3}{4} = \frac{15}{20}$ $1 \cdot \frac{3}{4} = \frac{15}{20}$ $\frac{3}{4} = \frac{3}{4}$

According to the a priori analyses in Table 5, we expect the first two items to be solved by substitution. Since solving the third item, by substitution, requires many CPs, it is not expected to be a successful method. The last item can be solved relatively easily by substitution; however, it requires the student to be familiar with fractions.

The test results show (Table 6) that the first two items were answered well. The last case shows that most students cannot multiply fraction (otherwise, indeed, substitution would work).

Table 6. Item and Associated Sum of Answers in the Test, Grade 7

Item	Item number	Correct	Incorrect	No answer
$2x = 10$ $x = \_$	2.5	44 64%	10 15%	15 21%
$2x = 16 + 2$ $x = \_\_\_$	3.6	29 42%	19 28%	21 30%
$7x - 7 = 13 - 3x$ $x = \_\_\_$	4.7	15 22%	20 29%	34 49%
$x \cdot \frac{3}{4} = \frac{15}{20}$ $x = \_\_$	7.6	17 25%	7 10%	45 65%

Almost half part of the students could solve the task type T: Solve first degree equation of the form $Ax = B$ when $A, B \in N$. But when $A, B \in Q$, only 25% of the Grade 7 students could answer the item correctly. The task type T: Solve first degree equation of the form $Ax + B = Cx + D$ seems to cause problems for the students, since only 22% of the students had a correct answer and almost half of the students did not answer the question at all. The test item 4.7 is halfway in the test, and the missing answers are not expected to be due to lack of time.
The picture is different for the small group of Grade 8 students – see Table 7.

Table 7. Item and Associated Sum of Answers in the Test, Grade 8

Item	Item number	Correct	Incorrect	No answer
$2x = 10$ $x = \_$	2.5	18 82%	0 0%	4 18%
$2x = 16 + 2$ $x = \_\_\_$	3.6	12 55%	3 14%	7 32%
$7x - 7 = 13 - 3x$ $x = \_\_\_$	4.7	3 14%	4 18%	15 68%
$x \cdot \frac{3}{4} = \frac{15}{20}$ $x = \_\_$	7.6	8 36%	1 5%	13 59%

Items 2.5 and 3.6 are of the form $Ax = B$, and were answered correctly by relatively more students in Grade 8 than in Grade 7.

The picture is less clear when it comes to items 4.7 and 7.6, where it seems that more students in Grade 8 master T: Solve first degree equation of the form $\mathrm{Ax} = \mathrm{B}$ when $\mathrm{A}, \mathrm{B} \in \mathrm{Q}$, than Grade 7 students. But the task type T: Solve the first-degree equation of the form $\mathrm{Ax} + \mathrm{B} = \mathrm{Cx} + \mathrm{D}$ seems to cause even more problems for the students in Grade 8 than in Grade 7. Students' test answers to item 4.7 can give us insight into types of errors. Figures 6 and 7 show two examples of failed attempts to use algebraic methods (of the type foreseen in the REM). The presence of the unknown on both sides of the equation makes it hard to guess a solution, even if 2 is among the usual candidates. According to our a priori analysis we expect the student to collect same type of terms and then reduce the coefficient of x. When addition of the opposite numbers and the distributive law fails, some students try to use arithmetic operations, but few succeed.

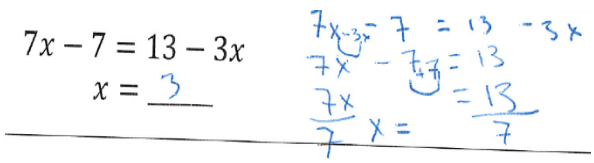


Figure 6. Student Answer

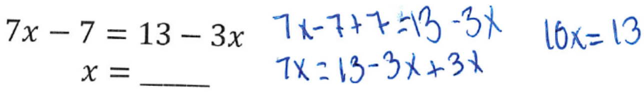


Figure 7. Student Answer

These observations are confirmed when we look at the results from similar tasks in the written exam after DLS.

1.1.1 Learned knowledge according to the written examination after DLS

In May 2018, 56,700 students completed the final exam after DLS (Table 8). In May 2019, 56,415 students did so (Table 9). We do not have access to the student responses and thus the actual techniques used to solve the tasks. Therefore, the percentage correct answers can only serve as an overall picture of the type of tasks students are not answering correctly, and not an insight into current praxeologies.

Table 8. Correct Student Answers of Items in Written Examination 2018

Item Year 2018	Task Solve the equations	Type of task	Correct answer
7.1	$6x + 4 = 28$ $x =$	T: Solve $Ax + B = C$	88%
7.2	$\frac{x}{2} + 1 = 5$ $x =$	T: Solve first degree operation wich include fraction	66%
7.3	$4 \cdot (x - 2) =$ $2x + 6,$ $x =$	T: Solve first degree equation with brackets	44%

Table 9. Correct Student Answers of Items in Written Examination 2019

Item Year 2019	Task Solve the equations	Type of task	Correct answer
11.1	$3x + 1 = 10$ $x =$	T: Solve $Ax + B = C$	89%
11.2	$5x - 3 = 2x + 18,$ $x =$	T: Solve $Ax + B = Cx + D$	61%
11.3	$\frac{2(x + 4)}{x} = 6$ $x =$	T: Solve first degree equation which includes fraction	46%

Algebra as a Modelling Tool

In the Danish common core for Grades 6 to 9, the aim of the RMO_2 'Formulas and algebraic expressions' is that 'the students can describe connections between simple algebraic expressions and geometric representations' ($LMO_{2,1}$)(Education, 2019b). This objective is transposed into the theme algebraic models in geometry according to the DEM. In the written national 2019 paper-and-pencil exam, two types of tasks appear. The first type of task is 'Determine the perimeter of the polygon', with the corresponding technique 'Add the side lengths of the polygon'. The second type of task is 'Determine the area of a polygon with all sides being either parallel or orthogonal'. To determine the area, the technique 'Calculate the area by using the formula of rectangle area together with the additive principle (the area of a disjoint union of polygons is the sum of the area of those polygons)’, is used.

The same types of tasks appear in the textbook material under a topic called ‘Geometry algebra’ and can also be located in $LMO_{2,1}$. The textbook points out that ‘you can use geometric figures to model arithmetic rules by letters’ (Hansen et al., 2016a).

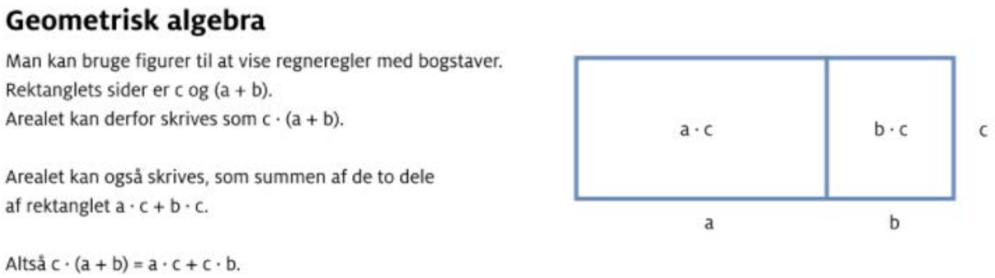


Figure 8. Geometry Algebra (Hansen et al., 2016a)

In the example Figure 8, a model of the distributive law is drawn and letters representing the side lengths of the rectangle are added. Algebra as a modeling tool according to the REM stays invisible to the student, who only needs to read the final model for the current calculation rule.

Figure 8 is an example of the dominant way of deducing an algebraic law, using the technique ‘Calculate the area using the formula of rectangle area together with the additive principle’ as mentioned earlier, for a particular geometric figure. The technique tells us that the area of the large rectangle is $a \cdot c + b \cdot c$, and can also be described by $(a + b) \cdot c$. The use of logos to explain this link is absent. The students are not introduced to ‘distributivity’, neither as an assumption nor as an axiom in algebra. The variation and generality of the distributive law also remain implicit, like it is not visible for the students that the example in Figure 8 is special (for instance, assume that $a, b > 0$).

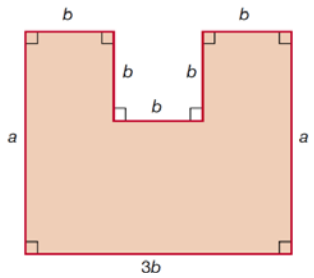


Figure 9. Irregular Octagon

The perimeter of the octagon is _________

The areal of the octagon is _________

In the written national paper-and-pencil examination after Grade 9 (Education, 2019a), 70% of the 56,700 students (aged 14–15 years old) calculated the perimeter of the irregular octagon in Figure 9 correctly. The students must apply the technique ‘Add the side lengths of the polygon’ to get $b + b + b + b + b + a + 3b + a$; then another algebraic technique ‘Collect equal terms’ can be used to get the final result $2a + 8b$.

Modelling the area of the octagon by dividing the octagon into rectangles can be done in several different ways. The model can be presented as $(a \cdot 3b) - b^2$ or $(a - b) \cdot 3b + b^2 + b^2$, which, by the technique ‘Collect equal terms’, of course both can be reduced to $3ab - b^2$.

Modelling the area of the octagon is much more difficult for the students in the examination after DLS. Only 26% of the students could determine the area correctly. The same picture appears in the diagnostic test, where only 17% of the students could model the perimeter of a regular triangle with the side lengths e to be 3e. In the last item of the diagnostic test, the students must determine the perimeter and the area of a rectangle with the side lengths s and 3s. Only 15% of the students could determine the perimeter of the rectangle to be 8s, and only 4% could model the area of the rectangle to be $3s^2$. This disappointing result can partly be explained by the fact that only half of the students answered the item within the allowed timeframe.

To model the perimeter of a regular polygon without a figural representation seems to have been even more difficult for the students. Less than 3% of the students could model the perimeter of a square with the side length $(a + b)$ to be $4(a + b)$ in the diagnostic test.

Algebra as a modelling tool in the DEM and the REM

The use of algebra as a modelling tool appears in different ways in the DEM and the REM. Like Chevallard (1989) we will group the intra- and extra-mathematical system into three common categories:

- Construct the algebraic model
- Rewrite the algebraic model
- Apply the algebraic model

In the REM, the construction of algebraic models includes defining the system we want to study, specifying and obtaining relevant data and assigning letters to variable. The construction of algebraic models in DEM is based on already defined systems and data set, and the specifications and variables are often assigned.

In the REM, rewriting algebraic models will include a reworking of the model that aims to look at new relations between the variables in the system and thus produce new knowledge about the system under study. In DEM, the rewriting of algebraic models occurs though targeted use of algebraic techniques, as in the initial example.

In the REM, applying algebra models enable us to study intra- and extra-mathematical objects and systems. In the DEM, the application of the models primarily consists of substitution of numerical values for one or more variables.

Geometry algebra is a common topic, according to the DEM. The previous examples from textbooks and national examination in DLS provide examples of the use of algebraic models in geometry. The modelling tasks are simple and can often be solved by inserting values in known or given formulae. The exercises are mostly routine tasks for the students because the tasks belong to types of tasks with well-known techniques.

In the DEM, the logos blocks appear informal and do not consist of ‘formal’ mathematical axioms and definitions. Where geometry algebra in the REM incudes the construction of algebraic models as well as rewriting and applying the models, and more generally connecting the intra- and extra-mathematical systems.

Discussion

How can the notion of the didactic transposition be used to give a status of school algebra? More specifically, how can the empirical construction of a DEM and the empirical and theoretical construction of a REM be used to extract the conditions and constraints which affect the didactic transposition of algebra in lower secondary schools? REM is used as a foundation for analysing DEM. Our case example from Danish lower secondary schools shows a difference between the intended techniques and theory according to the assignment text (DEM), and the variation of techniques to solve the tasks and the level of theory according to the REM. This is a specific constraint that affects the didactic transposition and argues for the construction of an explicit REM and DEM (Bosch, 2015).

Although the REM and the DEM are explicit, it is important to keep in mind that the REM is the researcher’s explicit model. According to (Bosch, 2015), the construction of the REM concerning school algebra provides opportunities to go beyond the assumptions held by the school institution itself (DEM). The written examination praxeologies discussed above are examples of the knowledge to be taught. However, the teaching episode described above and the result from the diagnostic tests show a frequent disconnectedness among praxeologies to be taught, those taught and those learnt. This disconnectedness of algebra praxeologies has been noted before, in a Danish context by (Jessen & Winsløw, 2017), and internationally by (Bolea et al., 1999).

Conclusion

The analyses of the conditions and constraints that influence the didactic transposition processes of school algebra practices highlight, from various angles, the disconnectedness of praxeologies to be taught, actually taught and learnt by students (Figure 1). Using the didactic transposition as a theoretical framework provides the opportunity to construct an epistemic reference model of school algebra. The REM is used to analyse the DEM and describe the distance between the models, to provide a status of current school algebra. The DEM of school algebra in a Danish context shows a rigidity of techniques (and praxeologies) and the atomisation of types of tasks (little or no explicit relation between praxeologies), due to an absence of coherent technological discourses to connect and compare the techniques, and of mathematical theory to justify the technologies that are actually presented.

The official national programme of DLS describes techniques without reference to concrete types of task. In the paper-and-pencil examination, the students have to apply techniques to concrete types of tasks, but they are not required to exhibit mathematical technology or theory justifying the techniques. In the textbooks, there are both technological and theoretical elements, which link algebra and geometry by using algebra as a modelling tool to determine the perimeter and area of a polygon with all sides being either parallel or orthogonal. These examples show, however, that the generalizing power of algebra to model intra- and extra mathematical systems is only punctually experienced and acquired by students. This weak presence of the interpretation of algebra as a modelling tool is related to the general phenomenon of the 'dis-algebraization' of school mathematics (Bolea et al., 2004b).

At the level of the domains, there is a somewhat blurred connection between arithmetic and algebra. The use of arithmetic in an algebraic way means that the first stage of algebraization not occurs while the leaner considers the CP only as a process and not as an object (Ruiz REF). This 'numerism' and concreteness are paramount and the difficulty in introducing the need for algebra (because it is already there) in the knowledge to be taught and taught knowledge, results in the disappearance of the *raison d'être* of algebra. This paramount 'numerism' and concreteness that exclude algebra as a modelling tool was already discussed by Chevallard in 1989. (Chevallard, 1989). In the case of DLS, there is also what (Bolea et al., 1999) define as an *atomization* of school algebra, into techniques for calculation with only vague theoretical discourse to connect them.

In line with Grugeon (Grugeon, 1997) the aim of this paper was to offer a multidimensional model of the didactic transposition to construct a comprehensive model of school algebra based on previous research on school algebra within the ATD. For operationalising of the model, we have used a rich empirical material of qualitative and quantitative data and used a diverse range of ATD tools to analyse the data. We have shown how the didactic transposition and the construction of an explicit reference model to analyse the dominant reference model can be combined and used as theoretical tool to produce a diagnosis of the state of school algebra within a given school system, using DLS as a case study. The methodology – involving many data sources, whose analysis is unified by the theoretical framework - is a further development of Grugeon's multidimensional model (Grugeon, 1997) in order to increase its generalisability and applicability in other national cases.

The present diagnosis outlines crucial features of the status of algebra in Danish lower secondary schools. An informed status, based on a sound diagnostic tool, is the best possible basis for meeting the challenges. Our findings confirm that the generalising power of algebra should be experienced and acquired by having students work in environments with a praxeological need for algebra (Strømskag & Chevallard, 2022).

References

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice: The case of limits of functions in Spanish high schools. *Educational Studies in Mathematics*, 59(1–3), 235–268. https://doi.org/10.1007/s10649-005-5889-z

Bolea, P., Bosch, M., & Gascón, J. (1998). The role of algebraization in the study of a mathematical organization. In I. Schwank (Ed.), *Proceedings of the First Conference of the European Society for Research in Mathematics Education* (Vol. II, pp. 135–145).

Bolea, P., Bosch, M., & Gascón, J. (2001). The didactic transposition of mathematical organisations in the process of algebraization: the case of proportionality. [La transposición didáctica de organizaciones matemáticas en proceso de algebrización: el caso de la proporcionalidad]. *Recherches en didactique des mathématiques,* 21(3), 247-304. (Translated by Deepl)

Bosch, M. (2015). Doing research within the anthropological theory of the didactic: The case of school algebra. In S. J. Cho (Ed.), *Selected regular lectures from the 12th International Congress on Mathematical Education* (pp. 51–69). Switzerland: Springer International Publishing.

Bosch, M., Chevallard, Y., Garcia, F. J., & Monaghan, J. (2020). Introduction: An invitation to the ATD. Bosch, M., Chevallard, Y., Garcia, F. J., & Monaghan, J. (Eds.) In Working with the anthropological theory of the didactic in mathematics education. *Routledge*. https://doi.org/10.4324/9780429198168

Bosch, M., & Cascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI bulletin*, 58(58), 51-65.

Bosch, M., Hausberger, T., Hochmuth, R., Kondratieva, M., & Winsløw, C. (2021). External didactic transposition in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, 7(1), 140–162.

Chaachoua, H., Bessot, A., Romo, A., & Castela, C. (2019). Developments and functionalities in the praxeological model. In *Working with the Anthropological Theory of the Didactic in Mathematics Education*, (1st ed., pp. 41–60). Routledge.

Chevallard, Y. (1985/1991). La Transposition didactique. Du Savoir savant Au savoir enseigné. [Didactic transposition. From learned knowledge to taught knowledge]. *La pensée Sauvage*. (Translated by Deepl)

Chevallard, Y. (1989). Le passage de l'arithmétique a l'algébrique dans l'enseignement des mathématiques au collège – Deuxième partie. Perspectives curriculaires: la notion de modélisation [The transition between arithmetic and algebra in the teaching of mathematics at secondary school level – Second part. Curricula approaches: The notion of modelling]. *Petitx*, 19, 45–75. (Translated by Deepl)

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic an attempt at a principled approach. *Hiroshima journal of mathematics education, 12*, 71-114.

Chevallard, Y., & Bosch, M. (2014). Didactic transposition in mathematics education. In: Lerman, S. (eds) Encyclopedia of Mathematics Education. Springer, Dordrecht. https://doi.org/10.1007/978-94-007-4978-8_48

Cosan, D. D., (2021). A diagnostic test for Danish middle school arithmetic. MSc dissertations. Department of Science Education, University of Copenhagen. Retrieved from https://www.ind.ku.dk/publikationer/studenterserien/studenterserien-publikationer/a-diagnostic-test-for-danish-middle-school-arithmetics/

Education D. M. (2019a). *National final examinations in mathematics. National Agency for Education and Quality.* Retrieved from https://www.xn--prvebanken-1cb.dk/proevematerialer/FAP/MAT/MATFP9

Education D. M. (2019b). Common objectives: Mathematics. *National Agency for Education and Quality.* Retrieved from https://emu.dk/sites/default/files/2020-09/GSK_F%C3%A6llesM%C3%A5l_Matematik.pdf

Garcia, F. J., Pérez, J. G., Higueras, L. R. & Bosch, M. C. (2006). Mathematical modelling as a tool for the connection of school mathematics. Zentralblatt für Didaktik der Mathematik 38, 226-246. https://doi.org/10.1007/BF02652807

Grønbæk N., Rasmussen A. B., Skott C. K., Bang-Jensen J., Fajstrup L., ….& Markvorsen S. (2017). The mathematic commission – reporting. *Danish Ministry of Education. 2017.*

Grugeon, B. (1997). Conception et exploitation d'une structure d'analyse multidimensionnelle en algèbre élémentaire. *Recherches en Didactique des Mathématiques (Revue)*, *17*(2), 167–210.

Hansen, N. J., Christensen, M. S., Lindhardt, B., & Thomsen, H. (2016). Kontext+ 8 Core book/Web. Alinea. Retrieved from https://kontextplus.alinea.dk/course/QRAF-kontext-8

Hansen, N. J., Johnsen, L. B., Lindhardt, B., & Thomsen, H. (2015). Kontext+ 7 Core book/Web. Alinea. https://kontextplus.alinea.dk/course/QRAU-kontext-7

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational studies in mathematics*, *27*(1), 59–78.

Jessen, B. E., & Winsløw, C. (2017). Matematikbroen: brobygning for elever gennem efteruddannelse for laerere. *Mona*, *3*, p. 39–59.

Jupri, A., & Drijvers, P. (2016). Student difficulties in mathematizing word problems in algebra. *Eurasia Journal of Mathematics, Science and Technology Education, 12*(9), 2481-2502.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). Reston: NCTM.

Loveless, T. (2013). The algebra imperative: Assessing algebra in a national and international context. *Washington, DC: Brookings.* Retrieved from https://www.brookings.edu/wp-content/uploads/2016/06/Kern-Algebra-paper-8-30_v14.pdf

Lucas, C., Fonseca, C., Gascón, J., & Schneider, M. (2019). The phenomenotechnical potential of reference epistemological models: The case of elementary differential calculus. In M. Bosch, Y. Chevallard, F. J. García, & J. Monaghan (Eds.), *Working with the Anthropological Theory of the Didactic in Mathematics Education* (pp. 77–98). Routledge. https://doi.org/10.4324/9780429198168

Lucas, C., Fonseca, C., Gascon, J., & Schneider, M. (2019). The phenomenotechnical potential of reference epistemological models. The case of elementary differential calculus. In M. Bosch, Y. Chevallard, F. J. García (Eds.), *Working with the Anthropological Theory of the Didactic in Mathematics Educatio: A comprehensive casebook* (pp. 77-98). Milton Park, UK: Routledge.

Ruiz-Munzón, N., Bosch, M., & Gascón, J. (2013). Comparing approaches through a reference epistemological model: the case of school algebra. In B. Ubuz, Ç. Haser & M. A. Mariotti (Eds), *Proceedings of the 8th Conference of the European Society for Research in Mathematics Education* (pp. 2870–2979). Middle East Technical University. http://cerme8.metu.edu.tr/wgpapers/WG16/WG16_Bosch.pdf

Ruiz-Munzón, N., Bosch, M., & Gascón, J. (2020). Un modèle épistémologique de référence pour la recherche sur l'algèbre élémentaire. [An epistemological reference model for research into elementary algebra] *Nouveaux cahiers de la recherche en éducation*, *22*(1), 123-144.

Strømskag, H., & Chevallard, Y. (2022). Elementary algebra as a modelling tool. A plea for a new curriculum. *Recherches en Didactique des Mathématique*, 42(3), 384

Paper II

A COMPARATIVE STUDY OF DIDACTIC MOMENTS IN A FIRST CHAPTER ON ALGEBRA IN DANISH AND JAPANESE MIDDLE SCHOOL TEXTBOOKS

Abstract

Teachers often base their teaching on textbook material. Textbooks play a role as mediators between official guidelines and teachers' work. Therefore, it is interesting to study the mathematical organization of material and its connection to curricula. This paper presents a comparative analysis of Japanese and Danish textbook material based on the foundation of the anthropological theory of didactics. Our analysis focuses on how the very first introduction to algebra is organized in Danish and Japanese textbooks for the middle school, and in particular, how the distributive law is treated as a central, specific element of algebraic theory. We more generally look at the roles of algebraic theory that textbooks can facilitate. One result is that the stepwise, modular progression in the Japanese curriculum is indeed reflected in the Japanese textbook material, which treats one mathematical subject area after the other, in a clear progression. The Danish competence-based curriculum with its spiral structure is also reflected in the Danish textbook material, where the content areas are revisited and expanded over the grades.

Key words: Algebraic expressions, school algebra, textbooks, anthropological theory of didactics, first moment of encounter

Introduction

Typical curriculum resources in mathematics consist of textbooks, official guidelines and digital resources such as interactive worksheets. In their daily work, teachers interact with curriculum resources, which includes selecting and modifying, for example, textbook material (Trouche et al., 2020). The form and content of the textbook material have implications for teaching and the learned knowledge. According to the anthropological theory of didactics (ATD), this didactic transposition of knowledge to be taught (curriculum) to taught knowledge is of special interest

(Chevallard & Bosch, 2014). In the transposition process, the textbook has a role as mediator between official guidelines and teachers' work, as a link between intention and implementation (Tesfamicael & Lundeby, 2019). Just as there are differences in the form and content of textbooks, there are also variations in teachers' and students' implementation of the curriculum. The way students 'use' the textbook depends on their mathematical knowledge and their knowledge of the material. For example, to find support for solving an exercise in the textbook, some students will look for help such as worked examples and the theoretical approach in the material (Pepin & Gueudet, 2020). In a systematic literature review of the potentials and limitations of the use of textbook materials in mathematics education, one of the findings was that there has been less emphasis on describing the textbook itself and the relationship between the textbook and the curriculum in relation to other themes such as teachers' use of textbooks and students' learning (Steen et al., 2020). The aim of this study is to describe and compare two different types of textbooks, Danish and Japanese, in terms of how they treat the first encounter with algebra and the connection between textbook and official objectives. The choice of algebra as a content area will be explained in more detail below.

To study the conditions and constraints of constructs of didactic phenomena, comparative studies are useful (Artigue & Winsløw, 2010), as they may highlight what depends on local contexts and what is more general. There is a certain variety of how and when algebra is introduced and operationalised in the school, according to different curriculum and teaching traditions (Eriksson, 2022). The purpose of this international comparison is to gain more knowledge about similarities and differences of curriculum, particularly the relationship between textbook content and national objectives. In this case, we are interested in how the transition from arithmetic to algebra is described in different textbook material, especially the first moment of encounter with algebraic expressions. An understanding of the diversity between curricula can assess the potential for transferring textbook material from one educational setting to another, to assist teachers in the teaching of school algebra.

School algebra

School algebra and hence the transition from arithmetic to algebra is one of the content areas where students and teachers in lower secondary school are particularly challenged (Kieran, 2007). This is also the case in Denmark, where Danish students continue to have major problems, throughout lower and upper secondary school, with numeracy and basic algebra (Education, 2022). This was also visible in the Trends in International Mathematics and Science

Studies (TIMSS) 2019 International Results in Mathematics, which indicated a worrying decline in mathematical performance among Danish grade 4 students (Kjeldsen et al., 2019). In the TIMSS 2019, Japan was one of the top five performing countries (Mullis et al., 2019). The objective is to gain insight into underlying reasons for the observed differences in student performance. Therefore, this paper attempts to analyse and compare curriculum materials from this country with those of Denmark. We chose Japanese textbooks because they are based on systematic empirical research, have a strong theoretical foundation and are translated into English. In particular, we examine lower secondary school algebra textbooks, focusing on the introduction to algebra and the encounter with algebraic expressions, as this is a fundamental aspect of basic algebra.

This study considers algebra as a modelling tool that models intra-mathematical systems and also as a tool to study systems in other disciplines, such as biology and physics (Bolea et al., 2001). Two of the most fundamental concepts in algebra are equivalence and variables. Equivalence and the use of the equal sign as expressing an identity is central for the transition from arithmetic to algebra. Students need to be familiar with algebraic symbols in order to engage with the concepts and to prepare them for further study in mathematics. One of the most powerful tools in arithmetic, and an important foundation for school mathematics, is the distributive property, along with the commutative and associative properties. According to Bruner (1960), these three properties are fundamental for working with equations. These properties are a central part of school algebra because they provide a foundation for exploration and generalizations in arithmetic and for the justification of generalizations (Schifter et al., 2008). In this context, the distributive properties are central for the level of algebraization, especially modelling the relationship between calculation programmes (Ruiz-Munzón et al., 2013). The importance of these fundamental properties has been known for many years but still remains highlighted as a contributor to the challenges of school algebra (Jessen & Winsløw, 2017).

Anthropological theory of didactic as a theoretical framework

Anthropological theories play an important role in understanding human societies, cultures, and behaviour. They offer frameworks and perspectives to analyse the complexities whining and across different institutions and cultural contexts. In this case, we use the anthropological theory of didactics (ATD), which has emerged as a theory of mathematics education, because we want to compare mathematical textbook material from two different cultures. In ATD, all human activities are considered as institutionally situated where human knowledge and practice are modelled by praxeologies. The notion of a praxeology was introduced as an fundamental implies of analyzing human activity (Chevallard, 2019). A praxeology is a general model that links the practical dimensions (the practice) and the theoretical dimensions (the theory) of any human activity (Barbé et al., 2005). A praxeology consists of types of tasks, techniques, technologies and theories (Bosch, 2015) and can be written as the quadruplet [T/τ/θ/Θ] (Chevallard, 2019). The simplest praxeology in mathematics, as in other disciplines, consists of a task of some kind that is solved by a corresponding technique. This means the "practical block" or praxis is formed by the type of task, denoted by T, and the corresponding technique, denoted by τ, used to solve T (Barbé et al., 2005). The 'theoretical block' or logos consists of technology, denoted as θ, (the discourse on the techniques, such as how they work and what tasks they can solve), and theory, denoted by Θ (the general discourse that unifies and justifies technologies, both formally and informally). In other words, techniques for carrying out tasks are explained and justified by a 'discourse on the technique' called technology. The technology is the rationale or justification for the chosen technique – why does it work and where does its effectiveness come from? Taking this discourse to an abstract level yields mathematical theory, which validates the technological discourse and connects the entire praxeology (Bosch, 2015). The anthropological approach assume that any task, or the resolution of any problem, requires the existence of techniques, even though the techniques are hidden or difficult to describe (Barbé et al., 2005).

A mathematical praxeology can be conceptualized as a type of mathematical organisation (MO), where an MO consists of one type of task T and the corresponding technique τ (Bosch & Gascón, 2006). When a set of punctual MOs is explained by using the same technological discourse, they form a local mathematical organisation (LMO), characterised by its technology.

In ATD, we usually use the term 'didactic moments' to describe discernible moments in the study process (Chevallard, 1999, as cited in Barbé et al., 2005, p. 238f; Bosch et al., 2020): The moment of the first encounter with the type of task T is the moment of exploration of T, with the emergence of a first technique τ used to solve T; the moment of constructing the technological and theoretical block [Θ/Θ]; the moment to work on the praxeology; and the moment of refining the technique(s) and the institutionalisation of the entire praxeology produced [T, τ, Θ, Θ]; and lastly, the moment to evaluate the praxeology (Barbé et al., 2005). In this case, the notion of didactic moments is used to look into the potential of the moment of first encounter with T as the foundation for analyses of the textbook material.

It is a crucial principle for ATD researchers, when analysing any process of teaching or learning, to relate explicitly and critically to the mathematical content involved, in terms of its rationales in different institutional contexts. In line with Bolea et al. (2001) we define algebra as a tool to model intra- and extra mathematical systems through an algebraization process. Ruiz-Munzón et al. (2013), define school algebra as a process of algebraization, a practical and theoretical tool to carry out modelling activity related to any school mathematical praxeology.

To detect what kind of school algebra the first moment of encounter offers, we can use the three-stage model of algebraization (Ruiz-Munzón et al., 2013). In the three-stage model of algebraization, arithmetic can be identified as the domain of calculation programmes (CP). The first stage of algebraization occurs as learners consider the CP as a whole and not only as a process. The second stage is introducing letters as parameters and unknowns, to model the relationship between CPs. The third and last stage of the algebraization process appears when the number of arguments of the CP is not limited and the distinction between unknowns and parameters is eliminated (Ruiz-Munzón et al., 2013). In this way, the three-stage model of algebraization can be used as a tool to detect and analyse general levels in the school algebra to be taught (Bosch, 2015).

Knowledge taught by immersion

Didactic processes can be organised in many other ways than by simply developing one praxeology at a time, following the order of the six moments. For instance, one could organise first encounters with several different types of tasks without pursuing any deeper technical work, and only later come back to a systematic approach. The meticulous pursuit of all moments for one praxeology would, by contrast, reflect a more structured progression, which in some cases could also be prescribed by official documents regulating the teaching in more or less detail.

Similarly, textbooks could support the implementation of didactic moments corresponding to different praxeologies with more or less structured progression. We find it helpful to think of the different approaches using the analogy of teaching a foreign language: one can proceed systematically to introduce phrase structures, grammatical rules and so on, one by one, or, at the other extreme, one can follow an ‘immersion’ method, where the students are simply exposed to spontaneous language use in situations with native speakers. The same approaches could also be taken in textbooks – with language as well with mathematics. At the one extreme, one praxeology is developed at a time, through all six moments. At the other end of the scale, one would have a more unstructured meeting with types of tasks, techniques, etc., in different and possibly distant ‘natural’ situations – such as the immersion approach to language teaching. We can then talk of textbooks that are more or less strongly structured, and textbooks that are less structured and use an ‘immersion strategy’ for the organisation of the various moments.

Research questions

In order to gain knowledge about the transposition from curriculum to textbooks and the organisation of algebra in Danish and Japanese textbooks for middle school students, it is necessary to begin by analyzing how the first encounter with algebra is presented. Furthermore, the manner in which this first encounter is developed, and the theory used is also central to a comparative analysis. Based on the above, the following research questions have been formulated:

- How is school algebra transposed from national objectives to textbooks in Japan and Denmark?
- How is the very first introduction to algebra organized in Danish and Japanese textbooks for the middle school – for instance, what tasks appear in the moment of first encounter?
- How do the three levels of algebraization appear in this first introduction?
- How does the distributive law, as a central, specific element of algebraic theory, appear?
- And what is the potential for achieving the moment of constructing the technological and theoretical block?

Context of the cases to be compared

The Japanese Ministry of Education, Culture, Sports Science and Technology (MEXT) prepares the curriculum guidelines for Japanese primary and secondary school, with the outlines of objectives and content of mathematics at each level. Japanese curricula for primary school and junior high school consist of two levels of official programmes, a general course of study in mathematics, Chugakko Gakushu Shido Yoryo, and a teaching guide for the course of study in mathematics, Chugakko Gakushu Shido Yoryo Kaisetsu Sansu-Hen (MEXT, 2023). The official program and teaching guide includes the basic act and general goals of mathematics education, as well as an outline of the contents for teaching mathematics in a stepwise progression. All schools in Japan are required to use textbooks that have been evaluated and approved by the Ministry of Education. The textbooks used in public schools are selected by the local education council. The Japanese textbook Junior High School Mathematics: 1 (Isoda & Tall, 2019) is one of these required textbooks for lower secondary school.

The Danish Ministry of Education publishes Common objectives (Education, 2019), which consists of the competence-based objectives for primary to lower secondary school. The common objectives is divided into three parts, primary school grades 1–3 (students aged 7–9), middle school grades 4–6 (students aged 10–12) and lower secondary school grades 7-9 (students aged 13–15). The competence-based learning goals are generally described in a spiral, integrated structure, where the mathematical content areas are introduced and re-introduced during primary, middle, and lower secondary school with increasing levels of depth and sophistication (Stein et al., 2007). The majority of Danish mathematical textbooks refer explicitly to the common objectives, but there is no systematic evaluation of textbook materials. Danish textbooks are primarily developed by mathematics teachers, based on their own didactic ideas and personal experience. One of the most commonly used textbook materials is KonteXt+ (Alinea, 2023) KonteXt+ is a series of materials for grades 0–9 (Alinea, 2023). For grades 4 to 9, KonteXt+ is a set of materials consisting of a core book and a workbook, checklists for the core book and workbook, extra sheets for help, useful links, evaluation forms, and a description of how the content of each chapter contributes to the achievement of the national competence objectives for each level. Mathematics teachers can access all this material online. The students primarily use the core book and workbook.

Methodology

Our investigation is based on the Japanese Junior High School Mathematics: 1 and the KonteXt+ core books. First, we identify where the moment of first encounter with algebraic expressions appears in the books. This is done by analysing the structure of the material by reviewing the books' table of contents and locating chapters where the introduction to algebraic expressions is indicated or specifically stated. Then we analyse the selected chapter by exploring what type of task appears in this first encounter with algebraic expression, and the expected following moment of T emergence of a first technique τ used to solve T. We look at the potential to achieve the moment of constructing the technological and theoretical block $[\Theta/\Theta]$, and finally the moment of refining the technique(s) and institutionalisation of the entire praxeology $[T/\tau/\theta/\Theta]$, if possible. Second, we use the three-stage model of algebraization to analyse the examples identified in the first part to detect the level of algebraization. Then we look at how the distributive property is applied in the selected chapters. In order to answer the question of how school algebra is transposed from the national objectives to the textbook, we will discuss how the first moment of encounter with algebraic expressions relates to the curriculum and to what extent the textbook supports the immersion approach.

Analysis of the Japanese textbook

Structure of the Japanese textbook

The introduction to the textbook, Junior High School Mathematics: 1, provides an overview of how each chapter is structured, with specified descriptions of the content elements and task types used in the book. There is also a section for parents, a description of the overall format of the book consisting of the main text in the chapter, the end of the chapter, and the end of the volume as an overview of the LMO. After ‘How to Use This Textbook’, there is ‘How to Use Your Notebook’, with an explanation of how the student’s personal notebook should be used for recording the student’s learning. The student’s notebook is a central part of the teaching practice in Japanese school institutions. The students’ private work in the notebook is open for inspection during teaching, and the teacher might select some of the students’ work to show and share different ideas and solutions on the blackboard (Shimizu, 1999). The last part of the introduction consists of ‘Ways of Thinking Mathematically’ and includes examples of analogical, inductive and deductive reasoning and review from elementary school. In Japan, algebraic expressions with letters are taught in the sixth year of primary school (MEXT, 2023). Chapter 2 in Junior High School Mathematics: 1 is entitled ‘Algebraic Expression’ and is divided into three levels of subsections. The headings of the first level of subsections are ‘Algebraic Expression’ and ‘Simplifying Algebraic Expression’. The structure of the chapter and the relation between the subsections are presented in Table 1. The titles of the sections are similar to the content used in the textbook.

Table 1. The structure of Chapter 2 in Junior High School Mathematics: 1

Chapter	Subsection Level 1	Subsection Level 2	Subsection Level 3
Chapter 2 Algebraic Expression	Algebraic Expression	Mathematical Expression Using Letters	
		How to Write Algebraic Expression	How to Express Products
			How to Express Exponentiation
			How to Express Quotients
			How to Express Quantities
			Expressing Quantities Using an Algebraic Expression
		Value of the Expression (substitution of symbols by numbers)	
	Simplifying Algebraic Expressions	Linear Expression	Terms and Coefficients
		Simplifying Linear Expression	Addition and Subtraction of Linear Expression
			Linear Expression and Multiplication of Numbers
			Division of Linear Expression by Numbers
			Various Simplifications
		Using Algebraic Expression with Letters	

The chapter begins with a mathematical problem, followed by fundamental questions for the problem as an introduction to the new content of the chapter. The Japanese term for such a 'motivating problem' is hatsumon, which means 'asking a key question that provokes students' and refers to the teacher's act of teaching (Shimizu, 1999, p. 109). Hatsumon is not directly mentioned in the textbook, but there are key questions which, supports the development of central elements of the contents. The textbook does not provide the hatsumon itself, but present problems and questions that can be used as 'material' for the hatsumon. In this way, the first mathematical problem presented forms the foundation of the problem-solving process that leads to the subsequent moment of first encounter with algebraic expressions, which is described in more detail below. The chapter closes with 'Summary Problems', consisting of tasks for reviewing and consolidating the learned knowledge and also to support 'deep learning', which is content to extend the students' understanding of the chapter's contents.

Text elements supporting didactic moments

The opening problem of the chapter, which is in fact repeatedly returned to throughout the chapter, is called 'How many straws do we need?' The context is that a rectangular pattern is formed by joining straws of the same length side by side. For example, one can form two squares by using seven straws. Students are asked how many straws are needed to make four and 10 squares. The technique to solve these tasks could be to draw a model of the particular cases and then count the number of straws, or (as intended) to create a simple mathematical expression to model the situation. We are told that Yui used the math expression 1+3×4 to find the number of straws needed to make four squares. Then we are asked to explain her idea and apply her method to find the number of straws needed to make five, six, and 10 squares. Then Takumi's mathematical expression of 4+3×(4−1) is presented to find the number of straws needed to make four squares, and we have to explain his idea. Next, we must suggest a method different from those of Yui and Takumi and explain the idea behind it (Isoda & Tall, 2019, p. 61). This inductive work leads to the question: 'Using the same method as above, can you make a mathematical expression that can be used to find the number of straws needed to make any given number of squares?' (Isoda & Tall, 2019, p. 61). This is the moment of first encounter with algebraic expression in the Japanese textbook Junior High School Mathematics :1. This is the moment where algebra is introduced as a modelling tool to model a series of mathematical expressions in a general way by using an algebraic expression, which constitutes the transition from arithmetic to algebra. This shows how to organize the moment of the first encounter with the task t: Express the relationship between the number of squares and the number of straws to build the squares, which is of course a more general type of task, in which some number pattern is described using algebra. The moment of exploration of this task leads to the emergence of a first technique τ used to solve T. In the case of t, the mathematical expressions from the previous introductory work are used to model the relationship by generalising arithmetic expressions, leading to the mathematical expression 1+3×(number of squares). Letting a represent the number of squares, we get 1+3×a. The moment of constructing the technological and theoretical block [Ѳ/Θ] begins with the written formula and the sentence 'Such mathematical expressions with letters are called algebraic expressions' (Isoda & Tall, 2019, p.62). Then the moment of refining the technique(s) appears when we have to write the other mathematical expression as an algebraic expression and get 4+3×(a−1). The two equivalent expressions, 1+3×a and 4+3×(a−1), represent two different ways of 'seeing' and describing number patters and suggests a need to develop and compare the technique(s). In particular, we need ways to describe and recognize equivalent algebraic expressions and the insight that 'algebraic expressions using letters serve as both the method to find the number of straws, as well as representing the result we want to find' (Isoda & Tall, 2019, p. 63). This will also be central at the moment of institutionalizing the entire praxeology.

Level of algebraization

The introduction to the algebraization process begins when the student is asked, 'How many straws are needed to make four squares?' The techniques to solve the task are based on repeated addition, with four straws to form the first square and then three straws to form the subsequently squares, written as 4+3+3+3. This is a CP based on arithmetic, and when one changes the mathematical expression for the four squares to 4+3×3, will see the CP in more condensed form.

This efficient form of notation is the hallmark of algebra. It can help us see connections which were previously impossible to see. In this case, it is a step towards generalisation and preparation for the first level of algebraization. The first level of algebraization appears at the moment of refining the techniques by introducing letters to model the relationship between the different CPs, which in this case are the various mathematical expressions for the number of straws to model the squares. The last step of the algebraization process emerge with the statement 'Algebraic expressions using letters enable us to find the number of straws needed regardless of how many squares there are' (Isoda & Tall, 2019, p. 63). Equivalence is also introduced with the two expressions $(a+1)+2a$ and $4a-(a-1)$ on pages 82 and 83 in Isoda and Tall (2019) by using the unknown a. The algebraic expressions model the main problem with squares of straws from the beginning of the chapter, and initiates the algebraization process. Ideas such as substitution and solving equations are introduced when connecting these expressions with concrete tasks such as 'find the number of straws needed to make 50 squares'.

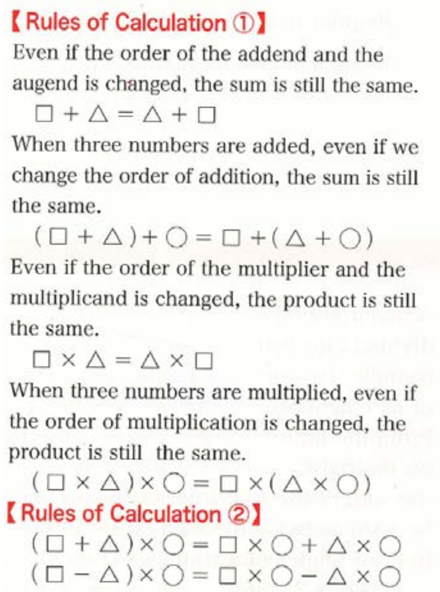
The idea of equivalence

In subsection 2, 'How to Write Algebraic Expressions' the aim is to learn how to express products and quotients as algebraic expressions by following the rules (Isoda & Tall, 2019, p. 65). How to express products is highlighted in a box entitled 'important'. The two important rules are that in algebraic expression one must remove the multiplication sign, and when multiplying numbers and letters, one must write the number in front of the letter (Isoda & Tall, 2019, p. 65). In this case, the rules express a convention. This explicit description of the algebra discourse is followed by a series of examples, such as $x \times (-4) = -4x$. In addition, there is a note that when multiplying two letters, one must write them in alphabetical order, for example $b \times a$

must be written as ab. In this context, rewriting the letter is related to the algebraic notation form, which is 'legal' because of the commutative property of addition. This is a situation where the construction of the technological block appears before the moment of the first encounter with the task T and the exploration of T through the selected examples. In that way, the explicit use of the algebraic condensed notation constitutes the institutionalization of the constructed praxeology, as the final moment. The explicit introduction of the algebraic rules continues with the statement that 'instead of writing 1a, remove the 1 and write a' (Isoda & Tall, 2019, p. 65). The explanation is followed by an additional frame with the equivalent expressions $1 \times a = a$, $(-1) \times a = -a$. On the one hand, this can be perceived as the moment of refining the technique(s) as part of the institutionalisation of the praxeology. On the other hand, it is an exploration of T, with the emergence of the first technique τ used to solve T. These equivalent expressions are necessary to deduce that $a + a^2 = (1 + a)a$ and $ab + a = a(b + 1)$, as an example. Then there is the exploration of T by solving the tasks $x \times 1$, $a \times (-1) \times b$, and $y \times (-0.1)$ by the corresponding techniques τ. This is a textbook example focusing on the logos part of the praxeology.

Distributive property

The first encounter with distributive property is part of previous teaching in arithmetic, namely the rules of calculation. In Junior High School Mathematics: 1, the calculation rules appear in 'Review – From Elementary School to Junior High School' – see Figure 1.



【Rules of Calculation ①】
Even if the order of the addend and the augend is changed, the sum is still the same.
□ + △ = △ + □
When three numbers are added, even if we change the order of addition, the sum is still the same.
(□ + △) + ○ = □ + (△ + ○)
Even if the order of the multiplier and the multiplicand is changed, the product is still the same.
□ × △ = △ × □
When three numbers are multiplied, even if the order of multiplication is changed, the product is still the same.
(□ × △) × ○ = □ × (△ × ○)
【Rules of Calculation ②】
(□ + △) × ○ = □ × ○ + △ × ○
(□ − △) × ○ = □ × ○ − △ × ○

Figure 1. Copy of Rules of calculation form Junior Hogh School Mathematics: 1, p. 10 (Isoda & Tall, 2019)

The rules are added that when numbers and quantities are expressed: one can use letters such as a or x instead of the square, triangle and circle symbols. The first encounter with distributive property was part of earlier work with arithmetic. In the first chapter, 'Positive and Negative Numbers', in Junior High School Mathematics: 1, the section 'Addition' contains the subsection 'Commutative and Associative Properties of Addition' (Isoda & Tall, 2019, p. 25). The aim of this subsection is to investigate whether the commutative property of addition and the associative property of addition rules for addition, learned in elementary school, also apply to positive and negative numbers. In section 3, 'Multiplication and Division' the commutative property of multiplication and the associative property of multiplication are explored (Isoda & Tall, 2019, p. 40). In subsection 3, the four operations are combined through calculus, and the distributive property, which holds for both positive and negative numbers, is explored (Isoda & Tall, 2019, p. 48). Figure 2 shows how the distributive property is modelled with equivalent expressions and a geometric figure.

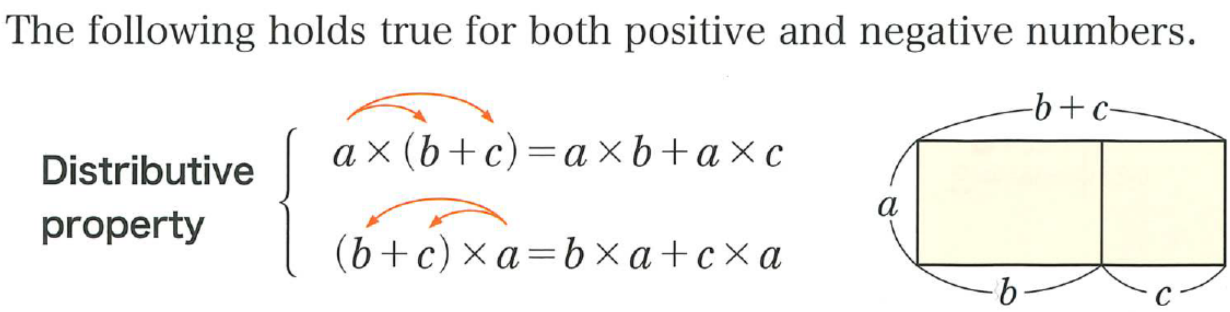


Figure 2. Copy of Distributive property in Junior Hogh School Mathematics: 1 (Isoda & Tall, 2019, p. 79)

This is the first encounter with distributive property in respect of both positive and negative numbers. In Chapter 2, 'Algebraic expression', the aim of the section 'Simplifying Algebraic Expressions' is to consider how to combine the terms of algebraic expressions (Isoda & Tall, 2019, p. 75). After an introduction to terms and coefficients, the moment of first encounter with the use of distributive property is to combine terms for the purpose of simply stating the algebraic expression. Then an exploration takes place with the example of $4x-6x=(4-6)x=-2x$, and the students must simplify the expressions as $5x+2x$ and $-y-4y$ (Isoda & Tall, 2019, p. 76).

Next, the technique to rearrange the terms and combine them with letters and numbers is introduced. The technique is used to simplify the task as $2x-12-6x+15$, among others. The section continues with the introduction to linear terms and expressions. In the subsection 'Linear

Expression and Multiplication of Numbers', the method of removing the parentheses using the distributive property is presented by reviewing Figure 2 (Isoda & Tall, 2019, p. 79). Then there is an exploration through examples and tasks where the student must simplify linear expressions, for example, $-2(4x+5)$ and $(1-6x)\times 3$, explicitly using algebraic notation form. This explicit way of applying the distributive property is the moment of constructing the technological and theoretical block [Θ/Θ]. The continuous expansion of the distributive property contributes to refining the technique and leads to the institutionalization of the entire praxeology, as the final moment.

The moment of first encounter with the distributive property takes place in elementary school. In Junior High School Mathematics: 1 there is an exploration of the property and the introduction of a corresponding technique by drawing knowledge from the distributive property learned in elementary school. Table 2 provides an overview of the stepwise progression of introducing distributive property in the Japanese textbook, Junior High School Mathematics: 1 (Isoda & Tall, 2019).

Table 2. Overview of the organization of the distributive property in Chapter 1 and 2 in the Japanese textbook.

Type of Task T	Technique τ	Technology θ	Theory Θ
Chapter 1			
Ex. p. 24 $(-1.2)+(-0.5)$ Ex. p. 25 Calculate the following a) and b) and compare the results. a)$(+5)+(-7)$ b)$(-7)+(+5)$	$(-1.2)+(-0.5)$ = - (1.2+0.5) = - 1.7 $(+5)+(-7)=-2$ $(-7)+(+5)=-2$	Addition of positive and negative numbers	Commutative Property of Addition $a+b=b+a$ $a,b \in Q$
T: Calculate a) and b) and compare the results. a) $a+b$ b) $b+a$	$\tau: a+b=b+a$		
Ex. p. 25 Calculate $(+11)+(-5)+(+9)+(-7)$	$(+11)+(-5)+(+9)+(-7)$ $=(+11)+(+9)+(-5)+(-7)$ $=(+20)+(-12)$	Change the order of the numbers using the commutative property. Find the sum	Commutative Property of Addition $a+b=b+a$ $a,b \in Q$ Associative Property of Addition

	$= +8$	of positive and negative numbers using the associative property.	$(a+b)+c = a+(b+c)$ $a, b, c \in Q$
T: Calculate $(+a) + (-b) + (+c) + (-d)$	τ: $(+a)+(-b)+(+c)$ $+(-d)$ $= (a+c) + (-c-d)$		
Ex. p. 32 Calculate $7+(-8)-5-(-4)$	$7+(-8)-5-(-4)$ $= 7+(-8)-5+(+4)$ $= 7-8-5+4$ $= 7+4-8-5$ $= 11-13$ $= -2$	The subtraction of positive and negative numbers is changing the sign of the number being subtracted and then adding it.	The commutative and associative property cannot be used for subtraction. However, by changing subtraction into an addition-only math expression, both commutative and associative property can be used.
T: Calculate $(+a) + (-b) - c - (-d)$	τ: $(+a)+(-b)-c$ $-(-d)$ $= a-b-c+d$		
Ex. p. 48 Calculate $(-5)\times\{(-4)+6\}$ Ex. p. 48 $12\times(\frac{1}{2}-\frac{1}{3})$	$(-5)\times\{(-4)+6\}$ $= (-5)\times(-4)+(-5)$ $\times 6$ $= -10$ $12\times(\frac{1}{2}-\frac{1}{3})$ $= 12\times\frac{1}{2}+12\times\left(-\frac{1}{3}\right)$ $= 6-4$ $= 2$	Calculate with positive and negative numbers, using the distributive property.	Distributive property of multiplication: $a(b+c) = ab+ac$ $a, b, c \in Q$
T: Calculate $(-a)\times\{(-b)+c\}$ $a, b, c \epsilon Q$	τ: $(-a)\times\{(-b)+c\})$ $= (-a)\times(-b)+(-a)$ $\times c$		
Chapter 2			
Ex. p. 79 Simplify $2(x+4)$	$2(x+4)$ $= 2\times x+2\times 4$ $= 2x+8$	Remove the parentheses, using the	Distributive property of multiplication: $a(b+c) = ab+ac$

T: Simplify linear expression $a(x+b)$	$\tau: a(x+b)$ $= a \times x + b \times x$ $= ax + b$	distributive property.	$a, b, c \in Q$
Ex. p. 76 Simplify $4x + 7 + 5x + 8$	$4x + 7 + 5x + 8$ $= 4x + 5x + 7 + 8$ $= (4+5)x + 7 + 8$ $= 9x + 15$	Rearrange the terms using the commutative property. Combine the terms with same letters and the terms with numbers using the distributive property.	Commutative property of addition: $a + b = b + a$ $a, b \in Q$ Distributive property of multiplication: $a(b+c) = ab + ac$ $a, b, c \in Q$
T: Simplify algebraic expression of the form $ax + b + cx + d$	$\tau: ax + b + cx + d$ $= ax + cx + b + d$ $= (a+c)x + b + d$		

Table 2 shows the step-by-step structure where techniques and theory from Chapter 1 are used in Chapter 2. In Chapter 2, letters are introduces to express the relationship between quantities by means of algebraic expressions. The explicit description in the textbook of the distributive property constitutes the moment of constructing the technological and theoretical block. By using the distributive property to simplify the algebraic expression, refining the technique is introduced. In this way, the praxeology of the distributive property forms a bridge between arithmetic and algebra, constituting the final moment of institutionalization.

Analysis of the Danish textbook

Locating the First Encounter with Algebraic Expression

To identify where the moment of first encounter with algebraic expressions appears in the KonteXt+ book series, we look at the headings of the chapters for the LMOs. In KonteXt+5 for grade 5, 11–12-year-old students, one of the headings is ‘Numbers and Letters’, indicating the first encounter with algebra. We will take a closer look at this chapter to locate the moment of first encounter with algebraic expressions. The chapter ‘Numbers and Letters’ is divided into sections related to various types of activities, and the subsections are in general named after their content. Table 3 presents the subsections of the chapter ‘Numbers and Letters’ in the Danish textbook KonteXt+5, according to the headlines and the online teacher’s guide to the material.

The headings ‘Introduction’, ‘Knowledge of’, ‘Exercises’ and ‘Reflection’ indicate what is central to each subsection. The sections ‘Scenarios’ and ‘Activities’ require more detailed description. A ‘scenario’ is a story or setting relating to the problem-based exercises. ‘Activities’ refer to mathematical problems, investigations and games which can be linked to the objective of the chapter. Below follows a more systematic review of the chapter.

Table 3. The structure of the chapter “Numbers and Letters” in KonteXt+5. The actual titles of the chapter and its sections are listed (except for the “Scenario” subsections which do not have headings).

Chapter	Subsection Level 1	Subsection Level 2
Number and Letters	Introduction	Picture and Classroom Conversation Activity Learning Goals
	[Scenario]	New Square in the Pedestrian Zone Thomsen’s Numbers Fencing
	Activities	Your Own Formula for Step Counting Figure Number The Angular Numbers Find Patters in the Numerical Table
	Knowledge of	Can you Calculate with Letters? Formulas and Arithmetic Expressions Formulas and Letters Number and Figure Patterns
	Exercises	
	Reflection	

In the introduction to the chapter, there is a picture with coloured balls and four questions related to the picture for class discussion. The first two questions are: ‘How many different coloured balls are in the photo?’ and ‘How would you name them (the balls) if you should use a letter?’

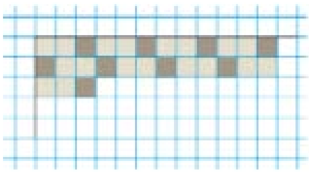
After the questions, there is a group activity, where four students are each given a card with information about a specific number; by using all four pieces of information, the students must determine the number. In the last part of the introduction to the chapter, a list of what the students will learn in the chapter are presented. The first four goals on the list are to learn that letters can represent different numeric values, to use formulas and arithmetic expressions, to calculate with letters as if they were numbers and to model simple everyday situations as equations (Andersen et al., 2019). The introduction provides insight into the diversity of task types and praxeologies included in the chapter; these will be further elaborated in the following sections.

Text Elements Supporting Didactic Moments

After the introduction, the chapter opens with a story, the scenario, about Anna who has a paving company and a job laying stones in a pattern.

Nyt torv i gågaden

Annas brolæggerfirma ALT I STEN har fået til opgave at lægge nye sten på Nytorv, som de har gjort det tidligere på Gammeltorv. Anna har valgt at blande lyse (L) sten og mørke (M) sten. Hun laver en model med 12 x 12 sten, som skal have sit eget mønster. Hun overvejer at lægge stenene på denne måde.



LLMLLMLLMLLM
MLLMLLMLL
LLM ...

Opgave 1

a. Tegn de fire første rækker af lyse og mørke sten på ternet papir.
b. Hvor mange mørke og lyse sten er der i hver af de 12 rækker?
c. Skriv rækkefølgen af lyse og mørke sten i første række med bogstaverne L og M fx LLMLL osv.
d. Skriv rækkefølgen af lyse og mørke sten i anden række med bogstaverne L og M.

Opgave 2

a. Hvorfor kan man skrive antallet af lyse og mørke sten i en række som 8L + 4M?
b. Hvis man kan skrive grundmønstret af lyse og mørke sten i første række som 2L + M, hvordan vil du så skrive anden række?
c. Hvis man kan skrive rækken af sten i første række som 2L + M + 2L + M + 2L + M + 2L + M, hvordan vil du så skrive anden række?
d. Hvis man kan skrive antallet af sten i første række som 4 · (2L + M), hvordan vil du så skrive anden række?

Figure 3. Copy of Model and collection ad tasks linked to the story of the new pedestrian square (Andersen et al., 2019, p. 128).

L represents the light-coloured stones, and M represents the dark-coloured stones. Anna’s model of the pattern consists of 12 x 12 stones. To extend the initially drawn model item 1.a, ask the students: ‘Draw on squared paper the first four rows of light and dark stones.’ Using the model of the stone pattern created in 1.a, the students can answer 1.b: ‘How many dark- and light-coloured stones are there in each of the 12 rows?’ by counting. In 1.c, the students are instructed: ‘Write the sequence of light- and dark-coloured stones in the first row, by using the letters L and M, e.g. LLMLL etc.’ This is the first encounter with the type of task T: Model a sequence by using letters. The model of the sequence is noted in the box on the left. The introduction of a first technique to solve T could be copying the list of letters from the box. The moment of exploration of T takes place when answering 1.d: 'Write the sequence of light- and dark-coloured stones in the second row, by using the letters L and M.' In this case, the model of the sequence is not complete, and a technique to solve T is required.

The question in item 2.a is: 'Why can you write the number of light- and dark-coloured stones in one row as 8L+4M?' This question is consistent with the type of task T: determine the number of elements (different stones) in a sequence. The technique to solve T could be first τ1, counting, and then τ2, use algebraic notation to write down the sum of the elements. Another technique to solve T could be τ3: write the sequence as addition, for example, L+L+M+L+L+M+L+L+M+L+L+M, or τ4: reduce the terms by applying the distributive property. The combination of τ1 and τ2 is expected to be the dominant technique, but central is that the exploration of T could lead to the introduction of different techniques to solve T. The moments of constructing the technological block, containing the algebraic notation form, and the theoretical block, by applying distributive property, follow more implicitly. To answer the initial question, 'Why can you write the number of light- and dark-coloured stones in one row as 8L+4M', the students have the opportunity to evaluate the entire praxeology.

In item 2.b, the basic pattern of the rows is modelled. In the box on the left, the first-row pattern is written as LLMLLMLLMLLM. Applying algebraic discourse and notation form, the sequence of letters would normally be interpreted as multiplication, with the product L8 M4 . In this case, the sequence is also modelled by addition to 2L+M+2L+M+2L+M, where 2L+M is the basic pattern. This is the first encounter with the type of task T: Model the pattern by an algebraic expression. We therefore consider that this is also the introduction to algebraic expressions. In this situation, the two types of tasks, T: Model a sequence by using letters, and T: Model the pattern by an algebraic expression, relate to different discourses on the technique and therefore different technologies.

The moment of exploration of T: Model the pattern by an algebraic expression, is through working with item 2.c: 'If you can write the sequence of stones in the first row as 2L+M+2L+M+2L+M, how would you write the second row?' The answer to the question is: M+2L+M+2L+M+2L+M+2L. If we apply the commutative property for addition, we will get the same solution as for the first row and have the opportunity to construct the theoretical blocks, logos.

The last item is 2.d: 'If you write the number of stones in the first row as 4(2L+M), how would you write the second row?' To answer the question, the students must accept that L and M are not numeric variables, but a 'stone unit' representing the colour of the stones. The basic pattern of the stones is written in the form of 2L+M, and the algebraic expression models the basic pattern repeated four times in a row. These different aspects are central for constructing the technological and theoretical blocks. The next page consists of variations of the previously presented types of tasks.

The Idea of Equivalence and Variables

The next two pages in the chapter relate to a story about a grocer called Thomsen. A person Jacob is sent to the grocer to buy apples, which cost 5 cents each, and the grocer Thomsen writes an equation on a piece of paper: $a \cdot 5 = 40$. In the first three items of exercise 1, the students must explain what 5, and 40 represent, but there is no explicit description of the form of notation in relation to current conventions. As noted previously, two of the most fundamental concepts in algebra are equivalence and variables. The verb ‘explain’ refers to the task to define the coefficient, the variable and the constant of the linear equation.

Item 1.d asks the question: ‘How many apples does Jacob buy?’ This is type of task T: solve linear equation of the form ax=b. This is the moment of first encounter with linear equations and the introduction of the exploration of T. The first technique to solve the task is expected to be τ: solve by substitution, according to the dominant epistemological model (Tonnesen, 2022). The exploration of T continues with the item 1.e: ‘What would the equation have looked like, if they had been bought for 1 dollar?’ (Andersen et al., 2019, p. 130). Here the student must change the constant from 40 to 100.

Exercise 2 is about a person named Inge who wants to buy pears. One pear cost 8 cents, and Inge has 48 cents in her purse. Item 2.a asks: ‘How many pears can Inge buy?’ This question can be solved by division. In item 2.b, the students is instructed: ‘Write the question as an equation’ (Andersen et al., 2019, p. 130). This is the moment of introducing the praxeology, despite the fact that constructing the technological and theoretical blocks remains. If we search for the moment of constructing the technological and theoretical blocks, we must go to the ‘Knowledge about’ section, where there is a description in Figure 4 of the notation form ‘when using letters’ (Andersen et al., 2019, p. 138).

1) Når man skriver 3a, mener man 3 · a – altså "tre gange a".
2) Når man skriver ab, mener a · b.
3) Når man skriver 2b + a, mener man 2b + 1a.
4) Når man skriver 2(a + b), mener man 2 · (a + b).

Figure 4. Copy of overview of the notational form (Andersen et al., 2019, p. 138)

Figure 4 presents several examples of correct algebraic notation. Next to the figure is written: 'One of the differences between calculation with numbers and calculation with letters is that you cannot find the result until you know which numbers should replace the letters.' This is followed by the statement: 'Many of the rules that apply to calculation with numbers also apply to calculation with letters' (Andersen et al., 2019, p. 138).

Bogstaveksempler	Taleksempler
$a + a + a = 3a$	$5 + 5 + 5 = 3 \cdot 5 = 15$
$a + a + b + b = 2a + 2b$	$4 + 4 + 7 + 7 = 2 \cdot 4 + 2 \cdot 7 = 22$
$2a + 5a = (a + a) + (a + a + a + a + a) = 7a$	$2 \cdot 9 + 5 \cdot 9 = (9 + 9) + (9 + 9 + 9 + 9 + 9) = 7 \cdot 9 = 63$
$5a - a = (a + a + a + a + a) - a = 4a$	$5 \cdot 6 - 6 = (6 + 6 + 6 + 6 + 6) - 6 = 4 \cdot 6 = 24$
$3(a + b) = 3a + 3b$	$3(5 + 12) = 3 \cdot 5 + 3 \cdot 12 = 51$

Figure 5. Copy of examples of calculation rules (Andersen et al., 2019, p. 138)

On the left side, examples with letters are presented, and on the right side, there are examples with numbers. The five examples cover variations of multiplication as repeated addition, bracket rules, the convention about notation and distributive property. The first example in Figure 5 is the same as the first example in Figure 4 and is an example of the consistent repetition form used throughout the material. Figure 5 also contains the first description of distributive property in the textbook for grade 5, based on an example. This diverse range of tasks, list of conventions in Figure 4 and examples of calculation rules in Figure 5 form the praxeological foundation for students' further development of algebra. To see how this foundation is developed and get insight into the progression, we will look at the textbook Kontext+7 for grade 7, students aged 13–14.

Level of algebraization

We will analyse the chapter 'Formula and Equations' in the textbook Kontex+7, which is organized in almost the same way as 'Numbers and Letters', shown in Table 3, with the addition of supplementary exercises, 'Calculate with Letters' and 'Solve an Equation'. The first scenario is named 'An Evening in Paris". It starts with a story about the mathematician François Viète and his search for 'a simple way to solve difficult calculation tasks' (Hansen et al., 2015, p. 92). This scenario introduces algebra as a tool to model arithmetic.

The problem in this scenario is as follows: ‘Three brothers, Oliver, René and Cyrano, must share 1025 silver coins. Oliver must have 275 coins more than Rene. Cyrano must have 150 coins less than Rene’ (Hansen et al., 2015, p. 92). Representing the share of René by x, the calculation becomes 1025=x+(275+x)+(x−150), which simplifies to 1025=3x+125. According to the context, Francois has the idea to calculate backwards to get: ‘Rene has 300 silver coins. Oliver has 575 silver coins and Cyrano has 150 silver coins. And “Viola [sic]! Francois invented the modern equation’ (Hansen et al., 2015, p. 92).

The problem requires a process of calculation. In this case, the first step of algebraization appears when the story about the brothers is modelled by an equation representing a relationship between the CPs. The model of the story provides an opportunity to consider the CP as a whole and not only as a process. In the modelling process, x is introduced as an unknown to model the relationship between the CP and then the equation is simplified to 1025=3x+125 which is the first level of the algebraization process (Ruiz-Munzón et al., 2013).

After this introduction, an example of how to solve an equation in three steps is presented, Figure 6, and the students must explain, in their own words, what happens in steps 1) to 3).

1) $1025 = 3x + 125$
2) $900 = 3x$
3) $300 = x$

Figure 6. Example of how to solve an algebraic expression in three steps (Hansen et al., 2015, p. 92)

To explain and defend each step, the technology of the coefficient, the variable and the constant of a linear equation is a part of the praxeology. Knowledge about equivalence and the use of the equal sign as expressing an identity represents the level of theory. Then, three linear equations, 87=12x+45, x+73,44=89,22, 3x+175=238, must be solved using the described technique and can be described as technical work.

Later in ‘An Evening in Paris’ it is stated that sometimes it is faster just to guess (by replacing the unknown by one or more numbers) to solve the equation. This is the moment of introducing substitution as a technique to solve linear equations. This is followed by eight equations to solve to consolidate the technique.

The distributive property

As shown in Figure 5, there is a description of distributive property in KonteXt+5. An almost identical Figure 7 is found in the section ‘Knowledge about’ in KonteXt+7. The first thing on the ‘Knowledge about’ page is the word ‘algebra’, which is defined as calculation with letters. In addition, we learn that ‘The calculation rules known from numbers can also be used when calculating with letters’ (Hansen et al., 2015, p. 102). This abstract theoretical statement is supported by the examples in Figure 7.

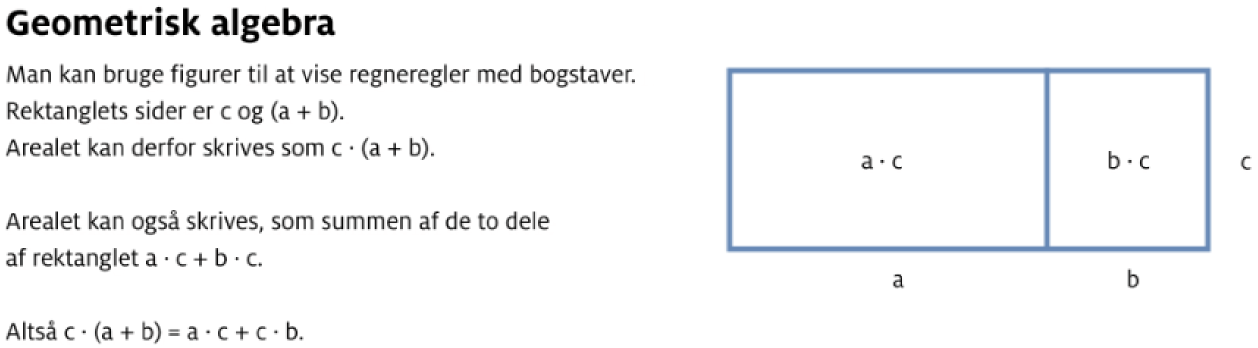
$a + a + a = 3 \cdot a$	$2 + 2 + 2 = 3 \cdot 2$
$a + b = b + a$ og $a \cdot b = b \cdot a$	$2 + 5 = 5 + 2$ og $2 \cdot 5 = 5 \cdot 2$
$a \cdot (b + c) = ab + ac$	$2 \cdot (5 + 7) = 2 \cdot 5 + 2 \cdot 7$
$a + (b + c) = a + b + c$	$2 + (5 + 7) = 2 + 5 + 7$
$a - (b + c) = a - b - c$	$2 - (5 + 7) = 2 - 5 - 7$
$1a = a$	$1 \cdot 7 = 7$

Figure 7. Copy of examples of calculation rules from KonteXt+ (Hansen et al., 2015, p. 102)

These six examples in Figure 7 cover a somewhat unstructured variety of algebraic identities: multiplication by 3 as addition by a number with itself three times, commutative properties for addition and multiplication, the distributive law, bracket rules (including a form of the associative law for addition) and a notational convention [1a means $1 \cdot a$ which is a]. The right column illustrates that these identities hold with numbers replacing the letters, and thus provides examples of the theoretical statement cited above (when reading from right to left). This is the first time the textbook presents a general form of distributive property.

In KonteXt+8 (grade 8), another example of distributive property is presented. In the chapter called ‘Formula and Equations’, under the section ‘Knowledge about’, algebra is described as the language of mathematics: ‘Working with letters as symbols for unknowns and variables is central in algebra’ (Hansen et al., 2016, p. 94). No further explanation of what defines unknowns and variables is given. A more general description follows: ‘The rules for calculation in algebra are often similar to the rules for calculation with numbers’ (Hansen et al., 2016, p. 94). Then a table with examples is presented, similar to the examples in Figure 5 and Figure 7. This demonstrates the repeated use of similarity between ‘rules’ in arithmetic and algebra and the presentation of distributive property exemplified in the same, but not exhaustive, way, throughout the grades.

To describe the rules for calculation, which include distributive property, in a more general way, geometric models are used in KonteXt+8.



Geometrisk algebra

Man kan bruge figurer til at vise regneregler med bogstaver.
Rektanglets sider er c og (a + b).
Arealet kan derfor skrives som c · (a + b).

Arealet kan også skrives, som summen af de to dele af rektanglet a · c + b · c.

Altså c · (a + b) = a · c + c · b.

Figure 8. Copy of geometry model of distributive property (Hansen et al., 2016, p. 94)

‘Geometrical Algebra’ is part of the section ‘Knowledge about’ and includes the following statement: ‘You can use geometric figures to show rules for calculating with letters’ (Hansen et al., 2016, p. 95). It is not pointed out that this is in a way a special example, since we have to assume that a,b,c>0. It is implicit that the area of the boxes is the product of the side lengths, and we make use of the fundamental property that areas are additive. Usually this is shown by using distributive property, not the other way around. From an algebraic point of view, it could be more correct to say that the distributive law agrees with the algebraic model of a geometric identity (of areas, also involving lengths). This method of modelling distributive property but also commutative and associative property by geometric models is repeated in the KonteXt+ series.

Comparison and discussion

The analysis of the Japanese and Danish textbooks can be organized according to three themes:

1. Identifying the moment of first encounter with algebraic expressions in the Japanese and Danish textbooks, and the relation between the LMO and curricula;
2. The mathematical praxeologies found in the material, which include the respective levels of algebraization and explicitness; and
3. The didactic approaches that seem to be suggested and supported by the textbooks.

Curriculum organization and the introduction of algebra

In the Japanese textbook, chapters and sections are consistently named with reference to their content and objectives. The analysed chapter, ‘Algebraic Expressions’, shows a systematic structure with clear mathematical sub-objectives, as presented in Table 1. In the Danish textbook, the name of the chapter also relates to the mathematical content, but the sections are divided according to a large variety of task types and activities, as presented in Table 3.

The differences in the structures of Tables 1 and 3, and the differences between the Japanese and Danish textbooks in general, reflect the structures of the respective national common objectives. The large variety of task types and activities presented in different contexts found in the Danish textbook represents an ‘immersion strategy’, where praxeologies are developed over time. This is in line with the spiral and integrated Danish curriculum, where mathematical competences and praxeologies are developed over several years while revisiting the same content repeatedly. The stepwise and structured approach in the Japanese textbook reflects the stepwise structure of the Japanese curriculum, with a clear progression with the outlines of the objectives and content of each mathematical level.

This connection between the organization of textbooks and the respective curricula can also be illustrated by our analysis of the introduction to distributive property. In the Japanese textbook, distributive property in arithmetic forms the basis for distributive property in algebra and constitutes an extension of theory. This means there is a theoretical progression and connection from arithmetic praxeologies to algebraic praxeologies. This stepwise modular approach has, according to Stein and Kim (2006), the implication that subsections cannot be separated and reconstructed into other configurations without losing efficiency in goal achievement. In this case, distributive property in arithmetic must be well established praxeology with [T/τ/ θ/Θ] before distributive property in algebra is presented. A reordering of the modular form could therefore lead to loss of theoretical coherence, by not developing praxeologies in their logical order.

The spiral structure of the Danish curriculum is reflected in the KonteXt+ series, where distributive property appears several times, in somewhat different forms, in the various grades. The distributive law is the only field axiom that links addition and multiplication, and consequently it is crucial in many ways in school arithmetic and algebra. In the Danish textbook, the introduction to distributive property is difficult to locate precisely, because it emerges in a variety of special cases or ‘rules’, which are listed and exemplified in several different sections and in different grades. In this way, the distributive law for arithmetic and that for algebra are intertwined, if not almost a merged praxeology. This gradual and, to some extent, repetitive approach is consistent with the spiral philosophy of the curriculum, according to which students are expected to acquire general principles such as the distributive law over time, as they appear in special cases and in task types of increasing difficulty. Stein and Kim (2006) argue that in spiral and integrated curricula, knowledge and skills (more or less, theory and techniques) are linked together, and because they are difficult to separate, they must be taught in similar ways over the years.

Level of algebraization and explicitness

In terms of the three-stage model of algebraization (Bosch, 2015), our analysis above demonstrates that only the first level of algebraization appear in the Danish and Japanese textbooks. A main difference is that hatsumon connects the algebraization process in high school mathematics, while the first level of algebraization in KonteXt are developed through different examples and exercises. This can also be linked to the difference between the curricula, as explained above.

In the Danish textbook, the term ‘rules’ is used to refer to both substantial properties and notational conventions. For instance, the fundamental commutative law ab=ba for multiplication is a level of theory Θ, where the convention to write a·x rather as ax is technology θ. In the Danish textbook, both are presented as ‘rules’, with no distinction made between technology and theory. By contrast, in the Japanese textbook, there are explicit distinctions between tasks, techniques, technologies and theories with a clear description of algebraic assumptions, conventions and results. As an example, there is a clear connection between distributive property in arithmetic and in algebra and the explicit description of the notation form as convention in ‘algebra discourse’. The Danish textbook material has a more implicit approach to central algebraic principles, such as distributive property. As illustrated above in Figures 5, 7 and 8, distributive property is applied in various examples, but its theoretical description and status remain implicit. This implicit approach can also be explained by the spiral and integrated structure of the curriculum, where the mathematical content is revisited and integrated over the years. This process is in line with Gravemeijer and Terwel (2000), who state that central algebraic assumptions on commutative, associative and distributive property might emerge as a part of mathematizing and the process of organizing the subject matter (Gravemeijer & Terwel, 2000). The same assumption about the mathematizing process may also apply to the idiosyncratic use of symbols in KonteXt+, where a repeated disposition for the conventional compact notation form might entail adaption by students, over time, to acquire important conventions.

The Japanese explicitness can also be seen in the headings of sections. This explicitness of content and learning goals is evident for the student during the learning process. The Danish textbook also contains learning objectives in the introduction to a chapter, but the connection between the type of tasks and corresponding techniques, and the level of theory, are present in a more implicit form.

It is also worth noting the structured focus on language in the Japanese textbook. There is an explicit description of ‘coefficient’ and ‘linear term’ before introducing ‘linear expression’; this is an example of the explicit stepwise development of the praxeology. The explicit use of the mathematical terms can be considered analogous to language learning by grammatical accuracy. The Danish textbook makes more use of commonsense terms, for example, ‘calculation rules’ in Figures 5 and 7, which are used to refer to both mathematical properties and conventions for algebraic notation.

The didactic approach

The didactic processes are organized in different ways in the Japanese and Danish textbooks. In the Japanese textbook, the introduction of algebraic expression is clearly located due to the LMO. In KonteXt+, the didactic moments include work on several types of tasks, where the (hatsumon) work with the initial problem has the potential to generate all six moments in the study process.

The use of the metaphor ‘Algebra – the language of mathematics’ in KonteXt+ describes the textbook’s different approach to the acquisition of algebra. Models for language learning can roughly be placed on a continuum, with content-driven models at one end and language-driven models at the other end (Snow, 2001). The prototypical content-based approach is the immersion model of foreign language education. If we use the same continuum for the Danish and Japanese textbooks, we could place KonteXt+ close to a pure immersion model, with ‘immersion through examples’, where the Japanese High School Mathematics: 1 is more like a pure ‘language-driven’ approach.

Conclusion

The analysis of Japanese and Danish textbook material shows clear links to the respective curricula. The Japanese curriculum uses a stepwise modular progression, where the content areas are built on previous mathematical foundation. The Danish curriculum is based on competences and has a spiral structure, where the content areas are revisited and expanded over the grades. A detailed praxeological analysis of the Danish curriculum is not possible, given that it provides only broad and vague guidance on the content that should (or rather can) be taught. The Japanese stepwise modular form of curriculum is also evident in the introduction to algebra in the Japanese textbook. The first introduction of algebraic expressions can be located quite precisely and consists of a task type where the students must model arithmetic relations using an algebraic expression. In the Danish textbook, the first encounter with algebraic expression is more difficult to locate. The transition from arithmetic to algebra is more fluid, and the introduction of letters in algebra is first linked to units and then to terms and variables in the KonteXt+ material.

Both textbooks include work with distributive property. The Japanese material has an explicit and theory based approach, whereas the Danish textbook material has a more implicit and example-based approach. This difference between the explicit and implicit approaches is also visible in the work with the algebraic notation form. The Japanese material has an explicit description of how to use the algebraic notation form properly, whereas the implementation of the algebraic notation form in the Danish textbook material is more implicit and presented at the same time as the work on tasks.

In their introduction to algebra, both textbook materials include exposure to the first level of algebraization. When applying the theory of didactic moments to look at the potential of the textbooks to support technico-technological moments, we see that the Japanese textbook material primarily builds one praxeology at a time, while the Danish textbook material – through its numerous activities – leads to work on several praxisblocks simultaneously, possible long before construction of the logos blocks. In this way, the Danish textbook material's approach to the introduction of algebra can be compared to the approach to language learning described as 'teaching through immersion', with exposure to a large variety of task types which can later be consolidated at the level of theory. The connection between curriculum structure and textbook material is clear when we look at the LMO, and it is therefore interesting to compare textbook material from countries with different curriculum structures.

References

Alinea, E. (2023). KonteXt+, Basic system grade 0. – 9. Retrieved from https://www.alinea.dk/matematik/grundsystemer-og-portal/produkt/kontext

Andersen, M. W., Dalsgaard, R. S., Hessing, S., & Lindhardt, B. (2019). *KonteXt+5, Kernebog/Web* (B. Lindhardt (ed.); 1. digital). Alinea. Retrieved from https://kitaboo.alineadigital.dk/reader/alinea_5.0/index.html

Artigue, M., & Winsløw, C. (2010). International Comparative Studies on Mathematics Education : a Viewpoint From the Anthropological Theory of Didactics. *Recherches En Didactique Des Mathématiques, 31*(1), 47–82.

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice: The case of limits of functions in Spanish high schools. *Educational Studies in Mathematics*, 59(1–3), 235–268. https://doi.org/10.1007/s10649-005-5889-z.

Bolea, P., Bosch, M., & Gascón, J. (2001). La transposición didáctica de organizaciones matemáticas en proceso de algebrización: el caso de la proporcionalidad. [The didactic transposition of mathematical organisations in the process of algebraization: the case of proportionality]. *Recherches en didactique des mathématiques,* 21(3), 247-304. (Translated by Deepl)

Bosch, M. (2015). Doing research within the anthropological theory of the didactic: The case of school algebra. In S. J. Cho (Ed.), *Selected regular lectures from the 12th International Congress on Mathematical Education* (pp. 51–69). Switzerland: Springer International Publishing.

Bosch, M., Chevallard, Y., Garcia, F. J., & Monaghan, J. (2020). Introduction: An invitation to the ATD. Bosch, M., Chevallard, Y., Garcia, F. J., & Monaghan, J. (Eds.) In Working with the anthropological theory of the didactic in mathematics education. *Routledge*. https://doi.org/10.4324/9780429198168

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin, 58*, 51–65.

Bruner, J. S. (1960). Introduction. In *The Process of Education* (pp. 1–32). Harvard University Press.

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education, 12*, 71–114.

Chevallard, Y., & Bosch, M. (2014). Didactic transposition in mathematics education. In: Lerman, S. (eds) Encyclopedia of Mathematics Education. Springer, Dordrecht. https://doi.org/10.1007/978-94-007-4978-8_48

Education, D. M. of. (2019). *Common objectives: Mathematics. National Agency for Education and Quality. Retrieved from* https://www.emu.dk/grundskole/matematik?b=t5)

Education, D. M. of. (2022). Joint development of mathematics. National Agency for Education and Quality. Retrieved from https://www.uvm.dk/aktuelt/nyheder/uvm/2022/sep/220921-ekspertgruppe-er-klar-med-anbefalinger-til-at-styrke-matematikfaget

Eriksson, H. (2022). Teaching algebraic thinking within early algebra—A literature review. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)* (pp. 481–488). ERME.

Gravemeijer, K., & Terwel, J. (2000). Hans Freudenthal: A mathematician on didactics and curriculum theory. *Journal of Curriculum Studies*, *32*(6), 777–796.

Hansen, N. J., Christensen, M. S., Lindhardt, B., & Thomsen, H. (2016). *Kontext+ 8 Kernebog/Web* (1. digital). Alinea. Retrieved from https://kitaboo.alineadigital.dk/reader/alinea_5.0/index.html

Hansen, N. J., Johnsen, L. B., Lindhardt, B., & Thomsen, H. (2015). *Kontext+ 7 Kernebog/Web* (1. digital). Alinea. Retrieved from https://kitaboo.alineadigital.dk/reader/alinea_5.0/index.html

Isoda, M., & Tall, D. (2019). *Junior High School Mathematics : 1*. Gakko Tosho.

Jessen, B. E., & Winsløw, C. (2017). Matematikbroen: brobygning for elever gennem efteruddannelse for laerere. *MONA - Matematik- Og Naturfagsdidaktik*, *3*, 39–59.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). NCTM.

Kjeldsen, C. C., Kristensen, R. M., Bindslev, S. G., & Christensen, A. A. (2019). Matematik og natur/teknologi i 4. klasse: Resultater af TIMSS-undersøgelsen 2019. *Aarhus Universitetsforlag*.

Ministry of Education Culture, Sports, Science and Technology (MEXT)(2023). *Chugakko Gakushu Shido Yoryo Kaisetsu Sansu-Hen* (2023, January)Retrieved fromhttps://www.mext.go.jp/en/index.htm

Mullis, V. S., Martin, M. O., Foy, P., Kelly, D. L. & Fishbein, B. (2019). TIMSS 2019 International Results in Matematics and Science. International Association for the Evaluation of Educational Achievement (IEA). Retrieved from TIMSS 2019 International Reports – TIMSS & PIRLS International Study Center at Boston College

Pepin, B., & Gueudet, G. (2018). Curriculum resources and textbooks in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 172–176). Springer. https://doi.org/10.1007/978-3-030-15789-0_40

Ruiz-Munzón, N., Bosch, M., & Gascón, J. (2013). Comparing approaches through a reference epistemological model: the case of school algebra. In B. Ubuz, Ç. Haser & M. A. Mariotti (Eds), *Proceedings of the 8th Conference of the European Society for Research in Mathematics Education* (pp. 2870–2979). Middle East Technical University. http://cerme8.metu.edu.tr/wgpapers/WG16/WG16_Bosch.pdf

Schifter, D., Monk, S., Russel, S. J., & Bastable, V. (2008). Early Algebra: What does understanding the laws of artithmetic mean in the elementary grades? In J. J. Kaput, D. W. Carraher, & M. L. Blaton (Eds.). *Algebra in the Early Grades* (2014th ed., pp. 413–449). Routledge.

Shimizu, Y. (1999). Aspects of Mathematics Teacher Education in Japan: Focusing on Teachers' Roles. *Jounal of Mathematics Teacher Education*, *2*, 107–116.

Snow, M. A. (2001). Content-Based and Immersion Models for Second and Foreign Language Teaching. In M. Celce-Murcia (Ed.), *Teaching English as a Second Foreign Language* (Third Edit, pp. 303–319). Heinle & Heinle.

Steen, N., Madsen, M. S., & Højgaard, T. (2020). Potentialer og begrænsninger ved anvendelse af lærebøger i matematikundervisningen: resultater fra et systematisk review. *Nordic Studies in Mathematics Education, 25(2)*, 5–28.

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences students learning. In *Second handbook on research of mathematics teaching and learning* (Vol. 1, Issue Febrar, pp. 319–370).

Tesfamicael, S.A., & Lundeby, Ø. A. (2019). A comparative study of Norwegian and Ethiopian textbooks: the case of relations and functions using anthropological theory of didactics (ATD). Universal Journal of Educational Research Vol. 7(3), pp.754-765. http://doi.org/10.13189/ujer.2019.070315

Tonnesen, P. (2022). Diagnostic test tool based on a praxeological reference model to examine students' technical and theoretical algebra knowledge. In Hodgen, J., Geraniou, E., Bolondi, G. & Ferretti, F. (Eds.) (2022). *Proceedings of the Twelfth Congress of the European Society for Reasearch in Mathematics Education.* (CERME12). Free University of Bozen-Bilzano and ERME.

Trouche, L., Gueudet, G., & Pepin, B. (2020). Documentational approach to didactics. In Lerman, S. (eds), *Encyclopedia of mathematics education* (pp. 237-247). Springer International Publishing. https://doi.org/10.1007/978-3-319-77487-9_100011-1

Paper III

Conditions and constraints for using a foreign textbook to support the transition for arithmetic to algebra.

Abstract

The transition from arithmetic to algebra is a major challenge in lower secondary school mathematics. Mastering algebra as a modelling tool and especially the use of algebraic expressions is necessary for students' further mathematical education. Textbook material is a central resource for mathematics teachers to teach school algebra. In this study, we investigate the conditions and constraints for using a translated textbook chapter on early algebra from Japan in a Danish lower secondary school, based on an earlier diagnosis of the state of algebra in the latter institution. The intervention suggests that explicit, step-by-step instructional material from Japan can indeed contribute to enhance both students' and teachers' knowledge of basic algebra. Through detailed classroom observations and teacher interviews, it becomes evident that while the Japanese material offer clear advantage in terms of conceptual clarity and structures progression, it is also demanding for teachers to integrate the material into an implicit, spiralling curriculum.

Keywords: Textbook impact, School algebra, Praxeology.

Introduction

The purpose of this study is to describe the conditions and constraints of using a Japanese textbook chapter in Danish Lower secondary school. What impact does the textbook have on teaching and learning school algebra?

Japan and Singapore are among the few countries that have consistently demonstrated top-10 levels of achievement in international mathematics assessments. This is also the situation in the PISA 2023 survey. The results reveal that Japan and Singapore are among the five OECD countries to have achieved both above-average results and positive progress between 2018 and 2022 in mathematics performance (OECD, 2023). A commonality between Singapore and Japan is the utilisation of mathematics textbook material, which is extensively based on empirical research, and to have a system of official quality screening of textbook material, as a condition for use in school (Yoshikawa, 2008; Cheng & Yeo, 2022). In these countries, as in other countries, textbook material remains the central resource for the teaching of mathematics (Gueudet et al., 2012; Cheng et al., 2021). It is generally acknowledged that teachers rely on textbook material to guide their teaching (Remillard, 2010). Therefore, it is tempting to investigate if textbook materials from countries like Singapore and Japan can contribute to improve teaching in other countries. Obstacles include the need for translation, and different curricula and institutional contexts. What are the implications of using textbook material from another national curriculum and school context than the one in which they teach?

Previous studies have examined the use of these thoroughly tested materials in other school contexts. (Jackiw et al., 2016; Lindorff et al., 2019). The aim was to ascertain whether the textbook material could contribute to raising students' overall level. The emphasis on the application of new curriculum material is also the primary strategy for improving mathematics education (Remillard, 2005). The efforts to implement "Singapore math" using its textbook material have been quite widespread in both the UK and United States, however with modest effect on student achievement (Jacci et al., 2016, Lindorff et al., 2019).

The objective of this study is to identify the conditions and constraints associated when applying a Japanese textbook chapter in Denmark to teach a specific mathematical content area (introductory algebra) which is known to be particularly problematic in Danish school. (Author, 2024a). The hypothesis is that by focusing on a single and problematic subject area, and by paying attention to official goals in that area, we might find more nuanced results concerning the possibility of using the material, beyond the evident obstacle of adopting an entirely new curriculum.

School algebra as an area of focus

From arithmetic to algebra: Transition challenges in Danish school context

The national "common core" for mathematics in Denmark consist of competence-based learning goals with a spiral integrated structure (Ireland & Monthan, 2020). This means that the mathematical content areas are introduced and repeatedly returned to throughout primary and lower secondary school, progressing in depth and sophistication (Author, 2024b). In the national curriculum for school mathematics in Denmark the overall goal for arithmetic and algebra in lower secondary school (from grade 6 to, and including, grade 9) is: “The students can use rational numbers and variables in description and calculations” (Education, 2024). Algebra as subject area is divided into three main sections: Equations, Formulae and algebraic expressions, and Functions. An epistemological examination of school algebra in Denmark after lower secondary school led Cosan (in press) to posit that there are three types of tasks represented in the national goals, textbooks and exams: setting up an algebraic model, based on numerical information, substituting in an algebraic model and rewriting an algebraic model. The difficulties which students experience when it comes to technical skills in algebra and encountering these three types of tasks, are an integral component of a recent diagnosis of school algebra in Denmark (Author, 2024a). It reflects an international trend of dis-algebraization and atomisation of school algebra (Bolea et al., 2003). In his early works, Chevallard described a state of affairs in French school, where the rules for manipulating algebraic expressions had become unmotivated drills, and rewriting becomes an end to itself, rather than a means to solve problems (Chevallard, 1985, 1989). Our previous analysis of the current situation in Denmark suggests that these skills are no longer extensively taught. Moreover, the weakness of theoretical dimensions in the teaching of arithmetic (in particular, fractions) also presents itself a challenge in the transition from arithmetic to algebra, as it leads to limited potential to draw upon general arithmetic properties in the introduction of basic algebra (Author, 2024a). The commutative, associative and distributive law are theoretical aspects of arithmetic, and may also serve as important foundations for school algebra, for instance when working with equations (Bruner, 1960). Modelling and applying such arithmetic properties are potentially a crucial part of school algebra because they provide the basis for exploration and generalisations of patterns and principles in arithmetic (Schifter et al., 2008).

Curriculum and textbook material as a resource

In general, the subject of mathematics, and school algebra in particular, is associated with the extensive use of textbook material. The field of research into the content and transition from primary to lower secondary school in school algebra has been growing since the 1990s (Kieran, 2007, 2022). The first encounter with algebra and especially the transition from arithmetic to algebra can be inferred form the curriculum and textbook material (Author, 2024b). Research has demonstrated that the way students are introduced to key aspects in their textbook - like properties of the equal sign - is of critical importance (Li et al., 2008). It is reasonable to posit that this also applies to other key aspects of school algebra: the use of symbols that are different in arithmetic and algebra (Kieran, 1990), how theoretical elements, as the commutative, associative and distributive law of algebra are introduced etc.

In Denmark, mathematics textbooks are produced by individuals with an interest in mathematics, and the material is not subject to quality control or requirements to follow national guidelines. Schools and municipalities can enter into purchasing agreements; in some cases, it is left to the individual teacher to select the material to be used in teaching. One of the most common textbook materials in Denmark is KonteXt. Each chapter has a theme where many types of task and associated techniques appear. The learning process entails the students practising each technique on numerous occasions in a variety of contexts. The books thus deploy a spiral and implicit approach to mathematical theory, where the repeated use of techniques provides an opportunity to experience the theoretical connections and structure (Steiner, 1987). This is in line with the national curriculum: after grade 3, the student should have discovered the rules of arithmetic and have knowledge of the relationship between the four operations (addition, subtraction, multiplication and addition, cf. Education, 2024). It is not explicitly described what relationships and rules the student should learn. By contrast, the Japanese curriculum has a more linear and stepwise approach (Ireland & Mouthaan, 2020).

This motivates our interest in Japanese textbook material and our choice of experimenting an excerpt of the chapter “Algebraic Expression” from *Junior High School Mathematics:1* (Isoda &Tall, 2019). The chapter begins with a “launch problem” which is frequently returned to in the chapter, to motivate and build up algebraic tools for problem-solving (Author, 2024b). The problem is “How many straws do we need?”, where a rectangular pattern is formed by joining straws of the same length side by side. The problem can be solved by students in multiple ways, the techniques to solve the tasks are introduced and justified one by one, in a stepwise structure. This approach also ensures that the foundation of arithmetic is established before the introduction of algebraic elements, and enables the construction of theoretical discourse, where algebra models arithmetic. To illustrate this, the basic algebraic properties, commutative, associative, and distributive law are explicit described in the first encounter with algebraic expressions and are linked to the properties in arithmetic (Author, 2024b).

We want to determine the extent to which explicit, and more linear step-by-step material can be used in a school context otherwise based on implicit and spiralling curriculum, to reinforce theoretical elements of teaching and thereby connections among otherwise fragmented elements of mathematical practice.

Praxeology and praxeological change as analytical framework.

In ATD, the production, acquisition and dissemination of knowledge is interpreted as human activity, modelled by praxeologies (Chevallard, 1999, 2019). Praxeologies furnish a general modelling tool which links a “practical block”, *praxis* and a “theoretical block”, *logos*. The practical block consists of *type of tasks* and the corresponding *technique* to solve the task type. The theoretical block consists of *technology* and *theory.* Technology is the discourse on the techniques, such as how they work and what tasks they can solve. The theory validates the technological discourse and thus confirm the entire praxeology (Bosch, 2015). In this way, a praxeology consists of the four T’s type of task, a technique, technology and theory (Bosch, 2015). This anthropological approach assumes that every task or solution to a problem is based on the use of techniques, even if these techniques are hidden or difficult to describe (Barbé et al., 2005). Praxeologies may also occur in larger systems in which multiple practice blocks share the same technology and theory (Chevallard, 1999). Mathematical praxeology is a useful model for describing mathematical activity, and in particular, algebra. Algebraic tasks and the corresponding techniques for solving them are linked to the technology used, as well as the specific compact notation form employed in algebra. This allows us to differentiate between discourses that are theoretically informed and those that are rooted in tradition. Praxeologies are constructed over time and constituted through activities. This implies that a praxeology does not possess a fixed form; rather, it is a dynamic entity that can be represented and modelled again. To describe this praxeological reconstruction, we can use the concept of *praxeological change* (Putra, 2019). The process of praxeological change requires students to redevelop or reconstruct larger complexes of mathematical practice, technologies and theories. This is not merely an individual endeavour; it is a process in which the praxis and knowledge blocks of all students must adapt over time to a new institutional standard of praxeologies. ATD postulates that human activity can be explained by conditions and constraints that appear or are made available in institutions, and that each person adapts to, adopts and develops these (Bosch & Gascón, 2014). Therefore, observable behaviour will consist of a mixture of personal and institutional components.

Research question.

To investigate what praxeological changes occur when using a textbook chapter from an explicit step-by-step curriculum into a school context based on implicit circular curriculum to help students and teachers to overcome the transition from arithmetic algebra, we ask the research question: *What praxeological change can be observed in Danish lower secondary school when mathematics teachers use adapted research-based resources from Japan to overcome the challenges in the transition from arithmetic to algebra?* We are also interested in the obstacles and potentials for praxeological change that appear.

Context and Methodology.

The intervention took place in three Danish grade 8 classes from September 7 to October 7, 2022. According to national statistics (Table 1) the intervention school can be described as an average school some respects. However, the students' test scores are clearly below the national average, particularly in mathematics, also when socioeconomic conditions are considered (Education, 2023).

	National	Intervention School
Average number of students at the school	295	276
Average number of students in the classes	21	23.2
The grade point average for the compulsory tests after grade 9	7.9	6.8
The average for the compulsory test in mathematics	7.3	6
The average for the compulsory test in mathematics without aids	7.7	6.3
The average for the compulsory test in mathematics with aids	6.8	5.7
Students with the highest well-being amount	89.9%	90.3%
Absence of students	8.1%	10.5%
Students' progression to secondary education	87.2%	87.2%

Table 1. National school statistics and data for the intervention school (Education, 2023)

The school operates with additional teachers who are present in some lessons to support specific groups, like high-performing or low-performing students in mathematics, or individual students at large. The school also employs a "mathematics teacher guide" (*matematikvejleder*) who supports and advice the teachers of mathematics in grade 6 to 9.

The team of grade 8 mathematics teachers meets once a week to prepare lessons together. The team thus teach the same mathematical themes at a given time throughout the year. The teachers do not use a fixed textbook system and take turns to choose material that everyone in the team uses for teaching. However, individual teachers sometimes select supplementary material for their teaching.

The textbook material Junior High School Mathematics:1 (Isoda & Tall, 2019), used for the intervention, was chosen for several reasons. First, it is based on systematic empirical research. The books are translated into English and have an explicit stepwise structure. We selected the introductory chapter on algebra. The chapter "Algebraic Expressions" (Isoda & Tall, 2019) was translated from English to Danish and handed over to the teachers without further instructions. The were informed that they were participating in an experiment to test a translated Japanese textbook chapter. The teachers, who volunteered to participate, were aware that the aim was to investigate whether and to what extent the use of the chapter could support the transition from arithmetic to algebra.

To answer the research question, we utilise multiple sources of qualitative data including classroom observations with fieldnotes, photos and video recording, students' notebooks and teacher interviews. During the intervention field notes and pictures of the students' work were taken throughout the maths lessons. As all three classes had mathematic lessons at the same time, fieldnotes were conducted in shifts in the classes, but video of all the lessons was recorded, to have the opportunity to examine and compare the work of all three classes throughout. The video camera was placed at the back of the classrooms and focused on the teacher and blackboard, to capture the teachers whole class management. After the intervention period, all students' notebooks were collected and copied, to gain insight into the students' individual work

The selection of teaching episodes for further analysis is based on those areas where the Japanese textbook it particularly differs from the typically used Danish textbook material:

1. The use of a challenging opening problem which is referred to throughout the chapter.
2. The explicit discussion of the use and meaning of notation, which us particularly important in a first encounter with algebra and algebraic expressions. For example, it is emphasised that $2n$ a shorthand for $2 \times n$.
3. The explicit and concise technological description and theoretical justification for the techniques that are introduced and exemplified (Author, 2024b).

The selection of episodes for further analysis aims to focus on these three key elements and are also a main focus in the teacher interviews following the observation. The interview is based on the photo elicitation interview method, where a printed transcript and photo of the episodes related to the three points outlined above are used to generate verbal discussion, thereby creating knowledge and data (Nissen et al., 2016). According to Glaw et al. (2017) “Different layers of meaning can be discovered as this method evokes deep emotions, memories, and ideas”.

Episode 1: The opening problem.

In the first lesson, the initial problem is introduced by the class mathematics teacher, Mary; the advanced mathematics teacher, Catrin, participates as co-teacher.

Outline of the episode

First, Mary asks the class what characterises a square. Students readily answer, “four equal lengths” and “all angles are 90 degrees”. Then they are provided with a small bag of coloured sticks and are instructed to construct a square using the sticks. Mary asks the class how many sticks they need to make a square. One student answer four. Mary draws a square and writes the student’s answer on the board. The students record this in their notebooks. Then, students should expand the figure by one, two and three squares adjacent squares, and record in their notebook the number of sticks employed. Mary draws the students’ answers, models and numbers, in a table on the whiteboard (Fig. 1).

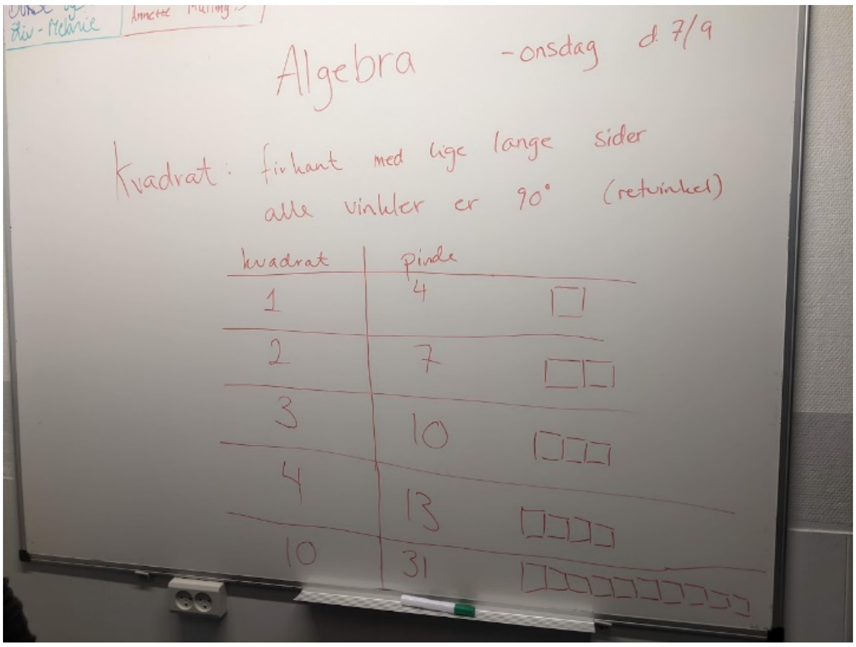


Figure 1. Photo of the whiteboard where the heading of the columns is “square” and “sticks”.

Mary now requests that the students determine the number of sticks required to construct a row of ten squares. The students work independently, and some have quiet conversations with their peers. Mary and Catrin observe the students' work. Students' utterances include statements such as "we just need to multiply", "ten times three", "there is one with four".

Twenty minutes into the lesson, Mary asks the class if they have constructed a 10-chain and the number of sticks used. A student replies "31, three times nine plus four". Mary records the student's response in both numerical and textual formats.

Mary then inquires whether they could determine the number of sticks needed for 15 squares and 100 squares. One student says, "It is still four for the first and three thereafter" and refer to the green text on the whiteboard (Fig. 2).

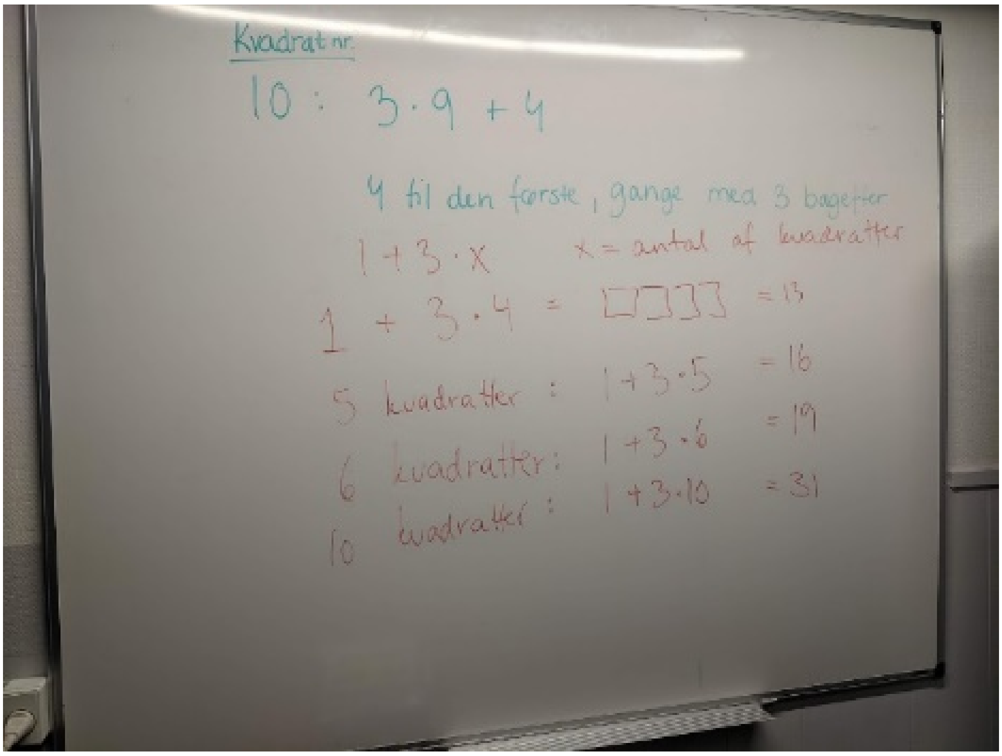


Figure 2. Photo of the whiteboard with Yui's expression and examples.

Mary now tells the class out that one student, Yui, has described the four connected squares with the following expression: $1 + 3 \times 4$. Yui is a fictional character who appears in vignettes throughout the textbook, for instance (as in this case) to suggest answers to a problem. Mary writes the expression on the whiteboard (Fig. 2), changing the multiplication sign from a cross to a dot, to align with the notation used in Danish school. Mary tells the students "I want you to spend three minutes thinking about what the numbers stand for. Write down your thoughts so you don't forget them." The students work individually on the assignment while Mary and Catrin circulate in the room, observing and quietly inquiring about each students' progress.

30-minutes into the lesson Mary asks different students to share their answers with the class. Four students reply consecutively:

S_1: "First there is one stick"

S_2: "You take three each time, but the first one needs to be added to make four"

S_3: "The first one is the one you build on and you do it four times"

S_4: "You add three each time"

Students then work on task 2, which is "Using Yui's method, what math expression do we use to find the number of straws needed to make 5 squares? 6 squares? What about 10 squares?" (Isoda & Tall, 2019 p. 61) Mary shares the students' answers to the task by writing the techniques to solve the task on the whiteboard (Fig. 2)

Mary inquires as to whether the method invented by Yui can be employed when the number of squares in the chain is 25 or 1000. The prevailing opinion is that this is indeed the case, provided that the value 4 in the first expression is replaced with 25 and 1000. One student adds "Just replace the last number with the number of squares in the chain". Mary continues asking whether it can be any number, and the class responds unanimously "Yes". Then one student says "x" and Mary follows up by asking "What is x?". The student answers "The number of squares". Mary writes "x = number of squares" on the whiteboard (Fig. 2)

Mary now introduces yet another method (in the textbook, suggested by the fictive pupil Takumi) where four squares are expressed by $4 + 3 \cdot (4 - 1)$. The pupils work on explaining this expression, just as for Yui's. Mary shares the students' work by writing the expressions and the technique to solve the tasks on the whiteboard (Fig. 3)

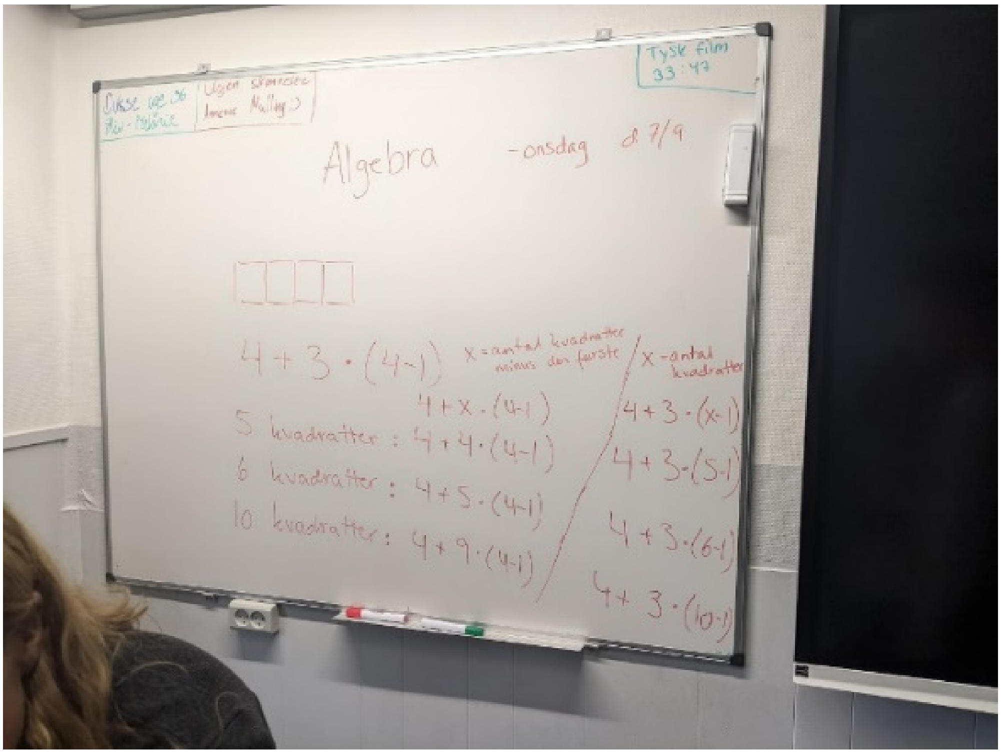


Figure 3. Photo of the whiteboard with Takumi expression and examples.

Then the students are given the next task from the book: "Using a method different from that of Yui and Takumi, make a math expression to find the number of straws needed. Explain your idea" (Isoda & Tall, 2019 p.61).

Once again, after a period of desktop work, Mary asks the students to share their results. Several of the students have coined the expression $4 + x \cdot (4 - 1)$ where x is the number of squares minus one (the first square in the cain). Mary writes the expression on the whiteboard along with examples, and compares it to Takumi's expression and examples (Fig. 3).

An hour has passed since the lesson began, and while the students are copying the expressions from the bord (Fig. 3) Mary and Catrin engage in a meta-discussion regarding the potential for further comparison of the expressions. They agree that it is an opportune moment to do something else, namely, to hand out the printed textbook material. The students are asked to answer Q1, Q2 and Q3 on page 63 (Isoda & Tall, 2019). In Q1 the students are required to determine the number of sticks needed to construct chains with 20 and 30 squares, while using Yui's methods. In Q3 the students must find the number of sticks needed to construct chains with 20 and 30 squares by using Takumi's methods. During their work, you can hear students exclaiming "It is the same" or "We get the same numbers". In plenary, the two algebraic expressions were compared, rewritten and reduced to conclude that Yui's expression $1 + 3 \cdot x$ and Takumi's expression $4 + 3 \cdot (x - 1)$ are in fact "equal".

Analysis: The importance of the opening problem for praxeological change.

The devolution, that is the phase of teaching where the teacher hands over the responsibility for the investigation process to the students (Brousseau & Warfield, 2020), proceeds without any need for repetition or clarification. The seamless devolution indicates that the type of task "determine the number of elements in a given pattern", as well as first techniques to solve the task, is known by students, who can draw on established techniques. The task does not require a praxeological change. The opening problem functions in the textbook as a motivation to introduce a variable x, which represents the number of squares in the chain. In the episode, a student spontaneously proposes the use of x as a variable, anticipating the textbook's introduction of algebraic expressions (Isoda & Tall, 2019 p.62). The use of an algebraic expression to encapsulate the calculation of sticks for different number of squares is explicit in the textbook material. The goal is to lead students to change from repeated arithmetic expressions for the number of squares in the chain, to the algebraic expression. Due to the Danish spiralling curriculum structure is not entirely new to the students observed, but also not fully familiar. In the last part of the episode where the students must use Yui's and Takumi's methods to find the number of sticks needed to construct chains with 20 and 30 squares, they realise that the methods produce the same result. The students hypothesise that Yui's and Takumi's methods are similar. Mary anticipates the task of comparing Yui's and Takumi's algebraic expression, which should appear later, according to the structure of the textbook. By rewriting Takumi's expression $4 + 3 \cdot (x - 1)$, Mary gets Yui's expression $1 + 3 \cdot x$. (see Episode 2). This validates and clarifies the students' hypothesis.

As we have seen, the pupils rather easily take this first step towards a praxeological change, from arithmetic to algebraic modelling of the pattern. The point that is explicitly described with the opening problem is that "algebraic expressions using letters serve as both the method to find the number of straws(sticks), as well as to represent the results we want to find" (Isoda & Tall, 2019 p.63), is not emphasised by the teacher in this episode.

Episode 2/1: Notation and explicitness

Outline of the episode

Mary guides the process of reducing the two algebraic expressions, to test the students' hypothesis that Yui's and Takumi's expression are equal. In Yui's expression, Mary rewrites the expression $1 + 3 \cdot a$ to $1 + 3a$ and explains that there is an invisible multiplication sign between 3 and a. To rewrite Takumi's expression $4 + 3 \cdot (a - 1)$, Mary refers to the "bracket rule" and writes $4 + 3a - 3$ on the whiteboard, then reorders and reduces to $1 + 3a$.

Analysis: Notation and theory about "bracket rules"

The textbook applies the multiplication sign $\times$. In this episode Mary is using the dot sign $\cdot$ for multiplication (Fig. 2, Fig. 3), as in Danish textbooks. The multiplication sign $\times$ is not unknown to the students as it appears on their calculators. Mary does not clearly distinguish technology (description of what is done, "removing parenthesis", "multiplying through" etc.) and theoretical elements (justifying discourse). The distributive law, which is the foundation of "bracket rules", is at the level of theory. In the episode described, both appear at the same level. Only later, with the explicit definition of algebraic notation (like $2x$ meaning $2 \cdot x$) and how to remove brackets by using the distributive law, the level of theory may become evident to some extent (see instance Episode 2/2).

The central point in the last section of the chapter with the heading "Using Algebraic Expression with Letters" (Isoda & Tall, 2019 p. 82-83), is that the two algebraic expressions for Takumi and Yui methods are equivalent. The students are to meet a new type of task, "Determine whether two algebraic expressions are equivalent", learn how to solve it by techniques (like "bracket rules") but also to relate these techniques to general and explicit theory previously presented in the chapter. In episode 2/1, Mary anticipates this task and therefore cannot draw on the theoretical elements presented throughout the chapter.

Upon further examination of the chapter's subsequent development, it becomes evident that the introduction of a variable to answer the introductory question is related to another point, which is never made explicit in Danish textbooks. In section 2, the aim is to "learn how to express products and quotients in algebraic expressions, following the rules" (Isoda & Tall, 2019 p. 65). The main points emphasised are:

1) In algebraic expressions, remove the multiplication sign $\times$.

2) When multiplying numbers and letters, write the number in front of the letter. (Isoda & Tall, 2019 p. 65).

Stating such rules may appear unnecessary to seasoned users of algebraic notation and discourse, but the textbook rightly does not assume this about students. They are to be introduced to algebraic discourse with a condensed notation system, governed by explicit conventions rather than more or less opaque ad hoc rules. However, we observe that the teachers consequently include the multiplication sign in the introductory phase (see Fig. 2 and Fig. 3) and therefore postpones this point.

The subsequent process of students' writing an algebraic expression for Takumi and Yui's method is structured by the textbook's step-by-step structure of questions with answers. This structure, where the textbook offers answers to the mathematical problem, is challenging for the teacher Mary. In a follow up interview, she says: "If I provide a document to the students, they will read it and then someone will begin copying what it says here (pointing to the printed material)". "They will not engage independently in the problem, but rather, they simply copy the material without considering their own potential solutions". Here, Mary seems to forget how she actually used the fictive characters' answers, to let the students validate and relate alternative methods, - concretely to set up several algebraic expressions which could subsequently be proved to be equivalent.

Episode 2/2: Notation and explicitness

Outline of the episode

The episode begins with the introduction to section two in the chapter about simplifying algebraic expressions. Based on the work with linear expressions the aim is to "consider how to combine terms of algebraic expressions" (Isoda & Tall, 2019 p. 75). Two rectangles are drawn in the book, one with side lengths 1 and 7, the other with side lengths a and 3. The teacher, Sarah, asks the class how they will write the area of the rectangles in algebraic expressions and adds "How to write the area?". On student replies that he will multiply length and width by each other. The teachers ask how it should be written up and the student answers $3a$. The teacher writes $3a$ on the whiteboard and ask the student if it is an algebraic expression. The student mumbles yes… because… and the teacher completes the sentence by saying that an algebraic expression is when we write it as a formula and removes the multiplication sign. Then the students find the area of the other rectangle, which is 7.

The teacher explains to the students that the book expresses the difference between the two areas by the expression $3a - 7$. She asks students to explain why. They discuss the meaning of difference and conclude that it means subtraction.

The teacher then tells the students that the difference between the two rectangles can also be expressed by $3a + (-7)$, even though (according to her) it looks “illogical”. Sarah makes the point that, regardless of whether the expression is written in the first or second way, there are two terms in the expression. One term is $3a$ and the other term is -7. And the number in front of the letter (pointing to $3a$) is called the coefficient. Some students tell the teacher that they do not understand “the system”. Sarah draws circles around the terms in the algebraic expressions on the whiteboard while repeating what she said about terms and coefficients. A boy reiterates that he does not understand Sarah’s “system” and refers to the coloured circles on the whiteboard. Sarah continues the lesson by saying that in the textbook, a girl claims that if we use what we have learned about positive and negative numbers, the terms are easier to see, as we change the algebraic expression into a sum (Isoda & Tall, 2019 p. 75). Sarah writes $-2x - 5 = -2x + (-5)$ and tells the students that the terms in this expression are $-2x$ and -5.

Analysis: Notation and the use of signs

In episode 2.2, the task is roughly of the type *T*: Describe the difference between the areas of two rectangles, whose side lengths are given as algebraic expressions, as an algebraic expression. The techniques used to solve the task are τ_3: Determine the area of the rectangle by multiplying the side lengths and τ_4: Determine the difference between two areas by subtraction. The techniques applied are familiar to students. The current task is about areas, and they are inherently positive.

The students can apply τ_3 and τ_4 to find the difference between the two rectangles. But it seems unclear to the students that the algebraic expression can be changed from $3a - 7$ to $3a + (-7)$ and why it needs to be rewritten. In the first expression, the minus sign between the two numbers represents the subtraction operation. In the second expression, the minus sign in front of 7 represents the operation where you do the additive inverse. This point remains implicit to the students. The textbook refers to previously learnt about negative numbers, justifying the rewriting of algebraic expressions to addition-only (Isoda & Tall, 2019 s. 75). Since the teacher and the students have not used the textbook material before, it is not possible to refer back to earlier chapters. On the other hand, negative numbers are not new to the students: according to the Danish curriculum they should know and be able to use negative integers after grade 6 (Education, 2019). In this episode, the connection to previously acquired knowledge about negative numbers is not established and the transition from arithmetic to algebra become fragmented. Moreover, the theoretical point regarding the exact meaning of “terms” and “coefficient” is not really motivated and remains opaque to the students.

The distinction between laws (like the commutative) and conventions (like writing $3a$ and not $a3$) is a key element of the textbook material but is not emphasised or even made explicit in this episode.

Episode 3: Negative exponents and level of technology.

In the Japanese textbook material, theoretical elements are integrated with problem solving and technical work, and not referred to isolated theoretical sections. In this episode the teacher May asks two students to take a closer look at the “close up” section, which are about the use of 0 and negative numbers as exponents. The textbook asks the question “Can we use a^1 or a^0 ? In general, the book’s “Close up” sections relates to further perspectives, which can be beyond the mandatory curriculum at stake (Isoda & Tall, 2019).

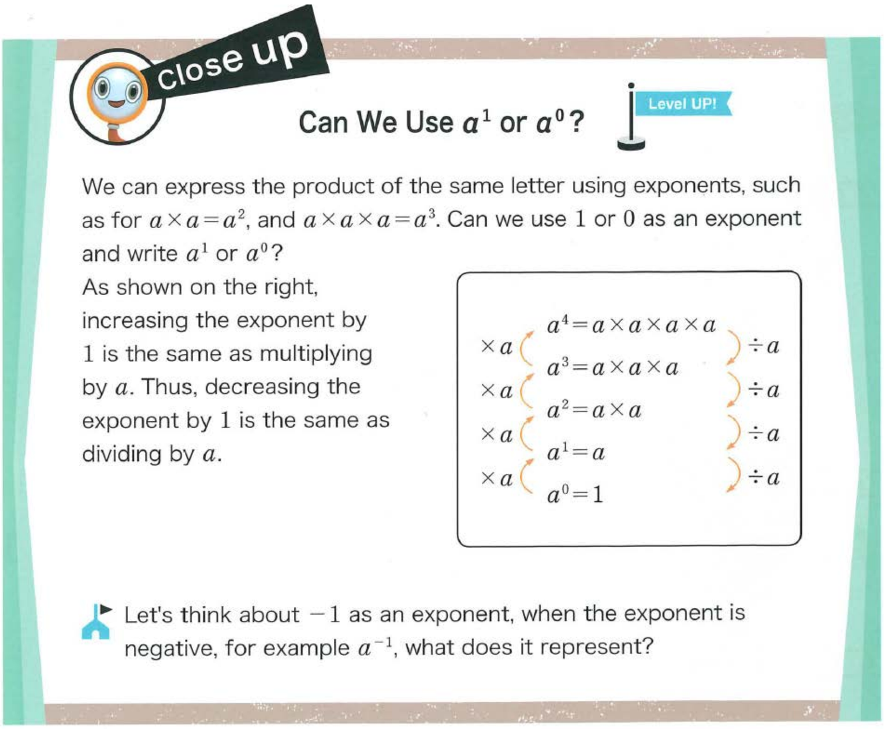


Figure 4. Copy of the negative power assignment in the textbook material (Isoda & Tall, 2019, p. 71)

Outline of the episode

The students in the class are working on questions about how to express quantities using algebraic expressions. The teacher notices that two boys have finished the questions and asks

them to work with ‘close up’ which is text and tasks related to further content and problems (Isoda & Tall, 2019 p. 3). The teacher May asks the students to find a^{-1} and a^{-2}.
The two students engage in a discourse on the pattern of how, when adding 1 to the exponent, an additional a is “appended”. The two students reason their way to $a^{-1} = -1$.

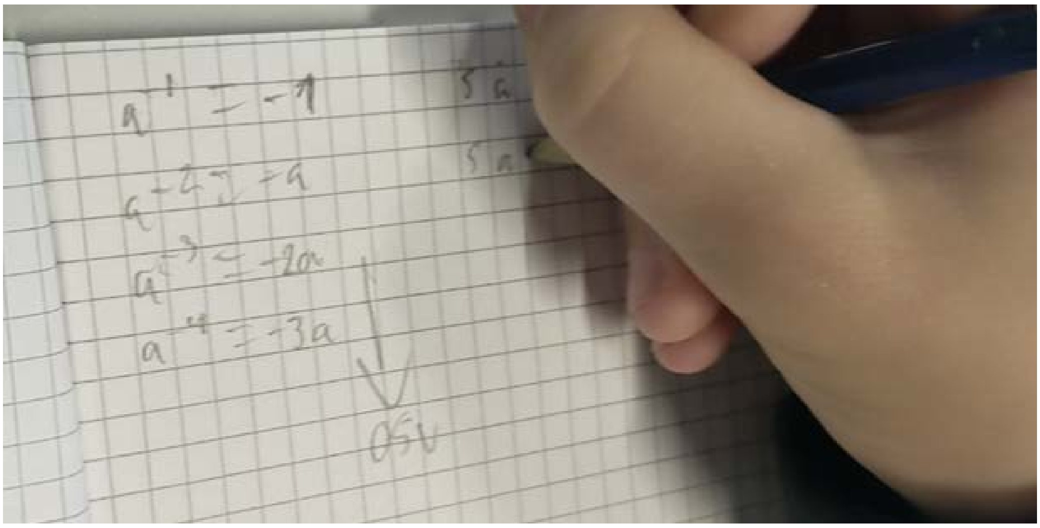


Figure5. Image of student answering the assignment “Close up” Figure 4.

The boys continue to infer $a^{-2} = -a, a^{-3} = -2a, a^{-4} = -3a, \ldots$ (Fig. 5). It is assumed by the students that they have solved the task, and they initiate contact with May. May, after seeing the answers, asks the students about their reasoning and looks at their notes. Then the teacher asks: “Have you tried plugging it into a calculator?” and follows up by suggesting the students to replace a by 2 and type $a^4,\ a^3,\ a^2, \ldots.$ The students input 2^{-1}into their calculator and are surprised to discover that the results is one half. They continue to enter 2^{-2}, 2^{-3} and 2^{-4}, and observe the pattern emerging $if\ a = 2, then\ a^{-1} = \frac{1}{2}, a^{-2} = \frac{1}{4}, a^{-3} = \frac{1}{8}\ , a^{-4} = \frac{1}{16}, \ldots$ The students contact May and present the following pattern “the number below the fraction line doubles in size”. May applauds this answer.

Analysis: The link between technology and theory

First, a is the base and $n = \{-1, -2, -3, -4, ..\}$ are the exponents. From the photo figure 5, and the video recording of the students’ response, it is not total clear what their reasoning was. But the students seems to reason that if the exponent decreases by one, it means subtraction of a. This method does not fit the step from $a^{-1}\ to\ a^{-2}$, and does not lead to discovering the meaning of negative exponents in general. Instead of pointing out that the subtraction is theoretically inconsistent with the pattern given in the book, May asks the students to enter the expression on the calculator with base $a = 2$, thus to re-explore the task based on automated arithmetic. Students then infer from this that $2^{-n} = \frac{1}{2^n}$. It remains an individual case and does not develop into a general model; moreover, the inference is essentially based on the authority of the calculator, rather than on reasoning from the pattern exposed in the book.

This book's setup has potential to initiate a praxeological change, from seeing exponentiation as "repeated multiplication" towards more general definitions, justified by allowing general rules to hold – here, essentially, the exponential rule $a^{n+m} = a^n a^m$. As such formality is beyond the level of the book, the choice of observing a pattern was made, to help students explain why $a^{-n} = \frac{1}{a^n}$ is a reasonable definition (for natural numbers n). The teacher's intervention replaces the pattern with the calculator, as foundation for the reasoning – and, as the calculator does not have symbolic computation, the intervention also reduces the problem from one that requires algebraic reasoning (modelling an algebraic pattern, in terms of algebra) and back to arithmetic reasoning (modelling an arithmetic pattern with algebra). Instead of exploring the students' faulty algebraic reasoning and helping them modify it, she thus reduces the problem to fit more familiar praxeologies, similar to the introduction problem. But she also deprives the problem of much of its intended potential for praxeological change.

Episode 4: The invisible level of theory

Episode 4 takes place in a classroom of students who are considered high performing and highly motivated. The other students are working with the same part of the textbook chapter "Linear expression and multiplication of numbers" (Isoda &Tall, 2019 p. 78-79), in their respective classes. The aim of the section is to consider how to simplify linear expressions.

Outline of the episode

The students have worked with the task type T: Simplify linear expressions of the form $ax \times b$, and subsequently worked individually with example 4 (Figure 6). After twenty-five minutes individually work, the teacher asks how the students have reduced the expression $2(x + 4)$. In the textbook, the example is followed by the distributive law, illustrated by a geometric figure (fig. 5).

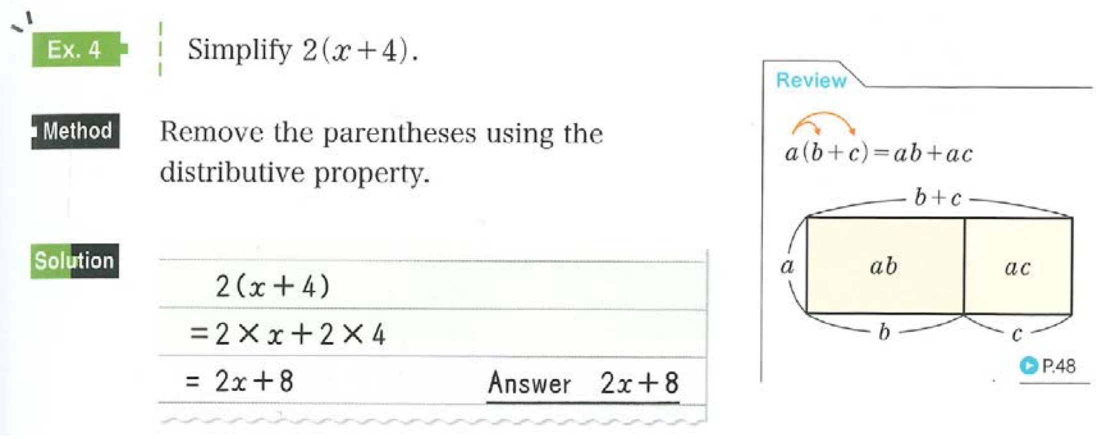


Figure 6. Copy of textbook example with method, solution and reference to theoretical foundation (Isoda & Tall, 2009 p. 79)

The teacher Tina asks the students what to do first. A student answers that he has “multiplied out the brackets”. The teacher follows up by saying “What is two multiplied by x and two multiplied by four?”, while writing $2(x + 4)$ on the whiteboard. A student replies, "two x and eight”. The teacher comments on the student’s calculation process in plenum: “you have skipped a step and I think some of you also do that when you both reduce and make equations… then you skip some steps… where you actually realise… that you have multiplied in (the bracket)… then I do not have to write two multiplied by x and two multiplied by four”. The teacher then posits that “the advantage of applying an additional calculation in which the x is multiplied by two, and four is multiplied by two, is that if the result is incorrect, it is possible to identify the specific point in the series of calculations at which the error originates”.

Analysis: The distributive law and “Bracket rules”

In this episode, the type of task is *T*: reduce a first-degree equation, by applying the techniques τ_1: apply $a(b + c) = a \cdot b + a \cdot c$, and τ_2: Simplify by collecting and reducing similar terms. The teacher employs a clear and systematic discourse on techniques, and the practice of documenting each step is based on the premise that it allows for later verification of the techniques used. I the follow-up interview, Tina emphasises that it is important to her that the students understand each step in the process of solving the task and are “aware of what they are doing”. Here, understanding means knowing what techniques to use, applying them correctly, and being able to verify that. The fact that τ_1 is a direct application of one general theoretical law, while τ_2 draws on both the commutative law and execution of arithmetic operations, as well as some kind of definition of “similar terms” (here, constant terms and first-degree terms), remains opaque. Although there is an explicit reference to the distributive law in the book’s assignment (Fig. 6), the level of theory remains implicit in the teaching, or is not considered any difference from the techniques which are amply explained. Thus, the explicitness is limited to technology, while the theoretical foundations remain implicit.

Discussion

The importance of the opening problem for praxeological change

Episode 1 shows the impact of an initial mathematical problem with an investigation process that give rise to the introduction of a variable. The episode shows the change from repeated arithmetic expression to the introduction of a general algebraic expression, which is the first step toward a praxeological change from arithmetic to algebra. According to Putra (2019), the process of praxeological change requires students to develop or reconstruct larger complexes of mathematical practices, technologies and theories. In this case the opening problem gives rise to work with central techniques to simplify linear equations, technological elements as how to use algebraic expressions with letters and key theoretical elements such as the distributive property.

The systematic and stepwise development of the problem throughout then chapter is both the potential and the challenge of the opening problem. When examples and points are presented in the intended order, a clear connection is made between praxis and logos. But when anticipating points like in episode 1 where the teacher reduces and compare algebraic expressions without the technological and theoretical foundation it reflects the international trend of dis-algebraization and atomisation of school algebra (Bolea et al., 2003).

Notation and explicitness in the Japanese textbook.

The explicit description of the use of algebraic notation form is a core feature of the material. I episode 2/1 and 2/2 this potential is not being realized. This may be because the meaning of the explicit notation form is not clear to the teacher and the reasons for when a change in notation is due to theory or conventions that teachers are not used to teach explicitly, and perhaps only partly aware of. But there are some signs that this potential can be realized gradually. When Catrin in the subsequent interview asserts that they (the teacher team) have become aware of the significance of the notation and have explored topics that they might have overlooked previously, it is a sign that praxeological change is emerging at least for the teachers.

The importance of distinction between laws and conventions.

The distinction between laws and conventions is central to the material used, and is also present in episode 2, 3 and 4. In episode 3 the teacher is aware of the step-by-step process in the reduction of a first-degree equation but although the textbook offers a description of the distributive law as a theoretical foundation, it is not applied by the teacher.

Episode 2 also exemplified that the textbook material contains tasks with potential to produce praxeological change, here to provide a symbolic rationale for the definition of negative powers. That potential is, however, not realised in the lesson, in part due to the teachers' suggestion to return to number patterns generated by calculator use, when the students fail to solve the task. There is no doubt teachers' lack of experience and habit of teaching algebraic theory (at the given level) is a considerable obstacle.

We can relate this to the distinction between technology and theory, which is central in school algebra as demonstrated by Chevallard (1989), Bolia et al. (2003) and Bosch (2015). While $x^{-n} = \frac{1}{x^n}$ is indeed a definition here, the book aims to make students realise a theoretical rationale – rather than simply to describe an exemplify and arbitrary "rule" or technique. However, it requires the teacher to realize this potential of the material so that the difference between technology and theory does not remain implicit, as is most often the case in Danish school algebra (Author, 2024b).

Conclusion

The aim of this study was to investigate conditions and constraints for realising praxeological change through the use of a textbook chapter from an explicit step-by-step curriculum, in a school context based on an implicit circular curriculum. The intention is to help students and teachers to cope with the transition from arithmetic to algebra. To investigate this, we asked the research question: What praxeological change can be observed in Danish lower secondary schools when mathematics teachers use adapted research-based resources from Japan to overcome the challenges of the transition from arithmetic to algebra?

The four episodes exemplified both potential and obstacles to adapting the textbook material from Japan in a Danish school context, in view of promoting praxeological change related to the transition from arithmetic to algebra. The teachers succeed to a large extent in managing the use of challenging opening problems that are pursued throughout the chapter, in view of showing how algebra provides real and clear solutions (rather than just new kinds of procedures to manage). The textbook also supports them in making notational conventions explicit as technology, and thereby less *ad hoc* to students. However, when it comes to distinguish algebraic conventions, laws and definitions, and teach them in the theoretically coherent way supported by the textbook, progress is at best slower and meets the obstacle of the teachers' limited didactic and probably also mathematical experience in this area. However, the interviews suggest that these obstacles could decrease with time, so that students would also benefit from the explicitness of theory provided by the text in question – to realise a praxeological change from algebra dominated by *ad hoc* technology to algebra based on few laws and rich theoretical connections.

References

Author (2014)(a)(b)

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice: The case of limits of functions in Spanish high schools. Educational Studies in Mathematics, 59(1–3), 235–268. https://doi.org/10.1007/s10649-005-5889-z.

Bolea, P., Bosch, M., & Gascón, J. (2001). La transposición didáctica de organizaciones matemáticas en proceso de algebrización: el caso de la proporcionalidad. Recherches en didactique des mathématiques, 21(3), 247-304.

Bosch, M. (2015). Doing research within the anthropological theory of the didactic: The case of school algebra. In *Selected regular lectures from the 12th international congress on mathematical education* (pp. 51-69). Springer International Publishing. https://doi.org/10.1007/978-3-319-17187-6_4

Bosch, M., & Gascón, J. (2014). Introduction to the anthropological theory of the didactic (ATD). In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of Theories as a Research Practice in Mathematics Education* (pp. 67-83). Springer International Publishing. https://doi.org/10.1007/978-3-319-05389-9_5

Brousseau, G., & Warfield, V. (2020). Didactic situations in mathematics education. *Encyclopedia of mathematics education*, 206-213. https://doi.org/10.1007/978-3-030-15789-0_47

Bruner, J. S. (1960). Introduction. In *The Process of Education* (pp. 1–32). Harvard University Press.

Cheng, L.P., Leong, Y.H., & Toh, W. Y. K. (2021). Singapore secondary school mathematics teachers' selection and modification of instructional materials for classroom use. In Kaur, B. & Leong Y.H. (Eds.), *Mathematics instructional practice in Singapore secondary schools* (pp. 205-230). Springer Nature Singapore. https://doi.org/10.1007/978-981-15-8956-0_11

Cheng L. P. & Yeo, K. K. J. (2022). Singapore school mathematics curriculum. In Lee Y.L. (Eds.) *Education in Singapore*. (Vol. 66, pp. 405-421). Springer Singapore Pte. Limited. https://doi.org/10.1007/978-981-16-9982-5_22

Chevallard, Y. (1985). Le passage de l'arithmétique à l'algèbre dans l'enseignement des mathématiques au collège – Première parie. L'évolution de la transposition didactique. *Petit x*, 5, 51-94.

Chevallard, Y. (1989). Le passage de l'arithmétique à l'algèbre dans l'enseignement des mathématiques au collège – deuxième partie. Perspectives curriculaires: la notion de modélisation. *Petit x*, 19, 43-75.

Chevallard, Y. (1999). El análisis de las prácticas docentes en la teoria antropológica de lo didáctico. *Recherches en didactique des mathématiques* 19(2), 221-266.

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education, 12*, 71–114. https://doi.org/10.24529/hjme.1205

Cosan, D. D., (in press). Praxeological differences in institutional transition: The case of school algebra. *Annales de Didactique et des Science Cognitives*.

Education, D. M. (2019). Common objectives: Mathematics. National Agency for Education and Quality. Retrieved form GSK_FællesMål_Matematik.pdf (emu.dk)

Education, D. M. (2022). Joint development of Mathematics. By expert group for mathematics. Retrieved from https://www.uvm.dk/-/media/filer/uvm/aktuelt/pdf22/okt/221011-ekspertgruppens-rapport-om-faelles-udvikling-af-matematik.pdf

Education, D. M. (2023). Education statistics. Ministry of education. Located May 3. 2023 at Statistik om Maribo Skole, Borgerskoleafdelingen | Børne- og undervisningsministeriet (uddannelsesstatistik.dk)

Grønbæk N., Rasmussen A. B., Skott C. K., Bang-Jensen J., Fajstrup L., …. & Markvorsen S. (2017). The mathematic commission – reporting. *Danish Ministry of Education. 2017.*

Gueudet, G., Pepin, B. & Trouche, L. (2012). Introduction. In Gueudet, G., Pepin, B., & Trouche, L. (Eds.) (2011). *From Text to 'Lived' Resources: Mathematics Curriculum Materials and Teacher Development* (1st ed., Vol.7) Springer Netherlands. https://doi.org/10.1007/978-94-007-1966-8

Ireland, J., & Mouthaan, M. (2020). Perspectives on curriculum design: comparing the spiral and network models. *Research Matters, A Cambridge Assessment Publication*, 30, 7-12.

Isoda, M., & Tall, D. (2019). *Junior High School Mathematics: 1.* Gakko Tosho.

Jaciw, A., P., Hegseth, W.M., Lin, L., Toby, M., Newman, D., Ma, B., & Zacamy, J. (2016). Assessing impact of math in focus, a "Singapore math" program. *Journal of Research on Educational Effectiveness*, 9(4), 473-502. https://doi.org/10.1080/19345747.2016.1164777

Jubri, A. & Drijvers, P. (2016). Student difficulties in mathematizing word problems in algebra. *Eurasia Journal of Mathematics, Science and Technology Education*, *12*(9), 2481–2502. https://doi.org/10.12973/eurasia.2016.1299a

Kieran, C. (1990). Cognitive Processes Involved in Learning School Algebra. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and Cognition: A Research Synthesis by the International Group* *for the Psychology of Mathematics Education* (ICMI Studies, pp. 96-112). Cambridge University Press. https://doi.org/10.1017/CBO9781139013499.007

Kieran, C. (2022). The multi-dimensionality of early algebraic thinking: background, overarching dimensions, and new directions. *ZDM-Mathematics Education*, *54(*6), 1131-1150. https://doi.org/10.1007/s11858-022-01435-6

Li, X., Ding, M., Capraro, M.M., & Capraro, R.M. (2008). Sources of differences in children's understanding og mathematical equality: Comparative analysis of teacher guides and student text in China and the United States. *Cognition and Instruction* 26(2), 195-217. https://doi.org/10.1080/07370000801980845

Lindorff, A.M., Hall, J. & Sammons, P. (2019) Investigating a Singapore-based mathematics textbook and teaching approach in classrooms in England. In Frontiers in education Vol. 4. https://doi.org/10.3389/feduc.2019.00037

Loveless, T (2013) The algebra imperative: Assessing algebra in a national and international context. Washington, DC: Brookings. Retrieved from Kern-Algebra-paper-8-30_v13.pdf (brookings.edu)

Nissen, S.K., Schmidt, M.C.S., & Tonnesen, P.B. (2016). *Photo elicitation interviews as a way of accessing primary school students' attitudes.*

OECD (2023). *PISA 2022 Results (Volume I): The State of Learning and Equity in Education*, PISA, OECD Publishing, Paris, https://www.oecd-ilibrary.org/education/pisa-2022-results-volume-i_53f23881-en

Putra, Z. H. (2019). Praxeological change and the density of rational numbers: The case of pre-service teachers in Denmark and Indonesia. *Eurasia Journal of Mathematics, Science and Technology Education, 15(5).* https://doi.org/10.29333/ejmste/105867

Remillard, J. T., Herbel-Eisenmann, B. A., & Lloyd, G. M. (2010). Teachers' use of curriculum materials: An emerging field. In Mathematics Teachers at work. *New York, UK: Routledge.*

Schifter, D., Monk, S., Russel, S. J., & Bastable, V. (2008). Early Algebra: What does understanding the laws of artithmetic mean in the elementary grades? In J. J. Kaput, D. W. Carraher, & M. L. Blaton (Eds.). *Algebra in the Early Grades* (2014th ed., pp. 413–449). Routledge.

Sfard, A. & Linchevski, L. (1994). The gain and the pitfalls of reification: The case of algebra. *Educational Studies in Mathematics, volume 26,* 191-228. https://doi.org/10.1007/bf01273663

Steiner, H. G. (1987): Philosophical and epistemological aspects of mathematics and their interaction with theory and practice in mathematics education. *For the learning of mathematics* 7, no. 1 (1987):7-13.

Straesser, R. (2011). Foreword. In Gueudet, G., Pepin, B., & Trouche, L. (Eds.) (2011). *From text to 'Lived' resources: Mathematics curriculum materials and teacher development* (Vol. 7) Springer Netherlands. https://doi.org/10.1007/978-94-007-1966-8

Strømskag, H. & Chevallard, Y. (2022). Elementary algebra as a modeling tool. A plea for a new curriculum. *Recherches en Didactique des Mathématiques*, 42(3), 384.

Yoshikawa, S. (2008). Education ministry perspectives on mathematics curriculum in Japan. In Usiskin, Z. & Willmore, E. (Eds.). (2008). *Mathematics Curriculum in Pacific Rim Countries – China, Japan, Korea and Singapore*. pp. 9-22. Charlotte, NC: Information Age.

Paper IV

Diagnostic test tool based on a praxeological reference model to examine students' technical and theoretical algebra knowledge

Abstract

The main aim of this paper is theoretical and methodological: to show how The Antropological Theory of the Didactics – in particular, the construction of a praxeological reference model – can be used as a foundation for developing a diagnostic test tool which examines students technical and theoretical knowledge of basic algebra (and related arithmetical knowledge on which initial algebra is based). The point is to provide explicit control on what is being tested, for instance, in relation to a given national curriculum, and in relation to teaching interventions.

Keywords: Anthropological theory of the didactics, school algebra, praxeological reference model, diagnostic test.

School algebra

In secondary school, basic algebra is a crucial bridge between arithmetic and more advanced subjects involving functions and analytic geometry. To set up algebraic models, it is necessary to master arithmetic and algebraic techniques by hand, at to know associated theory like the distributive law. But it is a longstanding and widespread problem that large groups of students seem to get stuck at this bridge between arithmetic and algebra (Herscovics & Linchevski, 1994). This has major personal and societal consequences because basic algebra as taught in lower secondary school plays a crucial role in upper secondary mathematics and hence for access to attractive higher education programmes. In that way school algebra is often described as a central gatekeeper (Loveless, 2013).

The last decades of algebra research have given more attention to the theoretical foundations of students' work. Algebraic transformations are not viewed only as procedures, but also as theoretical entities (Kieran, 2007). Kieran divides research in algebraic transformations into theoretical, technical, and practical elements. These elements are closely connected, and different institutions within the educational system manage them in subtly different ways. Early research on school algebra tended to make a sharp distinction among procedural and conceptual approaches. This dichotomy is currently challenged, and the potential of new theoretical and methodological approaches become essential to investigate the crucial *connections* between techniques and theory (Schneider & Stern, 2010). The Anthropological Theory of the Didactic offers, in particular, a promising new approach to this task.

To investigate the transition from arithmetic to algebra and to gain knowledge about what algebraic techniques and theory are particularly problematic for students in Danish lower secondary school, we ask the research question: "How can the Anthropological Theory of the Didactics (ATD) and the construction of a praxeological reference model (PRM) be used as foundation for developing a diagnostic test tool, to examine students technical and theoretical algebra knowledge?"

This paper will concentrate on the construction of the PRM and the derived diagnostic test tool with preliminary results from the pilot test. The PRM and the results of the diagnostic test will be used in a latter study to select research-based resources and design teaching interventions to support teachers teaching basic algebra.

1.2 ATD as theoretical foundation

The Anthropological Theory of Didactics, subsequently noted as ATD, has emerged as a theory of mathematics education. A central feature in ATD is the use of praxeology to model school mathematics activity. A praxeology compromise types of task, techniques, technologies, and theories (Bosch, 2015). The "practical block" or *praxis* is formed by the types of task and by the techniques used to solve them (Barbé et al., 2005). The "theoretical block" or *logos* consists of technology (discourse on techniques) and theory (more general discourse, based on deductive reasoning from definitions and the like). This means the techniques for carrying out tasks are explained and justified by a 'discourse on the technique' called technology; taking this discourse to a more abstract level yields mathematical theory, to validate the technological discourse and to connect entire praxeologies (Bosch, 2015).

The use of praxeology to analyse school algebra has been particularly successful, since the birth of ATD as a theoretical foundation for mathematics education research (Bosch, 2015). Bosch argues that the explicit reference praxeological models (PRM) concerning school algebra provides opportunities to ask research questions that go beyond the assumptions held by the school institution itself. At the international level, ATD research has led to significant new insights on the algebra problem, including the frequent disconnectedness of praxeologies taught and learnt.

According to Chevallard (2019), praxeologies are not static, but a dynamic system of institutionally situated activities. The explicit construction of a PRM will enable us to analyse what arithmetic and algebraic praxeologies are currently taught in the Danish lower secondary school according to curriculum, textbook material, and written examination. The PRM will also form the foundation for developing a diagnostic test tool, which can “diagnose” what algebraic techniques and theory are problematic for students (in our case, Danish grade 7). The result of the diagnostic test is analysed in terms of the PRM and may lead to revise the tool (e.g., if unexpected techniques appear). In a later study the diagnostic tool and the associated PRM will be used to design the intervention based on resources and to analyze the effects of the interventions. Thus, the model is the researchers’ explicit reference throughout the four-step research process shown in Figure 1.

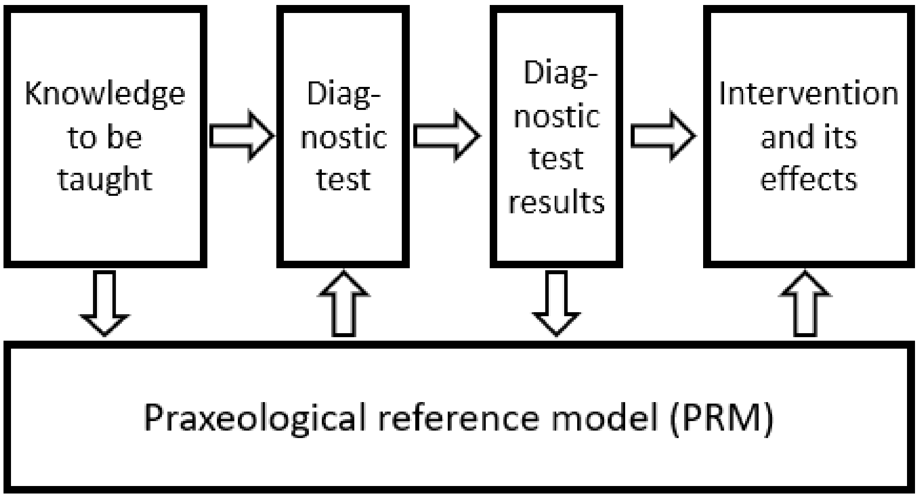


Figure 1. Construction and use of the praxeological model

Methods to construct the praxeological model

In Denmark, the so-called “common goals” for mathematics (Danish Ministry of Education, 2019), constitute the official directives for all primary and lower secondary school. They are divided into four parts: competences, number and algebra, geometry and measurement, statistics, and probability. The overall goal for algebra is that “the student can apply real numbers and algebraic expressions in mathematical investigations”. It is up to textbook authors and teachers to transpose the common goals into teaching practice; in addition to the official goals, they can also find some direction in the exercises appearing the national exam after grade 9.

Our first step in building the PRM was to identify types of tasks appearing in the 2019 national exam (Prøvebanken, 2021) and in the textbook series Kontext+ grade 7-9 (Lindhardt et al., 2021). We began by identifying the tasks in the material associated to arithmetic and algebra, understood as tasks solely focused on operations, equations and order relations involving numbers (arithmetic), or numbers and literal symbols (algebra). At the level of theory, operations are in these two cases obeying the axioms of ordered integral domains or fields. In this paper, we do not consider the use of CAS tools and instrumented techniques.

The second step is to analyse in terms of task type T_i and corresponding techniques τ_i used to solve T_i. Let us first take an example from arithmetic where students are asked to calculate the following tasks (Lindhardt et al., 2021, p.104).

a. $6 + (-7)$
b. $5 - (-7)$
c. $-3 - (-6) + (-4) - 2$

Tasks a. and b. are simple (they can be solved by one technique). Task a. is what we have named type T_4: addition of negative integer to a positive integer, with corresponding technique τ_4: $a + (-b) = a - b$; where b. is a task of type T_6: subtraction of negative integer from negative integer, with corresponding technique τ_6: $a - (-b) = a + b$. Task c. requires both techniques τ_4 and τ_6 and is thus a combination of more elementary tasks.

An example of a type of task from algebra is T_{15}: solve a first-degree equation. Tasks of this type appear for instance in the written national 2019 exam (Prøvebanken, 2021)

Solve the equations

a. $5x + 9 = 34$ $\quad x =$ _______
b. $3x + 4 = 6x - 5$ $\quad x =$ _______

They can be solved by the technique τ_{15}: involving addition, subtraction, multiplication, and division on both sides of the equal sign.

In general, exercises can contain several questions, where not all questions can be answered by a single technique. This means, that once a model of types of tasks and corresponding techniques has been established, more complex questions must be decomposed in tasks of the types established (Winsløw et al., 2013). It is relatively straightforward to identify types of tasks and techniques in arithmetic and algebra as shown above (cf. Wijayanti & Winsløw, 2017). The next step is to identify *themes* (groups of practice blocks unified by a technology) and *sectors* (groups of themes unified by a theory).

1.2.1 Example of PRM theme and sector

The problem of ordering two given fractions can, according to special cases, necessitate different techniques. It thus leads to group of practices which are taught together and are unified by a shared discourse about techniques, involving characteristics of the special cases, and descriptions of the techniques. Table 1 is type of task T_i and corresponding techniques τ_i from the analysed textbook material and written exam and gives an overview of the theme of PRM according to fractions.

Table 1. Theme of PRM based on textbooks (Kontext) and written exam (FSA) according to fractions

<table>
<tr><th>Type of task</th><th>Techniques</th><th>Kon-text+ 5</th><th>Kon-text+ 6</th><th>Kon-text+ 7</th><th>Kon-text+ 8</th><th>Kon-text+ 9</th><th>FSA 2020 Dec</th><th>FSA 2021 May</th></tr>
<tr><td>T_{19}: Given two-unit fractions $\frac{1}{a}$ and $\frac{1}{b}$, which is largest</td><td>τ_{19}: The fraction with the lowest denominator is largest i.e., $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$</td><td style="background:#c0c0c0"></td><td style="background:#c0c0c0"></td><td></td><td style="background:#c0c0c0"></td><td></td><td></td><td></td></tr>
<tr><td>T_{20}: Examine which fraction with like denominator and different numerators, $\frac{a}{c}$ and $\frac{b}{c}$ is largest.</td><td>τ_{20}: The fraction with the highest numerator is largest i.e., $if\ a < b \ \Rightarrow \frac{a}{c} < \frac{b}{c}$</td><td style="background:#c0c0c0"></td><td style="background:#c0c0c0"></td><td></td><td style="background:#c0c0c0"></td><td style="background:#c0c0c0"></td><td></td><td></td></tr>
<tr><td>T_{21}: Examine which fraction with like numeration and different denominators fraction $\frac{a}{b}$ and $\frac{a}{c}$ is largest.</td><td>τ_{21}: The fraction with lowest denominator is largest i.e., $if\ a < b \ \Rightarrow \frac{a}{b} > \frac{a}{c}$</td><td style="background:#c0c0c0"></td><td style="background:#c0c0c0"></td><td style="background:#c0c0c0"></td><td style="background:#c0c0c0"></td><td style="background:#c0c0c0"></td><td></td><td></td></tr>
</table>

It is characteristic of Danish textbooks that the same types of tasks reappear year after year, while the theme as a whole may still be relatively disconnected. This theme is a part of a larger “fraction sector”, unified by a theory which involves both informal and more formal representations and properties of fractions, from addition of simple fractions with pizza diagrams to calculation rules given with algebraic symbolism.

1.3 Methods to construct the diagnostic test tool

The aim of the diagnostic test tool is to detect what arithmetic and algebraic techniques are particularly problematic for Danish students at early lower secondary school, and to get insight in the students' theoretical knowledge especially in relation to basic algebra. The development of the diagnostic test tool is based on the PRM and was inspired by an earlier project by Cosan (2021) on middle school arithmetic.

Items designed to test techniques are straightforward to construct. And several items can be included to investigate how variations influence on success rates. For example, the item "Compute $6 + (-5)$" represents the type of task T_4: addition of negative integer to a positive integer, as defined above.

It is more difficult to design items that detect students' theoretical knowledge which includes technology to describe and explain techniques. The item "Explain why $a - (-a) = 2a$" requests from students a piece of discourse justifying the algebraic rule, which may appeal to more or less formal theoretical principles. For instance, some students may refer to "two minuses can be replaced by a plus" as an overarching principle in such contexts; this could, in fact, be part of a theory that some students hold. In the ATD sense such theory elements are empirical objects, to be discovered and traced.

Items can also simply request a description of a technique (i.e., a technology), e. g. "Explain how you would determine which of the fractions $\frac{2}{5}$ and $\frac{3}{5}$ is largest"

1.4 Results from the pilot test of the diagnostic test tool

The diagnostic test has been pilot tested by 25 grade 7 students (12–13-year-old) in lower secondary school in the capital of Denmark. The students got 45 min. to do the 67-item paper and pencil test.

Table 2. Sum of type of answers in the test

Correct answer	Incorrect answer	No answer	Sum
455	306	914	1675

Table 2 shows that more than half of the items has not been answered by the students. Despite this, it is possible to give some preliminary results. The following are examples of analyses of test answers in relation to the previously selected examples from the PRM.

Table 3. Item and associated sum of answers in the test

Item	Item number	Correct answer	Incorrect	No answer
$6 + (-5) =$	1.4	20	2	3
$7 - (-9) =$	1.5	8	16	1

Table 3 shows that almost all the students can solve the item of task type T_4: addition of negative integer to a positive integer. But only a third of the students could solve the item of task type T_6: subtraction of negative integer from negative integer. This result indicates that students know that adding opposite is the same as subtraction, but they cannot apply the rule $-(-a) = a$.

By varying the items given for a specific type of task, we can also discover specific features of type of task. For example, the test contains variations of the task type T_{15}: solve a first-degree equation (discussed above). These variations result in hugely different success rates.

Table 4. Test results of variations of the task type T_{15}

Item	Item number	Correct answer	In-correct	No answer
$36 - \_\_\_\_ = 29$	1.1	21	2	2
$8 + 4 = \_\_\_\_ + 5$	1.6	4	18	3
$\_\_\_\_ - 32 = 45$	2.1	23	1	1
$2x = 10$	2.5	10	1	14
$7x - 7 = 13 - 3x$	4.7	8	9	13

When comparing the results from item number 1.1, 1.6 and 2.1, item number 1.6 has significantly fewer correct answers. One possibly crucial difference between 1.6 and the other questions is the location of the unknown. In 1.6 the unknown is located on the right side of the equal sign. That this could make a big difference is confirmed by a longitudinal examination of how middle school students understand the equal sign and equivalence of equations (Alibali et al, 2007).

The purpose of item number 3.7 is to get insights in students' argumentation for which fraction is largest. And we get answers like: "If the denominators are the same, I just look at the numerators which are the highest" and "I look at the denominators and if the fractions have the same

denominator, then I will look at the numerator which one is the largest”. With such items, we can detect not only a (correct) technique but also what it is, and a level of technology.
In the following two examples, the student’s argumentation is based on pizza representations of fractions, which are also extensively used in Danish textbooks. Figure 2. “I want to draw”. Figure 3. “By seeing it as a circle and look where there are most fields that are filled”.

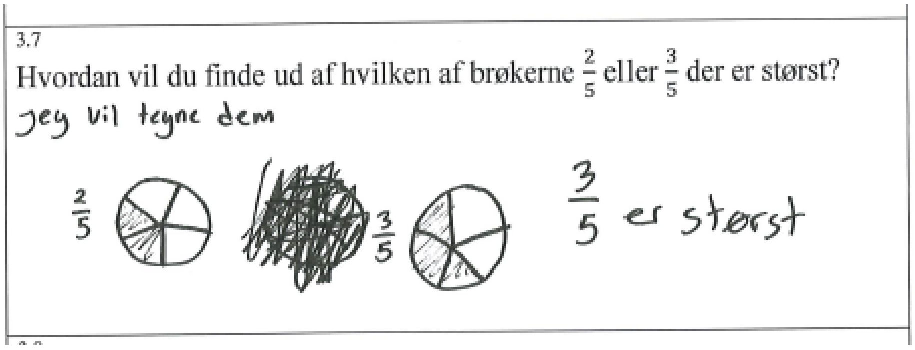


Figure 2. Student answer to question 3.7

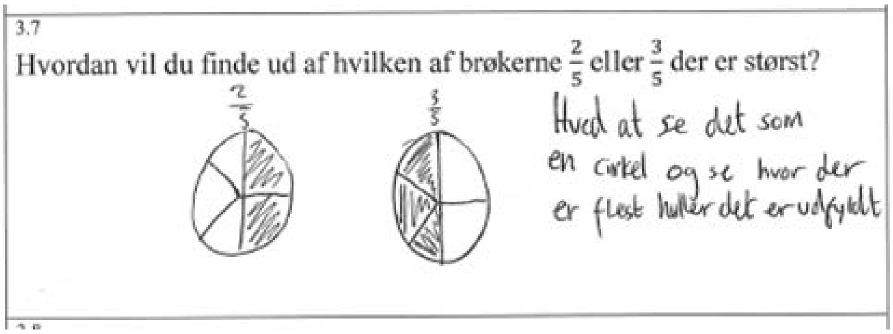


Figure 3. Student answer to question 3.7

In Figure 2, we can see the representation serves as an argument but in Figure 3, this wordless technology fails because of missing the usual convention, that the circle must be divided into equal parts. The diagrammatic representation has evident forces in giving meaning to fractions (between 0 and 1) for young children; but the division of a circle into five equal parts is also a task which contain other meanings (including angles etc.) that are in some sense irrelevant to the task. Varying this task to include fractions with large denominators or nominators would evidently also make this technology fail.

1.5 Conclusion

In this paper, we have shown by a few examples, how ATD and the construction of a PRM, can be used as a foundation for developing a diagnostic test tool, to examine students technical and theoretical knowledge. By first constructing the PRM based on textbook material, written examination, and common core. Then grouping the task and corresponding techniques in themes and sectors according to shared technology and theory. The PRM is then used to design the diagnostic test items in line with the tasks, techniques, and level of theory in the PRM. In that way the diagnostic test is aligned with the knowledge to be taught and provide explicit control on what is being tested.

Among the examples from the pilot test, we discussed the students' difficulties with relating subtraction and additive inverse, and especially with repeated additive inversion. These examples indicate that the diagnostic test, based on the PRM, can be used to identify significant obstacles. This is crucial for the next steps in our doctoral project.

The next step in the project is to apply the results from the pilot-test to inform and strengthen the PRM. Then to revise the diagnostic test tool and complete the final test in four classes, before and after the teaching intervention. The overall aim is to investigate the transition from arithmetic to algebra and to explore if and how interventions with research-based material can support teachers' efforts to teach basic algebra.

1.6 References

Alibali, M. W., Knuth, E. J., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. Mathematical Thinking and Learning, 9(3), 221–247. https://doi.org/10.1080/10986060701360902

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice: The case of limits of functions in Spanish high schools. Educational Studies in Mathematics, 59, 235–268. https://doi.org/10.1007/s10649-005-5889-z

Barbieri, C. A., Rodrigues, J., Dyson, N., & Jordan, N. C. (2020). Improving fraction understanding in sixth graders with mathematics difficulties: Effects of a number line approach combined with cognitive learning strategies. Journal of Educational Psychology, 112(3), 628–648. https://doi.org/10.1037/edu0000384

Bosch, M. (2015). Doing research within the anthropological theory of the didactic: The case of school algebra. In Selected Regular Lectures from the 12th International Congress on Mathematical Education (pp. 51–69). Springer International Publishing. https://doi.org/10.1007/978-3-319-17187-6_4

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. Hiroshima journal of mathematics education, 12, 71–114.

Cosan, D. D. (2021). A diagnostic test for Danish middle school arithmetic. MSc-dissertation, IND's studenterserie nr.102, University of Copenhagen. https://www.ind.ku.dk/publikationer/studenterserien/a-diagnostic-test-for-danish-middle-school-arithmetics/

Danish Ministry of Education (2019). Fælles mål matematik. Danish Ministry of Education. Located 10.12.2020 at https://emu.dk/grundskole/matematik/faghaefte-faelles-maal-laeseplan-og-vejledning

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. Educational studies in mathematics, 27(1), 59–78. https://doi.org/10.1007/BF01284528

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester (Ed.), Second handbook of research on mathematics teaching and learning (pp. 707-762). National Council of Teachers of Mathematics.

Lindhardt, B., Thomsen, H., Johnsen, L. B. & Hansen J. (2021). KonteXt+7, i-bog. Alinea. Located 01.03.2021 at https://kontextplus.alinea.dk/course/QRAT-kontext-7

Loveless, T. (2013). The algebra imperative: Assessing algebra in a national and international context. Brookings. https://www.brookings.edu/wp-content/uploads/2016/07/Kern-Algebra-paper-8-30_v13.pdf

Prøvebanken (2021). Folkeskolens afgangsprøve i matematik. Danish Ministry of Education. Located 10.06.2021 at https://www.xn--prvebanken-1cb.dk/

Schneider, M., & Stern, E. (20210). The developmental relations between conceptual and procedural knowledge: A multimethod approach. Developmental psychology, 46(1), 178.

Wijayanti, D., & Winsløw, C. (2017). Mathematical practice in textbooks analysis: Praxeological reference models, the case of proportion. REDIMAT, 6(3), 307-330.

Winsløw, C., Matheron, Y., & Mercier, A. (2013). Study and research courses as an epistemological model for didactics. Educational Studies in Mathematics, 83(2), 267–284. https://doi.org/10.1007/s10649-012-9453-3

Paper V

The role of algebraic models and theory in Danish lower secondary school.

Abstract

In this paper, we address the external didactical transposition of algebra in Danish lower secondary school. The theory of praxeology and especially the construction of a praxeological reference model enable us to analyse what elementary algebra praxeologies are currently to be taught in Danish lower secondary school according to curriculum, textbook material, and the written examination at the end of grade 9. To exemplify the set of conditions and constraints that affect the role of algebraic modelling and level of theory in praxeology we will use "algebra models in geometry" as a case.
Keywords: External didactic transposition, praxeological reference model, elementary algebra

Introduction and research question

In the written examination after lower secondary school, the majority of Danish students struggle with setting up an expression for the area and perimeter of a polygon with only vertical and horizontal sides, when given symbols for the relevant lengths, Figure 3. The lack of knowledge on how to use algebra as a modelling tool is a central issue in the current debate on the "algebra problem" in Danish lower secondary school (Østergaard, 2021). The generally modest algebra skills of students become visible not only at the written exam after lower secondary school. In secondary school, basic algebra is a crucial bridge between arithmetic and more advanced subjects involving functions and analytic geometry. According to Strømskag and Chevallard (to appear) the problem is not only a Danish one: also, in France and Norway, school algebra has become a set of formal exercises, rather than a modelling tool; they argue for "an imperative revitalization of the elementary algebra curriculum" (Strømskag & Chevallard, 2022, p. 1).

In Denmark, the so-called "common goals" for mathematics constitutes the official directives for primary and lower secondary school. The overall goal for algebra after lower secondary school (grade 9), is that "the student can apply real numbers and algebraic expressions in mathematical investigations" (Education, 2019). However, teachers base their teaching on textbooks and national exams, which are often more modest in their demands when it comes to algebra. Danish textbooks are crafted by “leading” math teachers, based on personal didactical ideas and experiences. The teachers also find some directions in the exercises appearing in the national exam after grade 9. To get insights in the set of conditions and constraints that affect the knowledge to be taught, we will look at the external didactic transposition (Bosch et al., 2021), and explore the research question: *What is the role of algebraic models and theory in Danish lower secondary school?*

Theoretical framework and methodology

As a teacher educator, textbook writer, and Ph.D. student, I am formed by the set of institutions I am acting in. However, the use of the Anthropological Theory of the Didactic as theoretical framework for my doctoral thesis gives me the opportunity as a researcher to detach myself from any specific institutional viewpoint (Bosch, 2015). To distance from common-sense models used within understand the institutions, we use *modelling* in the ATD sense as a didactic tool to structure and integrate modeling processes in a more general epistemological model of institutional mathematical activities (Garcia et al., 2006).

The overall aim of the doctoral project is to investigate the transition from arithmetic to algebra and to explore if and how research-based teaching materials can support and direct teachers’ efforts to teach basic algebra. In this first part of the study, we study only the external transposition to get insight into the set of conditions and constraints that affect the knowledge to be taught (Bosch et al., 2021).

To investigate the role of algebraic models and theory in Danish lower secondary school, we build a praxeological reference model (PRM) based on curriculum, textbook material, and written examination. The praxis is formed by type of tasks and by the techniques used to solve them, and logos consisting of technology and theory (Barbé, Bosch, Espinoza & Gascón, 2005). The explicit construction of a PRM will enable us to analyse what arithmetic and algebraic praxeologies are currently to be taught in Danish lower secondary school. The PRM includes themes such as and algebraic models in geometry. We will consider this theme as a case example and refer to the more comprehensive PRM (not presented here) by type of tasks T_i and corresponding techniques τ_i.

Algebraization in geometry

For the analyses one of the most common textbook material Kontext+ grade 5 to 9 (Lindhardt, Thomsen, Johnsen & Hansen, 2021) and the national 2019 written paper and pencil exam (Prøvebanken, 2021), are used. In the analyses two specific types of "algebra models in geometry" appear (among others). The first type of tasks is "Determine the perimeter of the polygon" denoted T_{27}, where the technique τ_{27}: Add the side lengths of the polygon, will solve this type of tasks. The second type of tasks is T_{28}: Determine the area of a polygon with all sides being either parallel or orthogonal, with the corresponding technique, τ_{28}: Calculate the area by using the formula of rectangle area together with the additive principle (the area of a disjoint union of polygons is the sum of the area of those polygons). At the level of algebraic theory, the distributive law appears.

We now analyze an example of each of this type of tasks and give an example of the theoretical approach. To analyze the examples, we distinguish between "arithmetical rules" and "algebraic formulas" and the essential notion of parameters in line with Strømskag and Chevallard (2022).

Geometric multiplication of binomials

The first example is from Kontext+8 a grade 8 textbook material.

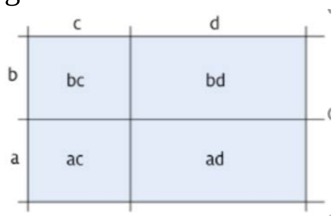


Figure 1. Floor with four rooms (Lindhardt et.al. 2021, p.87)

a) Show from the drawing, why $(a + b) \cdot (c + d) = ac + ad + bc + bd.$
b) How large is the area of the four rooms if $a = 5, b = 25, c = 10 \; and \; d = 40$?

To answer question a., we can τ_{28} with the given subdivision, and no subdivision. For the latter approach, the side lengths to be used are $l = a + b$ and $b = c + d$. Then we get that the area is $A = l \cdot b = (a + b) \cdot (c + d)$. To express the area of the four small rectangles we use the formula of the area of a rectangle for each of the small rectangles and sum these up. It means that can deduce the formula from algebraic model of area, with no need to manipulate the algebraic expression.

In question b. we must interpret what is being described as rooms to be the small rectangles. Then we can use the algebraic model of area of a rectangle, and by inserting the definite numerical values we get the numeric areas. This is a technique taught in grade 5, if following the textbook cited above.

Distributive law

The distributive law is the crucial links between addition and multiplication (and a field axiom) which forms part of the level of theory of both arithmetic and algebra in the PRM. In the textbook the point is made that "you can use geometric figures to model arithmetic rules by letters". In fact, what the example in Figure 2 does is to deduce an algebraic law by using τ_{28} on a particular geometric figure.

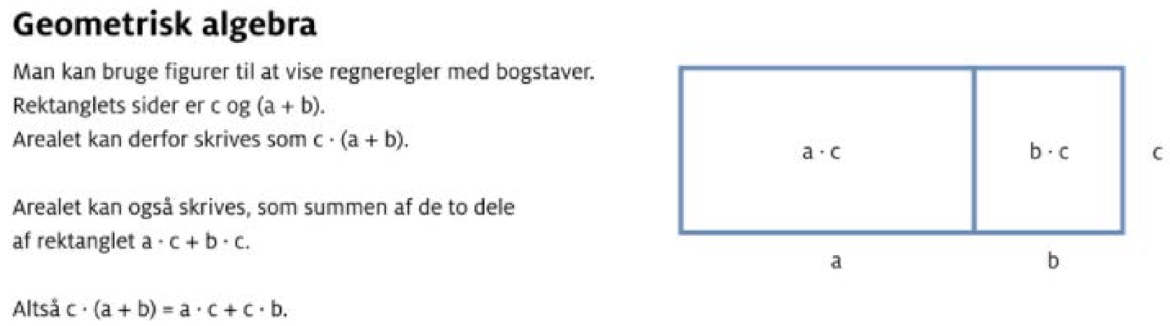


Figure 2. Geometric algebra (Lindhardt et.al. 2021, p.95)

Specifically, τ_{28} tells us that the area of the big rectangle is $a \cdot c + b \cdot c$, and it is also $(a + b) \cdot c$. The generality and variations of the distributive law stays implicit, and the students are not introduced to "distributivity" as an assumption or axiom in algebra. It is not visible for the student that the example is special (for instance, assumes that $a, b > 0$).

Perimeter and area of an irregular octagon

Our last example is from the national written paper and pencil examination after grade 9 (Prøvebanken, 2019). Figure 3 shows an octagon with marked sides of length a and b.

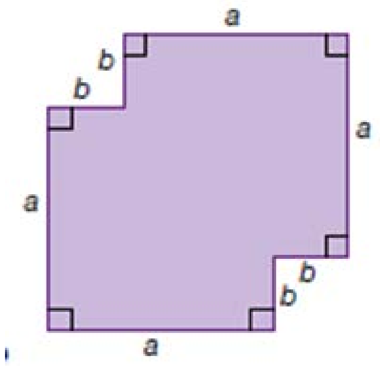


Figure 3. Irregular octagon

a) The perimeter of the octagon is ______________
b) The area of the octagon is _______________

Here, c. is of the type T_{27}, and τ_{27} yields that the circumference is $a+b+b+a+a+b+b+a$; then, another algebraic technique (τ_{15}: collect equal terms) can be used to get the final result 4a+4b.

Question d. is again of type T_{28}. Here must model the area, by dividing the octagon into rectangles, which can be done in several different ways, resulting in expressions such as $(a+b)^2-2b^2$ or $a^2+ab+b(a-b)$, which of course can both be reduced to a^2-b^2+2ab by τ_{15}.

Discussion and conclusion

In the Danish curriculum for grades 6 to 9, the aim of the topic "Formulas and algebraic expressions" phase one is that "the student can describe connections between simple algebraic expressions and geometric representations" (Education, 2019).

The previous textbook examples provide examples of the use of algebraic models in geometry. Types of tasks in the theme "Algebra models in geometry" are mostly formal exercises solved by inserting values in known or given formulae. The level of theory is implicit and algebraic manipulation occurs rarely (the examples above are thus somewhat special). When algebra turns into a modelling tool, it is often in relation to simple geometric situations. And even though such modelling exercises do occur in the textbook material, the results from the written examination, where only 9.6% could answer questions c. and d. (above) properly, suggest that algebraic models and theory are not really a part of the students' topos (Chevallard, 2019).

Reference

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic Restrictions on the Teacher's Practice: The case of limits of Functions in Spanish High Schools. Educational Studies in Mathematics, 59, 235–268. https://doi.org/10.1007/s10649-005-5889-z

Bosch, M. (2015). Doing research within the anthropological theory of the didactic: the case of school algebra. In Selected Regular Lectures from the 12th International Congress on Mathematical Education (pp. 51–69). Springer International Publishing. https://doi.org/10.1007/978-3-319-17187-6_4

Bosch, M., Hausberger, T., Hochmuth, R., Kondratieva, M., & Winsløw, C. (2021). External didactic transposition in undergraduate mathematics. International Journal of Research in Undergraduate Mathematical Education, 7(1), 140-162. https://doi.org/10.1007/s40753-020-00132-7

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. Hiroshima journal of mathematics education, 12, 71–114.

Danish Ministry of Education (2019). Fælles mål matematik. Danish Ministry of Education. Located 10.12.2020 at https://emu.dk/grundskole/matematik/faghaefte-faelles-maal-laeseplan-og-vejledning

Garcia, F. J., Gascón, J., Ruiz Higueras, L. & Bosch, M. (2006) Mathematical modelling as a tool for the connection of school mathematics. ZDM, 2006, Vol. 38(3), pp. 226-246. http://doi.org/10.1007/BF02652807

Lindhardt, B., Thomsen, H., Johnsen, L. B. & Hansen J. (2021). KonteXt+7, i-bog. Alinea. Located 01.03.2021 at https://kontextplus.alinea.dk/course/QRAT-kontext-7

Prøvebanken (2021). Folkeskolens afgangsprøve i matematik. Danish Ministry of Education. Located 10.06.2021 at https://www.xn--prvebanken-1cb.dk/

Strømskag, H., & Chevallard, Y. (to appear). Elementary algebra as a modelling tool: A plea for a new curriculum [Manuscript submitted for publication].

Strømskag, H., & Chevallard, Y. (2022). Conditions for revitalizing the elementary algebra curriculum. [Paper presentation] 12th Congress of the European society for Research in Mathematics Education, 2-5 February 2022, Bozen-Bolzano, Italy.

Østergaard, K. (2021). Private conversation in relation to a presentation of the results of the national written examination.