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Epistemic Value Pluralism in the Practice of Stochastic Analysis

Case study in the dynamic approach

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"*One belief, more than any other, is responsible for the slaughter of individuals on the altars of the great historical ideals - justice or progress or the happiness of future generations, or the sacred mission or emancipation of a nation or race or class, or even liberty itself which demands the sacrifice of individuals for the freedom of society. This is the belief that somewhere, in the past or in the future, in divine revelation or in the mind of an individual thinker, in the pronouncements of history or science, or in the simple heart of an uncorrupted good man, there is a final solution.*"

Isaiah Berlin, 1969

"*Mathematicians can and do fill in gaps, correct errors, and supply more detail and more careful scholarship when they are called on or motivated to do so. Our system is quite good at producing reliable theorems that can be solidly backed up. It's just that the reliability does not primarily come from mathematicians formally checking formal arguments; it comes from mathematicians thinking carefully and critically about mathematical ideas.*"

William P. Thurston, 1994

Preface

This thesis is the culmination of my three distinct academic programs: a BA in philosophy, a BSc in mathematics, and a MSc in insurance mathematics. I firmly believe that this project, in its present form, would have been impossible without the unique combination of the three disciplines.

My naïve understanding of mathematics, shaped during my Bachelor of Arts (Philosophy), was soon disrupted when I encountered mathematics in practice. I had assumed formalism to be the essence of mathematics, an ontology in itself. Instead, I discovered that formalism is a tool, not an end. Practical considerations often take precedence over strict adherence to rigor and precise definitions. What I once perceived as "categorical mistakes" turned out to be deeply embedded features of mathematical thinking.

This realization was a turning point. The illusion of analytical philosophy gave way to a richer understanding: intuition is not an error, but an essential part of mathematical practice. Formalism is merely *one* epistemic tool among many; it does not define the entirety of mathematics, nor should it be its only ideal.

In this thesis, I explore one of the clearest examples I have found of epistemic pluralism in mathematics: *the dynamic approach*. This approach exemplifies the tension between rigor and intuition. My hope is to show that they are not mutually exclusive but complementary aspects of the same mathematical practice. By embracing both, one can deepen one's understanding of mathematics as a discipline of both precision and utility.

Acknowledgements

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I am deeply grateful to Frederik R. Klausen for persistently challenging me to question whether I was pursuing trivial insights through speculative methodologies, a potential pitfall in both philosophy and mathematics. I am both surprised and satisfied that you found this endeavor enlightening.

I am grateful to *the Study Group in the Philosophy of Mathematical Practice* for the opportunity to present this project in its early stages. A special thanks goes to Martin P. Speirs for prompting reflections on broader implications beyond the case of stochastic analysis and to Henrik K. Sørensen for highlighting the historical significance of the topic.

Thank you to the organizers of *the European Summer School on the Philosophy of Mathematics* at *the University of Vienna* in September 2024 for granting me admission to the program. The lectures by Yanic Hamami and Matthew Inglis were particularly influential in shaping the philosophical focus of this project, especially regarding mathematical rigor and its epistemic dimensions.

I owe immense gratitude to my two supervisors:

To Jesper L. Pedersen, thank you for introducing me to the dynamic approach within the context of stochastic processes in life insurance and for guiding me in refining the scope of the case study presented here.

To Mikkel W. Johansen, it is difficult to overstate your influence on my academic journey into the philosophy of mathematical practice. Your unwavering support as a teacher, employer, and supervisor has been instrumental in my development as a philosopher. Your encouragement, critical feedback, and belief in my abilities have profoundly shaped not only this project but also my broader academic perspective.

Abstract

This thesis is a case study of the dynamic approach in stochastic analysis. As a response to *rigor pluralism* in Tanswell (2024), I present two proofs from the practice of mathematics to argue for *epistemic value pluralism*.

The contemporary discussion of mathematical rigor will be presented through advocates of opposing views. I argue that *the standard view of rigor* presents an operational conception that enables a clear demarcation of rigorous and unrigorous proofs. The success and failure in ascribing rigor present the possibility to address other valuable qualities of unrigorous proofs. Unlike rigor pluralism, which risks collapsing into *semantic relativism*, epistemic value pluralism better accommodates the diverse epistemic functions of proofs within mathematics.

The analysis and discussion are anchored in a case that presents two proofs of the same theorem in stochastic analysis: one by *the dynamic approach* and the other by *the monotone class theorem*. Although the proof by the monotone class theorem is rigorous, the proof by the dynamic approach, though unrigorous, demonstrates significant epistemic value by fostering intuition. I argue that such unrigorous proofs, although lacking *precision*, have an indispensable *utility* for mathematics.

Furthermore, I explore the ontological implications of *semantic externalism*, emphasizing how *externalization* of mathematical objects beyond their definitions allows complementary characterizations. By this, producing deeper familiarity with the concept.

In conclusion, epistemic value pluralism provides a robust framework for analyzing mathematical practice, demonstrating how unrigorous proofs, such as the one by the dynamic approach, offer indispensable utility by fostering intuition.

Contents

1 Introduction

This thesis explores epistemic considerations within the practice of stochastic analysis, focusing on the *dynamic approach* in martingale theory. By examining two proofs of the same theorem, one by the *dynamic approach* and one by the *monotone class theorem*, the aim is to highlight the unique epistemic functions of the dynamic approach.

The motivation for this analysis stems from a response to *rigor pluralism* as presented by Fenner S. Tanswell (2024). Although I share the goal of addressing the tension between the articulated standard of *rigor*[1] in mathematical practice and insights from the philosophy of mathematical practice regarding proofs' plurality of purposes, I argue that *semantic relativism*, as proposed by Tanswell, falls short of providing the conceptual tools necessary to fully analyze this duality. Instead, I propose *epistemic value pluralism* as a more effective framework to resolve this tension.

To contextualize this argument, I review competing perspectives on rigor in relation to Tanswell, drawing on Yacin Hamami and Jody Azzouni, both of whom are explicitly addressed by Tanswell and classified as proponents of *the standard view of rigor*. Although the standard view of rigor represents an orthodoxy in the philosophy of mathematical practice (Hamami, 2022, p. 409), it serves as a suitable classification tool for this analysis. However, I diverge from Tanswell and Hamami by interpreting Azzouni's work as representing a *view of proofs* rather than strictly adhering to the standard view of rigor. That said, Azzouni's characterization of proofs is a foundational assumption of the standard view, which warrants its inclusion in my exploration of competing theories in Chapter 2. In addition, Azzouni presents *the puzzle of informal mathematics* that will function as a guiding tool for the exploration of rigor in Chapter 2 as well as a motivation to present the case in Chapter 3.

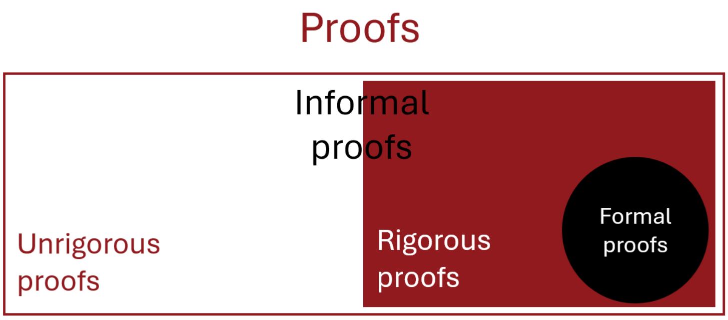
Before presenting the case study in Chapter 3, I introduce the empirical findings of Matthew Inglis and Andrew Aberdein. This inclusion reflects the spirit of the philosophy of mathematical practice, which seeks to complement conceptual speculation with empirical insights from mathematical practice (Carter, 2024, pp. 2–3). Moreover, I refrain from offering normative conclusions about

[1] English: Rigour

what mathematicians *should* do. Instead, my aim is to uncover the implicit epistemology within the practice, asking: What *worldview* does an epistemic agent in mathematical practice implicitly endorse?

In Chapter 3, I dive into the practice of stochastic analysis as presented by Per K. Andersen, Ørnulf Borgan, Richard D. Gill and Niels Keiding. This case study functions as my primary empirical source. After presenting the general results in Inglis and Aberdein (2015), I zoom in on one particular subfield of mathematics to identify the use of unrigorous tools as well as to justify their use. Andersen *et al.* (1993), which explicitly employs heuristic tools, serves as a rich resource to understand unrigorous mathematics. This aligns with the notion that edge cases can reveal smaller tendencies in broader practice (Flyvbjerg, 2006, p. 229). To fully illustrate the case, I actively reconstruct two proofs: one by the dynamic approach using conceptual tools from Andersen *et al.* (1993), and another by the monotone class theorem with techniques from Schilling (2005). Although reconstruction risks misrepresentation, it is necessary to compare these proofs effectively. To maintain fidelity to the source, I directly refer to Andersen *et al.* (1993) throughout Chapter 3.

One challenge encountered in this case study is the inconsistency in terminology between sources. For example, "formal" and "informal" proofs are used differently in various contexts. In Chapter 2, I harmonize these terms where possible, defining a "formal proof" as one adhering to the formal definition and categorizing all other proofs as "informal". A "rigorous proof" is then a subset of proofs that includes some informal proofs and all formal ones, as illustrated below. This contrasts with the terminology of Andersen *et al.* (1993), where "formal" and "informal" align more closely with "rigorous" and "unrigorous". To avoid misrepresenting Andersen *et al.* (1993) or engaging in circular reasoning - concluding that unrigorous proofs are instances of unrigorous mathematics - I preserve their original terminology in Chapter 3 and ask the reader to remain aware of these changes in language.



In Chapter 4, I apply Hamami's framework to analyze the two proofs from Chapter 3, concluding that the proof by the dynamic approach is unrigorous, while the proof by the monotone class theorem is rigorous. Drawing on the four-dimensional matrix of terms of evaluation presented in Inglis and Aberdein (2015), I argue that both proofs have value for mathematical practice, an insight more effectively captured through epistemic value pluralism than through the semantical relativism of rigor pluralism.

Finally, in Chapter 5, I extend the discussion to ontological considerations, examining the *externalization* of mathematical objects from their definitions within the framework of *semantic externalism*, a contemporary position in the philosophy of language. This discussion broadens the implications of my case study, shedding light on the epistemic roles of characterizations in mathematics. Although I do not take a definitive position on the ontological implications of these findings, I outline potential directions for future research.

This case study highlights the various epistemic functions of proofs in stochastic analysis, ultimately advocating epistemic value pluralism as the preferred framework for capturing this plurality. By presenting a concrete case from mathematical practice, this paper contributes to the ongoing discussion of rigor at the intersection of a case from mathematical practice and an analysis from philosophy.

2 Epistemic Value Pluralism

In exploring two fundamentally different conceptions of *liberty* — *negative liberty* as freedom from interference, and *positive liberty* as the freedom to be one's own master or legislator — Isaiah Berlin introduces the ideas of *value monism* and *value pluralism*. The former, rooted in ancient Greek philosophy, posited that the good, the right, and the beautiful ultimately refer to the same thing. It asserted that no values can ever conflict with one another; otherwise, the world would be chaos, not cosmos.

The metaphysical tradition, from Plato to Hegel, rejected the idea that maximizing one value might necessitate abandoning other goals. For these philosophers, the coexistence of multiple values was regarded as a formal contradiction: a "metaphysical chimera" (Berlin, 1969, p. 28).

The reduction of all questions of value to a single principle defines value monism. Its counterpart is value pluralism, that recognizes that human goals are numerous and diverse in nature. Value pluralism acknowledges the complexity and multifaceted nature of our normative concerns, often necessitating prioritizing one value over another.

Value monism can result in conceptual equivocation, forcing diverse ideals to conform to a single framework. Berlin highlights how positive liberty is sometimes prioritized at the expense of freedom to serve other fundamental needs, such as equality, happiness, or security (Berlin, 1969, p. 172). Consequently, he rejects positive liberty in favor of negative liberty, while advocating for value pluralism — where negative liberty becomes one among many values pursued in a just society.

In this chapter, I will explore the diverse needs of mathematical practice. *Epistemic value pluralism* rejects the reduction of normative concerns to a single guiding principle, contrasting with *epistemic value monism*, which I critique in the form that elevates rigor as the sole standard:

> "[M]athematics ought to be done in a way to meet the standards of rigour" (Tanswell, 2024, p. 7).

This represents a specific form of epistemic value monism, where rigor is regarded as *the only value*. While other values could theoretically occupy this role, this particular emphasis on rigor is the most prominent. It is presented in the works of Hamami (2022) and Tanswell (2024), discussed in Section 2.2 and Section 2.3. Accordingly, I will treat epistemic value monism as synonymous with the view that rigor serves as the sole standard for mathematical practice.

However, scholars have argued that proofs fulfill a range of epistemic purposes (Dutilh Novaes, 2020, p. 205). Their creation, evaluation, and application are shaped by the contexts in which epistemic agents engage with them. For instance, the didactic aims of classroom instruction contrast sharply with the exploratory endeavors of individual mathematicians. I will argue in Section 2.4 and Chapter 3 that purposes are not merely reducible to context; even within the same context, different proofs may serve distinct epistemic objectives.

If unrigorous proofs better achieve specific epistemic goals, this reflects a prioritization of other valuable qualities in proofs. The essence of epistemic value pluralism lies in acknowledging that while rigor is undoubtedly *one value* of mathematical practice, it is not the sole criterion for evaluating proofs.

2.1 Mathematical Proofs

Students of mathematics are introduced to an explicit definition of a proof. At *University of Copenhagen*, this takes place in an undergraduate course on discrete methods in mathematics (Universitet, 2019, p. 4).

A *formal proof* is defined as a specific form of valid inference, that is, a deduction from a finite set of premises $p_1, p_2, \ldots, p_n$ to a conclusion q. For such a deduction to qualify as a proof, each premise p_i must be an axiom of the theory, a previously proven proposition, or derivable from the premises $p_1, p_2, \ldots, p_{i-1}$ (Lützen, 2019, pp. 15–16).

This aligns with David Hilbert's view of a *formalized proof* as a concrete string of symbols (Hilbert, 1926, p. 186). However, since Immanuel Kant, philosophers have criticized this interpretation of proofs (Azzouni, 2009, p. 9).

Mathematical practice cannot be reduced to *analytic a priori* derivations from axioms and lemmas; instead, an "explanatory gap" exists in the constructive nature of mathematics (Azzouni, 2009, p. 10). Kant emphasized this when he argued that mathematics is guided by *intuition*[1] — epistemic agents' perception of space and time (Kant, 1781/2002, pp. 477–478). However, mathematics has evolved beyond its traditional domains of geometry and arithmetic as concrete descriptions of the world. In this sense, Kant's account appears less convincing for modern mathematics, which often involves objects constructed independently of the perception of space and time (Azzouni, 2009, p. 9).

Translating informal mathematics into formal language does not resolve the epistemic challenge of how informal proofs generate conviction and foster greater understanding than their formal counterparts (Azzouni, 2009, pp. 20–21). Despite this, the formal proof remains the articulated norm (Azzouni, 2009, p. 23), even though it may not be the internalized norm. Mathematicians often engage in informal practice, which raises *the puzzle of informal mathematics*: How do informal proofs generate conviction, and how are they related to formal proofs?

2.1.1 Burgess's Standard

A solution to the puzzle of informal mathematics is presented by John P. Burgess (2015), who asserts that a proof is genuine if it is *rigorous*. I will call this *Burgess's standard*:

> "The quality whose presence in a purported proof makes it a genuine proof by present-day journal standards, and whose absence makes the proof spurious in a way that if discovered will call for retraction, is called *rigor*" (Burgess, 2015, p. 2).

This introduces a new quality distinct from formality. Unlike formality, which is a simple matter when to ascribe — either inferences are explicitly written in *modus ponens*, or they are not — the necessary and sufficient conditions for ascribing rigor are more ambiguous. As I will present in Section 2.3, there are multiple interpretations of what rigor means.

One interpretation of Burgess's standard is that whatever is published and not retracted is rigorous, since it is judged by the mathematical practice as fulfilling the criterion of good mathematics. This is *the standard-as-rigor*

[1] German: Anschauung

interpretation, since it assigns the title of rigor to all proofs that live up to a standard established by mathematical practice.

Alternatively, Burgess's standard can be interpreted as suggesting that rigor is a specific quality, and if a proof possesses it, the proof meets present-day journal standards. This is *the rigor-as-standard interpretation*, which treats rigor as the defining quality of good mathematical practice.

Both interpretations are forms of epistemic value monism, but differ in their conception of rigor. In the standard-as-rigor interpretation, rigor denotes a composite concept that encompasses the qualities found in actual proofs accepted by mathematical practice. In contrast, the rigor-as-standard interpretation treats rigor as a singular quality that all good proofs should possess, imposing an imperative for mathematical practice to fulfill this criterion.

Burgess's solution to the puzzle of informal mathematics is rigor. In the standard-as-rigor interpretation, the descriptive account of what rigor is seems problematic. Objects possess qualities independently of whether or not they are judged to have them. Nevertheless, this interpretation comes up implicitly in discussions of rigor as I will show in Subsection 2.3.5. However, the rigor-as-standard interpretation prescribes how epistemic agents should do mathematics; it does not explain what rigor is or how it relates to formal proofs.

2.1.2 The Derivation-indicator View

Azzouni has another approach to the puzzle of informal mathematics. Contrary to how he is portrayed in Hamami (2022) and Tanswell (2024), Azzouni does not present a theory of rigor. Instead, he interprets a distinction introduced by Yehuda Rav between *proof* and *derivation*. These are defined as, respectively, the informal mathematical discourse with irreducible semantic content and the purely syntactical objects within formal systems (Azzouni, 2004, p. 82). A proof is an argument that *indicates* the existence of a derivation (Azzouni, 2004, p. 88). Not a mere abbreviation of a derivation (Hamami, 2022, p. 412); nor does it explicitly specify which derivation is indicated (Azzouni, 2004, p. 99). Instead, a proof plausibly suggests the existence of a derivation.

Although *formalists* like Hilbert believed that formalizations of proofs into derivations had epistemic advantages (Azzouni, 2004, p. 101), mathematical practice often prioritizes proofs over derivations (Azzouni, 2004, p. 96). *The derivation-indicator view* explains this by highlighting the semantic richness of informal mathematics, which resists reduction to the purely syntactic frameworks of logic and set theory. This insistence on nonreduction preserves the holistic nature of mathematical concepts (Azzouni, 2004, p. 99).

In Azzouni's discussion of informal proofs, he does not explicitly address rigor. The link to rigor comes from Azzouni quoting Donald MacKenzie, who uses "rigorous-argument proof" as proof and "formal proof" as derivation (Azzouni, 2004, pp. 85–86). However, this broad application of "rigor" risks diluting its meaning. In Azzouni's framework, the term "rigor" appears to apply to all informal proofs (Azzouni, 2004, p. 96), leading to the conclusion that all proofs are rigorous on MacKenzie's account. This aligns with the standard-as-rigor interpretation. However, in the rigor-as-standard interpretation, Azzouni's account ceases to address rigor directly and becomes merely a theory of what constitutes a proof. To fully address the concept of rigor, I must look further. In Section 2.2, I turn to Hamami's explicit treatment of rigor, before exploring Tanswell's broader reflections in Section 2.3.

2.2 Rigorous Proof

In Hamami (2022), a precise formulation of the so-called *standard view of rigor* is presented by breaking it down into three components.

> **The descriptive part:** An empirical claim of how and when epistemic agents in mathematical practice judge the proofs to be rigorous.
>
> **The normative part:** A conceptual account specifying the necessary and sufficient conditions for a proof to be rigorous.
>
> **The conformity thesis:** A relation between *the descriptive part* and *the normative part*.

Hamami states that the standard view has a pragmatic motivation. The ideal of formal proof and the fact that it is unattainable in practice present a tension. The standard view presents a middle ground, defining a proof as rigorous if it is *routinely translated* into a formal one (Hamami, 2022, p. 414). This solution to the puzzle of informal mathematics is simple: Informal proofs are convincing to the extent that they are related to formal ones by being routinely translatable to them. Let us dive into what that means now.

2.2.1 Descriptive Part of Rigor

When a proof is presented to a mathematician, she will eventually judge it as valid or invalid. This process, referred to as *verification*, should not be confused with the verification of a proposition in empirical science. By verification, Hamami designates the analysis of a proof — i.e., an argument — rather than a proposition.

A proof can be *decomposed* into *inferences*, which are classified as either *immediate* or *intermediate*. An inference is immediate if it can be verified without introducing additional steps; otherwise, it is intermediate. Using this terminology, Hamami introduces the *DV schema* as the structure of any descriptive part of rigor. It consists of two processes: *decomposition* and *verification*. The former involves breaking intermediate inferences into smaller steps, while the latter verifies inferences as valid or invalid.

Decomposition Processes

Verify an inference of the form:

$$p_1,\ p_2,\ \ldots,\ p_n \vdash q,$$

where the inference is intermediate, one need to prove the proposition:

$$p_1,\ p_2,\ \ldots,\ p_n \Rightarrow q.$$

Decomposition is as *proof search processes*[2]. Importantly, the goal is not to find just any proof to fill intermediate gaps but rather to identify proofs or lemmas that bridge *enthymematic gaps*.

An enthymeme is an argument with implicit premises. Thus, a competent reader could fill in an enthymematic gap by background knowledge in a "reasonable amount of time" (Hamami, 2022, p. 424). Some "explanatory gaps" as addressed in Section 2.1 might not be enthymematic. This is what makes some informal proofs unrigorous on Harmami's account.

Consequently, what counts as a decomposition is contingent upon the mathematical practice, specifically the shared background knowledge of competent and what constitutes a "reasonable amount of time"[3].

[2]Here, I have adapted Hamami's notation, replacing his mixed formalism with the standard symbols $\vdash$ (syntactic entailment) and $\Rightarrow$ (material implication) (Hendricks and Pedersen, 2011, pp. 8–9).

[3]Hamami loosely suggests that "a few days" is considered too long (Hamami, 2022, p. 423).

Verification Processes

Verification by contrast, takes immediate inferences as input. If rigorous proofs were formal ones, verification would just require checking for *modus ponens*:

$$P,\ P \Rightarrow Q \vdash Q.$$

However, this approach does not reflect how mathematical practice actually comprehends immediate inferences. Epistemic agents verify proofs without necessarily reducing them to axioms and primitive rules of inference.

Hamami introduces two concepts to define the scope of verification: *higher-level rules of inference* and *rule certificates*.

An epistemic state comprises the propositions an agent knows and the higher-level rules they possess. Initially, an epistemic state is limited to a set of axioms and primitive inference rules learned during early training (Hamami, 2022, p. 426). Over time, the agent updates their epistemic state via conservative extensions. These involve deriving new propositions from the known ones or acquiring new higher-level rules. When an agent adopts a new higher-level rule, they are said to have acquired a rule certificate.

Although this model is a highly idealized representation of an agent's epistemic state, it provides a conceptual framework for describing how agents judge proofs as rigorous. In particular, the description of the DV schema does not refer to routine translation or formal proofs. These terms will take center stage in the following explanation of the normative part of rigor.

2.2.2 Normative Part of Rigor

In contrast to the descriptive part, the normative part of rigor is the ability to routinely translate a proof into a formal proof. The key concept is the notion of *routine translation*, which Hamami interprets *algorithmically*.

Proofs are categorized into four levels of *granularity*: (Hamami, 2022, p. 429)

> **Vernacular-level**: Informal proofs as presented in standard mathematical texts, contain both intermediate and immediate inferences.
>
> **Higher-level**: Informal proofs that only involves immediate inferences, relying on high-level rules known to epistemic agents.
>
> **Intermediate-level**: Informal proofs that only depend on axioms and primitive rules of inference.

Lower-level: Formal proofs that are relying exclusively on primitive rules of inference and axioms from a formal deductive system.

Routine translation [*RT*] is a transformation algorithm from one level to the next. For example, going from the vernacular to the higher level decomposes intermediate inferences into immediate ones (Hamami, 2022, p. 431).

The normative part of rigor is defined as the success of RT. "Routinely" can then be given a specific interpretation that does not depend on the actual routines of the mathematicians. The descriptive and normative parts of rigor are here conceptually independent of each other. It is conceptually possible for a proof to have the descriptive part without the normative, and vice versa.

2.2.3 The Conformity Thesis

The conformity thesis posits that the descriptive part implies the normative part of rigor. In simpler terms, if a proof is judged as rigorous, then it is rigorous (Hamami, 2022, p. 33). By contraposition: If a proof is unrigorous, it will be judged as unrigorous.

This relationship is asymmetric. The thesis does not state the reverse: that all rigorous proofs are judged as such. In some cases, rigorous proofs might not be recognized by an epistemic agent. An error of judgment could occur, for instance, if the agent does not invest the time to decompose each intermediate inference or to verify each immediate inference in the proof.

To demonstrate that a proof is not rigorous in the normative sense, it would not suffice to show that mathematical practice does not judge it as rigorous. Instead, one would need to show that RT fails to work at all levels. This failure would occur if one of the algorithmic translations was impossible.

Hamami's model has shortcomings. For example, the descriptive part presents a *tabula rasa*. In mathematical practice, knowledge is much more iterative in its appropriation. Undergraduate students know that $0 < 1$ before they prove that the neutral element for addition is smaller than the one for multiplication. This challenges the faith in the conformity thesis if the DV schema is an inadequate representation of practice. Furthermore, the conformity thesis is equivalent to stating that the probability of mathematicians making a false positive in verification is zero. Presented in this way, it seems ludicrous.

However, the normative part of rigor presents an operational conception that can function as a clear demarcation criterion for rigor. In Section 4.2 I will conclude by this that the proof by the dynamic approach is unrigorous.

I now turn away from this presentation of one conception of rigor to address both a critique of the standard view of rigor and presenting other competing conceptions. Hamami's simple solution to the puzzle of informal mathematics explains the convincing nature of informal proofs as an indirect feature of formal proofs. The following characterizations will address why informal proofs seem to have a convincing quality beyond the point of formal ones.

2.3 Rigor Pluralism

The tension between epistemic value monism and the plurality of proof purposes is what leads Tanswell to advocate *rigor pluralism* (Tanswell, 2024, p. 6). He identifies an implicit assumption of the "true essence" of rigor, critiquing the notion that "one concept fulfills all" (Tanswell, 2024, p. 7). Instead, rigor pluralism proposes the use of different models of rigor tailored to the evaluation of various proofs (Tanswell, 2024, p. 8). To illustrate this perspective, Tanswell presents four distinct models of rigor and proof.

Tanswell contends that each of these four models highlights different aspects of rigor. This section presents each model and presents the underlying assumptions in their descriptions. In Subsection 2.3.5, I summarize with a critique of Tanswell and address his implicit assumption of epistemic value monism.

2.3.1 The Standard View of Rigor

The standard view of rigor asserts that a rigorous proof maintains a connection to formal proofs, e.g., through *RT* as presented in Section 2.2.

While Tanswell critiques various aspects of the standard view, one key issue is *correctness* (Tanswell, 2024, p. 12). Proponents of the standard view could accept the existence of correct but unrigorous proofs. However, Tanswell takes the standard view to assume that rigorous proofs must always be correct.

This does not seem right. For example, a proof sketched on a napkin after a conference talk or a drawing on a blackboard might be the correct choice of communication in that setting, although they are unrigorous proofs.

"Correct" is not defined in Tanswell (2024). This term is surprisingly ambiguous in the context of proofs. Let me clarify why: Proofs, as objects[4], cannot be inherently correct or incorrect. Instead, correctness refers to actions: the context-dependent suitability of presenting a particular proof.

Tanswell is right in questioning that a correct proof is one that can be RT. However, this implication of the standard view arises only in conjunction with epistemic value monism. Proponents of the standard view may argue that a rigorous proof is translatable into formal ones, but they may also recognize contexts where a rigorous proof is not desirable. The proponent of the standard view does not need to endorse epistemic value monism. One could both hold the standard view and epistemic value pluralism.

2.3.2 The Proof as Dialogue View

The second model Tanswell introduces conceptualizes proofs as dialogues (Tanswell, 2024, p. 30). This model emphasizes the dialectic nature of mathematics (Tanswell, 2024, p. 37), acknowledging that different contexts require varying levels of rigor. Tanswell references empirical findings from Davies et al., which demonstrate that mathematicians assess the rigor of proofs based on contextual factors. He concludes:

> "Likewise, as the quote [see below] suggests, the optimum level of detail for rigour might vary substantially by context" (Tanswell, 2024, p. 41).

However, this conclusion misinterprets the study by Davies et al.:

> "There are times when you give them examples in class that have a certain level of rigor, but as time goes on and as the students become more mature in their own abilities to reason and argue . . . you can make statements and assumptions that they have an obligation to sit down and figure out what the details are or know what the implied statements are within that." (Davies et al. 2021, p. 9, quoted in Tanswell, 2024, p. 41).

Davies et al. show that the *degree of rigor* varies with context — not the level of detail required for a proof to be rigorous. This conflation stems from Tanswell

[4]Wether proofs are abstract or concrete objects are a point of disagreement. Although they seem abstract, formalists like Hilbert have argued that: "A formalized proof, like a numerical symbol, is a *concrete* and visible object" (Hilbert, 1926, 187, my italics). However, this is not a disagreement about whether or not proofs are objects.

implicitly assuming the standard-as-rigor interpretation of Burgess's standard. I will come back to this in Subsection 2.3.5.

2.3.3 The Proof as Recipe View

The third model, like the second, focuses on what a proof is rather than what rigor is. Here, Tanswell highlights the active role of proofs, conceptualizing them as *recipes* or *practical knowledge* (Tanswell, 2024, p. 43).

Tanswell draws on Gilbert Ryle's distinction between "knowing that" and "knowing how" as respectively theoretical and practical knowledge. He portrays proofs as a set of *epistemic actions*. Published proofs often contain instructions, which Tanswell interprets as intrinsic elements of a proof, rather than rhetorical devices. According to this view, rigor entails an error-free process that explicitly clarifies dependencies on definitions and lemmas to the extent it is in need (Tanswell, 2024, p. 54).

But how much explicitness is needed? While Tanswell asserts that "formal logic does not have a monopoly on rigour" (Tanswell, 2024, p. 55), this claim does not follow from the conceptualization of proofs as recipes. Formal grounding might always be required for rigor; what the recipe model demonstrates is that proofs can serve as epistemic actions tailored to the needs of an epistemic agent. Once again, Tanswell implicitly assumes that proofs serving an agent's needs are automatically rigorous — an assumption valid only under the standard-as-rigor interpretation of Burgess's standard.

2.3.4 The Rigor as Virtue View

The fourth model addresses rigor as an evaluation of the epistemic agent who presents the proof. The ascriptions of rigor reflect not only the properties of the proof but also *the prover*. This view is inspired by *virtue epistemology*. Just as virtue ethics evaluates actions based on the character of the agent, the rigor of a proof is tied to the agent's epistemic virtues (Tanswell, 2024, p. 58):

> "[T]he rigorous mathematician would be both careful not to be overconfident and sensitive to multiple sources of evidence" (Tanswell, 2024, p. 62).

Rigor as openness to evidence and caution in judgment seems more aligned with the virtue of *humility* than rigor. In addition, Tanswell also remarks:

> "[R]igour is just one among many relevant intellectual virtues for mathematics" (Tanswell, 2024, p. 62).

This is epistemic value pluralism in a virtue epistemic framework. This does not go well with the characterization of rigor in the previous quote. The epistemic virtue of humility hardly connects to rigor. This is once again the standard-as-rigor interpretation of Burgess's standard. Certainly, both humility and rigor are virtues, but they are not the same virtue. And neither do they have to be if one takes up epistemic value pluralism.

Let us now turn to an overall evaluation of Tanswell's argument for rigor pluralism. The argument is only implicit throughout his book, but the contour of an argument is present (Tanswell, 2024, p. 66).

2.3.5 The Problem of Rigor Pluralism

Tanswell's four models of rigor are categorically misaligned: the first and fourth concern rigor directly, while the second and third address proofs, and by this have implications for what rigor is. This inconsistency is irrelevant under the assumption of epistemic value monism, where rigor is the sole evaluative standard for proofs. However, epistemic value pluralism makes this categorical mismatch problematic.

Although Tanswell's aspiration for pluralism in mathematical practice is praiseworthy, his multi-model approach in rigor pluralism is unconvincing. His implicit argument assumes that no single concept of rigor captures all aspects of mathematical practice. That is a denial of the rigor-as-standard interpretation of Burgess's standard. By this, he argues that mathematical practice is in need of different models of rigor suited to different epistemic objectives. That is an endorsement of the standard-as-rigor interpretation of Burgess's standard, where "rigor" has different meanings depending on context.

Tanswell's semantic relativism undermines his critique of the standard view. Why should rigor encompass all features of a proof? Tanswell suggests:

> "[R]igorous proof is about one way of doing *good* mathematics" (Tanswell, 2024, p. 8).

Yet elsewhere he states:

> "[M]athematics ought to be done in a way to meet the standards of rigour" (Tanswell, 2024, p. 7).

This equivocation suggests that Tanswell's concept of rigor is shaped more by implicit epistemic value monism than by genuine pluralism. This renders his solution to the puzzle of informal mathematics by addressing different epistemic tasks deficient in the opposite direction to Hamami. It places too much importance on the first part of the puzzle: "How do informal proofs generate conviction?" Tanswell's answer is that informal proofs are rigorous$_1$, rigorous$_2$, rigorous$_3$ or rigorous$_4$. This addresses only the second part of the puzzle: "how are informal proofs related to formal proofs" when the informal proof is rigorous in a standard view sense.

The version of pluralism I will consider in Chapter 4 and Chapter 5 rejects rigor pluralism as an equivocation of rigor. Instead, it recognizes multiple epistemic values by epistemic value pluralism and engages in context-specific debates about their prioritization. This avoids the pitfalls of Tanswell's shifting definitions and will provide a simpler language for evaluating mathematical practice. I will use the standard view of rigor in my ascription of rigor to the two proofs of Chapter 3. Following the spirit of Tanswell, in Section 4.4, I argue that the property of RT is not all there is to a proof.

Let us now address the pluralism Tanswell sets after through conceptual means by empirical inquiry. This, in contrast to the previous addressing of the puzzle of informal mathematics, will not be a conceptual exploration of rigor. I will present how mathematicians actually use the adjective "rigor".

2.4 Appraisal of Mathematical Proofs

Different proofs serve different purposes. At times, mathematicians seek those that are *useful* for exploring unfamiliar concepts. At other times, they value proofs for their *beauty*, appreciating the artistry within a tradition that spans more than 2,500 years. In yet other contexts, *precision* or *intricacy* take precedence, ensuring that even the most intricate edge cases fit into the foundation of mathematics.

This diversity of purposes suggests that evaluating mathematics could be a highly complex task. Some mathematical works might excel in one dimension at the expense of others. Terrence Tao, for instance, argues that students develop an intuitive sense for identifying mathematics that fosters further productive work (Inglis and Aberdein, 2015, p. 88). His framework provides a way to judge recently developed mathematics that has not yet demonstrated its generative potential. However, Tao's speculative notion of intuition may be unnecessary if the dimensionality of mathematical qualities is lower than he conjectures (Inglis and Aberdein, 2015, p. 88).

In contrast, Inglis and Aberdein propose a model to describe how mathematicians evaluate proofs along four dimensions: aesthetics, intricacy, utility, and precision (Inglis and Aberdein, 2015, pp. 99–100). Their empirical study challenges the classic assumption that mathematical beauty implies simplicity (Inglis and Aberdein, 2015). By examining correlations in how mathematicians describe proofs, they reveal a more nuanced landscape of appraisal.

In their study, 255 mathematicians were asked to think of a proof of previous interest to them. After their choice, they were presented with 8 adjectives drawn from a list of 80 adjectives used to describe mathematical proofs. For each adjective, the mathematicians were asked to carefully evaluate how well it described their chosen proof and grade this on a five-point Likert scale (Inglis and Aberdein, 2015, pp. 95–96).

By two different correlation tests[5], they motivated the use of an exploratory factor analysis to describe the application of adjectives. Five different factors were extracted and together they explained 44.7% of the variance (Inglis and Aberdein, 2015, p. 96). One of the factors was systematically used for inaccurate descriptions of the proofs the mathematicians had chosen, such as: "careless" and "shallow". The authors concluded that this factor was not a true dimension and continued only with four dimensions (Inglis and Aberdein, 2015, p. 99). I only address these five factors as four dimensions as well.

In addition, the authors also calculated the Spearman rank correlation for "beautiful" (Inglis and Aberdein, 2015, p. 100). The results showed that proofs deemed "beautiful" were often also described as "profound" (Inglis and Aberdein, 2015, p. 97). However, contrary to popular belief, there was no correlation between "beautiful" and "simple" significantly different from zero (Inglis and Aberdein, 2015, p. 100). This finding undermines the assumption that beauty and simplicity are inherently linked in mathematical practice.

While this lack of correlation does not directly refute the *metaphysical possibility* of a necessary relationship between beauty and simplicity, it casts doubt on it. Any persistence of such a claim would require re-conceptualizing these

[5] Two test were conducted. One by computing the Kaiser-Meyer-Oklin value and comparing it with a threshold value that was exceeded. This is not a hypotheses test, but an evaluation in descriptive statistics. Further they ran Bartlett's Test of Sphericity with high significance ($p < 0.001$), which reasonably rejects the null hypotheses of no corelation.

qualities beyond their actual use. For now, skepticism about their connection seems justified unless contrary evidence emerges.

Further analysis revealed that "beautiful" was correlated *positively* with adjectives such as "elegant" and *negatively* with "ugly", while "simple" was correlated *positively* with "obvious" and *negatively* with "difficult". This introduced two distinct dimensions: aesthetics and intricacy (Inglis and Aberdein, 2015, pp. 99–100). Similarly, correlations between adjectives like "practical" and "informative" on the one side, and "precise" and "rigorous" on the other, supported the construction of utility and precision as separate dimensions.

In my exploration of rigor, observations concerning the adjective "rigorous" are noteworthy. Unfortunately, the Spearman rank correlation was only done with "beautiful", so I can only relate "rigorous" to the four dimensions extracted and not directly to any other adjective. "Rigorous" associates strongly with the precision dimension but weakly with the other three. Focusing solely on rigor in mathematical practice could produce mathematics that excels in one dimension while neglecting others. If mathematics is a multidimensional practice, it must balance these competing qualities.

In Chapter 4, I will address the diverse goals of the proofs in Chapter 3. To frame this discussion, I will distinguish between the *figurative*[6] and *epistemic* qualities. The aesthetic and intricacy dimensions pertain to the way mathematics is presented, focusing on form. In contrast, the utility and precision dimensions align more closely with epistemic pursuits. The current model provides strong evidence that mathematical appraisal can be described in a few dimensions, rather than a high-dimensional framework, as Tao suggested.

2.4.1 Goals of Mathematical Practice

Different agents of mathematical practice may pursue different goals. When describing mathematics as a *practice*, it remains unspecified what kind of practice mathematics is (Carter, 2024, p. 11). Viewing mathematics as an *art form* emphasizes aesthetics as the goal. If mathematics is treated as an *abstract system* built for its own sake, the focus might shift to the intricacy of the system. If mathematics serves as a *tool* for other disciplines, such as physics or finance, utility becomes the standard of evaluation. Lastly, when viewed as a *descriptive science*, the objective is the precise description of its own abstract objects.

[6] I choose to stay away from the term formalistic, to avoid confusion with formalism.

The objectives of a mathematical proof are not inherent in the concept of proof itself but depend on the overarching goals of mathematical practice. If mathematics is considered an *epistemic practice*, its objectives are epistemic. This does not exclude an interest in figurative qualities like beauty and intricacy, but as epistemic agents, mathematicians prioritize *epistemic goals*.

Both utility and precision appear to be *epistemic qualities*. Epistemic agents value information as well as unambiguity, as reflected in their associations with utility and precision dimensions, respectively. Choosing one quality at the expense of the other is not feasible, since the dimensions by definition are orthogonal. So, maximizing one does not entail the maximizing of the other.

"Rigorous" is associated strongly with precision but weakly with utility. Assumed that the utility of proofs is an epistemic concern, then rigor cannot be all there is to a proof. Although it may be impossible for some proofs to excel in both utility and precision, this only underscores the need for multiple proofs addressing different aspects of mathematical practice.

Above I commented on the metaphysical limitations on a correlation analysis of how mathematicians appraise proofs as "beautiful" and "simple". The same methodological consideration is applicable to this attention on how mathematicians use "rigorous". Maybe mathematicians are not actually denoting rigor when they say "rigor". In Hamami's language, this is a denial of the conformity thesis from Subsection 2.2.3. As presented in Chapter 1, the methodological assumption in this thesis follows the philosophy of mathematical practice. I am uncovering the implicit worldview of the practice: for this, actual language is the best — if not only — source for addressing ontological questions.

The empirical findings in Inglis and Aberdein (2015) present further motivation for epistemic value pluralism. In Sections 4.4 and Section 5.2 I revisit proof appraisals in relation to the proofs of Chapter 3.

It is now time to present the dynamic approach in Chapter 3. The case presents us with an epistemic activity from stochastic analysis. In Chapter 4 I will use the theoretical considerations in Chapter 2 to argue that unrigorous proofs are present in mathematical practice and valuable for epistemic agents. This challenges epistemic value monism. In terms of Burgess's standard, the first point contradicts the standard-as-rigor interpretation, and the second contradicts the rigor-as-standard interpretation.

3 Practice of Stochastic Analysis

I now turn to *the dynamic approach* in martingale theory. By presenting two proofs for the same theorem in stochastic analysis, this chapter bridges the general theory established in Chapter 2 and the concrete analysis and discussion conducted in Chapter 4 and Chapter 5. The theorem, which combines predictability and the martingale property, serves as a key tool in stochastic integral theory and has applications such as life insurance analysis.

First, in Section 3.1, I will provide the necessary background, including key definitions and intuition, culminating in the statement of Theorem 1. In Section 3.2, I will present a proof by the dynamic approach, followed by a proof by the monotone class theorem in Section 3.3. Section 3.4 concludes with a comparison of the two proofs, highlighting their distinct characteristics. This sets the groundwork for Chapter 4, where I analyze the implications of these different approaches based on the epistemic insights from Chapter 2.

All notation is from Andersen *et al.* (1993), specifically *Chapter II, Sections 1* and *2* (Andersen *et al.*, 1993). Theorem 1 corresponds to a weak version of *Theorem II.3.3* (Andersen *et al.*, 1993). The dynamic approach draws on heuristics from *Section II.1*. The second proof is based on the sketch in *Section II.3*. I will refer to the results of Schilling (2005) and Hansen (2021) translated into the formalism of Andersen *et al.* (1993) to ensure consistency. Although I would have preferred to use Schilling (2005) to present the definitions of concepts left undefined in Andersen *et al.* (1993), I was in need of a source that defined stochastic processes in continuous time. Schilling (2005) does not do this since it handles *countable* sequences of measurable functions. To be able to handle a more general concept of stochastic processes, I stick to Hansen (2021), although it is unpublished lecture notes.

This chapter is not a critical evaluation of proofs as strings of symbols as they are presented in practice. Rather, I engage in the construction of proofs to highlight and address their epistemic characteristics. This is a reconstruction of mathematical practice, not an analysis of direct observations in the form of a literature review. Although the stylized presentation of these proofs idealizes certain aspects of the practice, it also runs the risk of misrepresenting the actual practice. To remain faithful to Andersen *et al.* (1993), I will include quotations that illustrate the motivations behind the conceptual tools.

The purpose of this chapter is to present an example of both a rigorous and an unrigorous proof from mathematical practice. This sets the stage for the analysis in Chapter 4 and Chapter 5, where I analyze the proofs to elucidate their distinct epistemic benefits and assess their implications for the broader conceptualizations in the practice of stochastic analysis.

3.1 Formal Frame

In this section, I develop the formal framework for stochastic processes X. A stochastic process is as a sequence $\mathbb{X}_1, \mathbb{X}_2, ...$ of random variables $\mathbb{X}$, defined on a probability space $(\Omega, \mathscr{F}, P)$ (Hansen, 2021, p. 37). This concept of a stochastic process is a random phenomenon evolving in *discrete* time.

I will have to construct martingales and predictable processes in *continuous* time to serve the approaches discussed in Section 3.2 and Section 3.3. So I will need a more general definition of a stochastic process than just an expansion of the familiar concept of the random variable $\mathbb{X}$ (Hansen, 2021, p. 445).

A stochastic process:

$$X : \Big(X(t) : t \in [0, \infty)\Big),$$

is a family of random variables indexed by t on the interval $[0, \infty)$.

A real-valued random variable $\mathbb{X}$ maps an element $\omega \in \Omega$ from the probability space $(\Omega, \mathscr{F}, P)$ to the real numbers $\mathbb{R}$. I denote its realization as $\mathbb{X}(\omega)$.

A stochastic process X for fixed t is a random variable: $X(t) : \Omega \to \mathbb{R}$. On the other hand, for fixed ω, it is a map $X(\omega) : [0, \infty) \to \mathbb{R}$. This function, known as a *sample path*, is the trajectory of the process over time. If the sample paths

$$X(\omega) : \Big(X(t, \omega) : t \in [0, \infty)\Big)$$

are *right-continuous* with *left limits* for *almost all* ω, it is called *càdlàg*.

To study how a process evolves, I introduce the concept of its history, called a *filtration* (Andersen *et al.*, 1993, p. 60):

$$\big(\mathscr{F}(t) : t \in [0, \infty)\big).$$

The filtration $\big(\mathscr{F}(t)\big)$ is an increasing family of sub-σ-algebras of $\mathscr{F}$, representing the information at time t. For all $t \in [0, \infty)$ the filtration satisfies:

1. Increasing: $\mathscr{F}(s) \subseteq \mathscr{F}(t) \subseteq \mathscr{F} \quad$ for $s < t$,
2. Right continuous: $\bigcap_{s<t} \mathscr{F}(s) = \mathscr{F}(t)$.

Throughout this chapter, I am going to use the filtration $\mathscr{F}(t-)$, as the smallest σ-algebra containing all $\mathscr{F}(s)$ for $s < t$.

A process X is *adapted* to a filtration $\big(\mathscr{F}(t)\big)$ if it is $\mathscr{F}(t)$-measurable for all $t \in [0, \infty)$.

I now define the key processes of interest: the martingale. A process M is a martingale if it, for all $t \in [0, \infty)$, satisfies: (Andersen *et al.*, 1993, p. 64)

1. $M(t)$ is $\mathscr{F}(t)$-measurable.
2. $E\Big[|M(t)|\Big] < \infty$.
3. $E\Big[M(t) \Big| \mathscr{F}(s)\Big] = M(s) \quad$ for all $s < t$.

By convention, the initial value of a martingale is set to zero $M(0) = 0$.

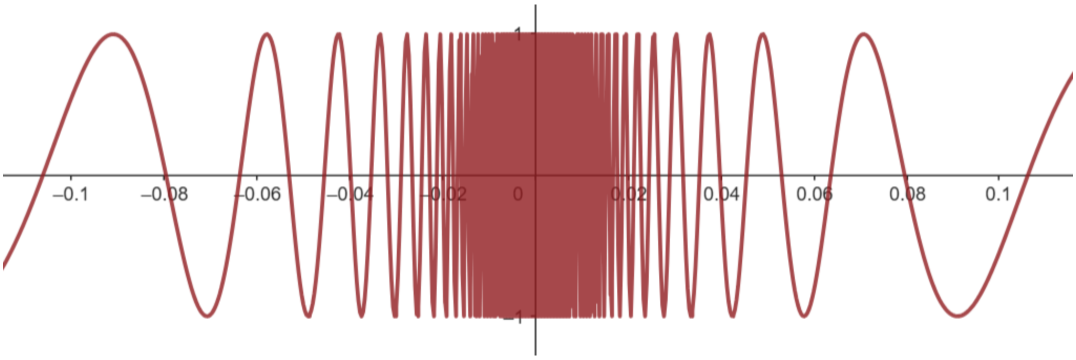
This is a *global* characterization of the process. In Section 3.2, I will give a *local* characterization using dynamics.

A stochastic process H is *predictable* if it is measurable with respect to the σ-algebra on $[0, \infty) \times \Omega$ generated by all left-continuous, adapted processes (Andersen *et al.*, 1993, p. 66). This ensures $H(t)$ is known at time $t-$.

In the following, I will restrict myself to processes Y that have *càdlàg* sample paths with *finite variation* [FV], meaning (Andersen *et al.*, 1993, p. 64):

$$\int_{[0,t]} |dY(s)| < \infty \quad \text{for all } t \in [0, \infty) \text{ and almost all } \omega \in \Omega.$$

The variation process $\int |dY|$ is an expression of how much the function $Y(\omega) : t \to \mathbb{R}$ oscillates. For $Y(t, \omega) = \sin\left(\frac{1}{t}\right)$ graph swings between -1 and 1 faster and faster as $t \to 0$. There is no explicit way to calculate the integral



$\int_{[0,t]} |dY(s)|$, but the *linear variation* for a sample path $Y(\omega) : t \to \mathbb{R}$ on the interval $[0, T] \subset [0, \infty)$ is given as: (Hansen, 2021, p. 478)

$$V_{0,T}\left(Y(\omega)\right) = \sup\left\{ \sum_{i=1}^{n} \left| Y(t_i, \omega) - Y(t_{i-1}, \omega) \right| \middle| 0 = t_0 < t_1 < \ldots < t_n = T \right\}$$

As seen in the figure above $t \mapsto \sin\left(\frac{1}{t}\right)$ is an example of unbounded variation. I only consider processes Y such that $V_{0,T}\left(Y(\omega)\right) < \infty$.

A stochastic integral is an integral where both the integrand and the integrator are stochastic processes. For a given $\omega \in \Omega$, a new stochastic process is defined as (Andersen *et al.*, 1993, p. 64):

$$t \mapsto \int_{[0,t]} X(s)\, dY(s),$$

such that for given (t, ω),

$$\int_{[0,t]} |X(s)|\, |dY(s)| < \infty.$$

In the following, I will only consider predictable and bounded processes as integrands.

I now present a weaker version of *Theorem II.3.1* from (Andersen *et al.*, 1993, p. 71), focusing on its conceptual aspects to better highlight the differences between the approaches in Section 3.2 and Section 3.3.

Theorem 1. *Suppose M is a FV martingale, H is a predictable process, and H is bounded. Then $\int_{[0,t]} H(s)\, dM(s)$ is a FV martingale.*

Intuitively, this result states that scaling a martingale M by a predictable and bounded process H produces a new martingale, provided H is bounded. The boundedness of H ensures that $E\left[\left|\int_{[0,t]} H(s)\, dM(s)\right|\right] < \infty$.

Let us now turn to the two proofs for this result. First, a proof by the dynamic approach, followed by a proof by the monotone class theorem.

3.2 Proof by Dynamic Approach

The increments of a function $f : A \to B$ at a point $x \in A$ are thought of as a *differential quotient*, which captures the limit of a *difference quotient*: the ratio between the change of the function and the change of its variable.

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The differential quotient is not well suited to handle the discontinuities of the jumps in a stochastic process. Furthermore, it is only well defined for a fixed ω with a sample path $X(\omega) : [0, \infty) \to \mathbb{R}$. For an unfixed $\omega \in \Omega$, the differential quotient cannot be translated directly into the language of limits. Randomness is essential, as stochastic processes are defined on $[0, \infty) \times \Omega$.

To address this, I introduce the concept of *dynamic*, which is incommensurable with the differential quotient. Dynamics are defined in terms of *stochastic differential equations* to describe the behavior of stochastic processes.

For a stochastic process X, its dynamic is the increment:

$$dX(t) = d\left(\int_{[0,t)} dX(s) \right).$$

This represents the change in $X(t)$ over a small time interval $[t, t + dt)$. Informally, for a càdlàg process it is expressed as:

$$dX(t) = X\Big((t + dt)-\Big) - X(t-).$$

This is an informal notation, so the concrete interpretation is heuristic. In Andersen *et al.* (1993) they present this as an "infinitesimal time interval $[t, t + dt)$" (Andersen *et al.*, 1993, p. 89), not a time interval with infinitesimal length $[t, t + dt]$. In this regard, it is important to note that the interval $[t, t+dt)$ is arbitrarily small, since dt can be seen as an infinitesimal contribution to t. However, Andersen *et al.* (1993) are not interested in the interval $[t, t + dt]$ itself, as the right-hand endpoint is left open due to the presence of $(\cdot-)$ in the construction of dX. The authors are interested in the value of the process immediately before an infinitesimal contribution to t.

Using the framework of dynamics, a martingale can be characterized in a new way. Instead of the definition in Section 3.1, *informally* a martingale is characterized as a process M with dynamic dM that satisfies:

$$E\Big[dM(t)\Big|\mathscr{F}(t-)\Big] = 0.$$

This local characterization reflects that a martingale is a process with no expected increment in an infinitesimally small time interval $[t, t+dt)$. In other words, the expectation for the value of $M(t)$ is $M(t-)$ given the information $\mathscr{F}(t-)$, the value of the process just before time t.

Heuristically, this is equivalent to the global characterization:

$$E\Big[M(t)\Big|\mathscr{F}(s)\Big] = M(s) \quad \text{for all } s < t.$$

One sees this by adding up the increments over small sub-intervals $[u, u+du)$ that partition $[s+ds, t+dt) = (s,t]$ (Andersen *et al.*, 1993, p. 52):

$$\begin{aligned}
E\Big[M(t)\Big|\mathscr{F}(s)\Big] - M(s) &\overset{1}{=} E\Big[M(t) - M(s)\Big|\mathscr{F}(s)\Big] \\
&\overset{2}{=} E\Big[\int_{(s,t]} dM(u)\Big|\mathscr{F}(s)\Big] \\
&\overset{3}{=} \int_{(s,t]} E\Big[dM(u)\Big|\mathscr{F}(s)\Big] \\
&\overset{4}{=} \int_{(s,t]} E\Big[E\Big[dM(u)\Big|\mathscr{F}(u-)\Big]\Big|\mathscr{F}(s)\Big] \\
&\overset{5}{=} 0.
\end{aligned}$$

Thus, obtain:

$$E\Big[M(t)\Big|\mathscr{F}(s)\Big] = M(s).$$

Step-by-step clarification:

1. Follows from linearity of conditional expectation.
2. Follows from the construction of stochastic integral.
3. Follows from conditional Fubini's theorem (Schilling, 2005, p. 354) only through informal notation.
4. Follows from the tower property (Hansen, 2021, p. 355), as $u > s$.
5. Follows from the dynamic characterization of a martingale.

Even though one intuitively grasps the content of the local characterization of a martingale, it is not a well-formed[1] property. The dynamic is an increment in an infinitely small interval $[t, t+dt)$. The concept of conditional expectation $E\Big[\cdot \Big| \mathscr{F}(t-)\Big]$ is not defined for such arguments (Hansen, 2021, p. 341), which is why this is a heuristic based on the informal notation $dX(t) = X\Big((t+dt)-\Big) - X(t-)$. However, Andersen *et al.* (1993) is transferring their intuitions of construction of conditional expectations into the domain of dynamics. I will return to this point in Chapter 4.

I now prove the result of Theorem 1 using the dynamic approach.

Let M be a FV martingale and let H be predictable and bounded.

Define:

$$Y(t) = \int_{[0,t]} H(s)\, dM(s).$$

The dynamic of Y is formally given by:

$$dY(t) = d\left(\int_{[0,t)} dY(s)\right).$$

Informally, the dynamic of Y is:

$$\begin{aligned}
dY(t) &\stackrel{1}{=} Y\Big((t+dt)-\Big) - Y(t-) \\
&\stackrel{2}{=} \int_{[0,t+dt)} H(s)\, dM(s) - \int_{[0,t)} H(s)\, dM(s) \\
&\stackrel{3}{=} \int_{[t,t+dt)} H(s)\, dM(s) \\
&\stackrel{4}{=} H(t)\, dM(t)
\end{aligned}$$

Step-by-step clarification:

1. Follows from the informal expression of dynamics.
2. Follows from the definition of Y.
3. Follows from the construction of stochastic integral.
4. Follows from the informal interpretation of dynamics.

[1] I use "well-formed" in the tradition of formal logic to ascribe sequence of symbols generated from formal rules (Hendricks and Pedersen, 2011, p. 22). No interpretation is allowed.

By taking expectation conditioned on the history up to time $t-$, I see that:

$$\begin{aligned}
E\Big[dY(t)\Big|\mathscr{F}(t-)\Big] &\overset{1}{=} E\Big[H(t)dM(t)\Big|\mathscr{F}(t-)\Big] \\
&\overset{2}{=} H(t)\cdot E\Big[dM(t)\Big|\mathscr{F}(t-)\Big] \\
&\overset{3}{=} H(t)\cdot 0 \\
&= 0
\end{aligned}$$

Step-by-step clarification:

1. Is an informal application of conditional expectation on dynamics.
2. Follows from $H(t)$ being predictable.
3. Follows from $M(t)$ being a martingale and the informal characterization.

Furthermore, since $H(t)$ is bounded, I have the following:

$$\begin{aligned}
E\Big[|Y(t)|\Big] &\overset{1}{=} E\Big[\Big|\int_{[0,t]} H(s)dM(s)\Big|\Big] \\
&\overset{2}{\leq} K\cdot E\Big[\Big|\int_{[0,t]} dM(s)\Big|\Big] \\
&\overset{3}{=} K\cdot E\Big[|M(t)|\Big] \\
&\overset{4}{<} \infty
\end{aligned} \tag{3.1}$$

Step-by-step clarification:

1. Follows from the definition of $Y(t)$.
2. Follows since H is bounded, so there exist a $K \in \mathbb{R}_+ \forall s < t : K \geq H(s)$.
3. Follows from the construction of stochastic integral.
4. Follows from $M(t)$ being a martingale.

Therefore, Y is a FV martingale.

This completes the proof of Theorem 1 by the dynamic approach. This reasoning is informal, as presented in (Andersen *et al.*, 1993) who concludes the chapter with: "So, it is quite justified to leave the matter in its present informal state: We have described a valuable heuristic tool, not presented a formal mathematical theory" (Andersen *et al.*, 1993, p. 109).

I will now turn my attention to the proof by the monotone class theorem.

3.3 Proof by Monotone Class Theorem

I now turn to the proof of Theorem 1 using the monotone class theorem. Let M be a FV martingale, and let H be a predictable and bounded process.

I first consider a subset of H, defined as $\tilde{H}(t) = X \cdot \mathbb{1}_{(u,v]}(t)$, where X is a bounded and $\mathscr{F}(u)$-measurable random variable, and $u, v \in [0, \infty)$ are fixed time points with $0 \leq u < v$.

Define

$$\begin{aligned} \tilde{Y}(t) &= \int_{[0,t]} \tilde{H}(s)\, dM(s) \\ &= \int_{[0,t]} X \cdot \mathbb{1}_{(u,v]}(s)\, dM(s) \\ &= X \cdot \big(M(t \wedge v) - M(t \wedge u)\big). \end{aligned} \tag{3.2}$$

To show that $\tilde{Y}$ is a FV martingale, I prove the equality:

$$E\Big[\tilde{Y}(t)\Big|\mathscr{F}(s)\Big] = \tilde{Y}(s), \quad \text{for } 0 \leq s < t.$$

Equivalently, to demonstrate that

$$E\Big[\tilde{Y}(t)\Big|\mathscr{F}(s)\Big] - \tilde{Y}(s) = 0.$$

Expanding this expression:

$$\begin{aligned} 0 &= E\Big[\tilde{Y}(t)\Big|\mathscr{F}(s)\Big] - \tilde{Y}(s) \\ &\overset{1}{=} E\Big[\tilde{Y}(t)\Big|\mathscr{F}(s)\Big] - E\Big[\tilde{Y}(s)\Big|\mathscr{F}(s)\Big] \\ &\overset{2}{=} E\Big[\tilde{Y}(t) - \tilde{Y}(s)\Big|\mathscr{F}(s)\Big] \\ &\overset{3}{=} E\Big[X\big(M(t \wedge v) - M(t \wedge u)\big) - X\big(M(s \wedge v) - M(s \wedge u)\big)\Big|\mathscr{F}(s)\Big] \\ &= E\Big[X \cdot \big(M(t \wedge v) - M(t \wedge u) - M(s \wedge v) + M(s \wedge u)\big)\Big|\mathscr{F}(s)\Big]. \end{aligned}$$

Step-by-step clarification:

1. Follows from $\tilde{Y}(s)$ being $\mathscr{F}(s)$-measurable.
2. Follows from linearity of conditional expectation.
3. Follows from equation (3.2).

I now evaluate this for the five possible cases: $(s < t \leq u < v)$, $(s \leq u < t \leq v)$, $(u < s < t \leq v)$, $(u < s \leq v < t)$, and $(u < v < s < t)$.

Case 1: Let $s < t \leq u < v$. So

$$\begin{aligned}
&E\Big[X \cdot \Big(M(t \wedge v) - M(t \wedge u) - M(s \wedge v) + M(s \wedge u)\Big)\Big|\mathscr{F}(s)\Big] \\
&= E\Big[X \cdot \Big(M(t) - M(t) - M(s) + M(s)\Big)\Big|\mathscr{F}(s)\Big] \\
&= E\Big[X \cdot 0\Big|\mathscr{F}(s)\Big] \\
&= 0.
\end{aligned}$$

Case 2: Let $s \leq u < t \leq v$. So

$$\begin{aligned}
&E\Big[X \cdot \Big(M(t \wedge v) - M(t \wedge u) - M(s \wedge v) + M(s \wedge u)\Big)\Big|\mathscr{F}(s)\Big] \\
&= E\Big[X \cdot \Big(M(t) - M(u) - M(s) + M(s)\Big)\Big|\mathscr{F}(s)\Big] \\
&= E\Big[X \cdot \Big(M(t) - M(u)\Big)\Big|\mathscr{F}(s)\Big] \\
&\overset{1}{=} E\Big[E\Big[X \cdot \Big(M(t) - M(u)\Big)\Big|\mathscr{F}(u)\Big]\Big|\mathscr{F}(s)\Big] \\
&\overset{2}{=} E\Big[X \cdot E\Big[M(t) - M(u)\Big|\mathscr{F}(u)\Big]\Big|\mathscr{F}(s)\Big] \\
&\overset{3}{=} E\Big[X \cdot \Big(M(u) - M(u)\Big)\Big|\mathscr{F}(s)\Big] \\
&= E\Big[X \cdot 0\Big|\mathscr{F}(s)\Big] \\
&= 0.
\end{aligned}$$

Step-by-step clarification:

1. Follows from the tower property (Hansen, 2021, p. 355) since $s \leq u$.
2. Follows from X being $\mathscr{F}(u)$-measurable.
3. Follows from M being a martingale.

Case 3: Let $u < s < t \leq v$. So

$$\begin{aligned}
&E\Big[X \cdot \Big(M(t \wedge v) - M(t \wedge u) - M(s \wedge v) + M(s \wedge u)\Big)\Big|\mathscr{F}(s)\Big] \\
&= E\Big[X \cdot \Big(M(t) - M(u) - M(s) + M(u)\Big)\Big|\mathscr{F}(s)\Big] \\
&= E\Big[X \cdot \Big(M(t) - M(s)\Big)\Big|\mathscr{F}(s)\Big] \\
&\stackrel{1}{=} X \cdot E\Big[M(t) - M(s)\Big|\mathscr{F}(s)\Big] \\
&\stackrel{2}{=} X \cdot \Big(M(s) - M(s)\Big) \\
&= X \cdot 0 \\
&= 0.
\end{aligned}$$

Step-by-step clarification:

1. Follows from $u < s$ and X being $\mathscr{F}(u)$-measurable.
2. Follows from M being a martingale.

Case 4: Let $u < s \leq v < t$. So

$$\begin{aligned}
&E\Big[X \cdot \Big(M(t \wedge v) - M(t \wedge u) - M(s \wedge v) + M(s \wedge u)\Big)\Big|\mathscr{F}(s)\Big] \\
&= E\Big[X \cdot \Big(M(v) - M(u) - M(s) + M(u)\Big)\Big|\mathscr{F}(s)\Big] \\
&= E\Big[X \cdot \Big(M(v) - M(s)\Big)\Big|\mathscr{F}(s)\Big] \\
&\stackrel{1}{=} X \cdot E\Big[M(v) - M(s)\Big|\mathscr{F}(s)\Big] \\
&\stackrel{2}{=} X \cdot \Big(M(s) - M(s)\Big) \\
&= X \cdot 0 \\
&= 0.
\end{aligned}$$

Step-by-step clarification:

1. Follows from $u < s$ and X being $\mathscr{F}(u)$-measurable.
2. Follows from the martingale quality for any fixed point v.

Case 5: Let $u < v < s < t$. So

$$\begin{aligned} & E\Big[X \cdot \big(M(t \wedge v) - M(t \wedge u) - M(s \wedge v) + M(s \wedge u)\big)\Big|\mathscr{F}(s)\Big] \\ &= E\Big[X \cdot \big(M(v) - M(u) - M(v) + M(u)\big)\Big|\mathscr{F}(s)\Big] \\ &= E\Big[X \cdot 0\Big|\mathscr{F}(s)\Big] \\ &= 0. \end{aligned}$$

In all cases, I have shown that:

$$\begin{aligned} & E\Big[\tilde{Y}(t)\Big|\mathscr{F}(s)\Big] - \tilde{Y}(s) \\ &= E\Big[X \cdot \big(M(t \wedge v) - M(t \wedge u) - M(s \wedge v) + M(s \wedge u)\big)\Big|\mathscr{F}(s)\Big] \\ &= 0, \end{aligned}$$

which is equivalent to:

$$E\Big[\tilde{Y}(t)\Big|\mathscr{F}(s)\Big] = \tilde{Y}(s).$$

Thus, $\tilde{Y}$ is a FV martingale since:

$$\begin{aligned} E\Big[|\tilde{Y}(t)|\Big] &\stackrel{1}{=} E\Big[\Big|X \cdot \big(M(t \wedge v) - M(t \wedge u)\big)\Big|\Big] \\ &\stackrel{2}{\leq} K \cdot E\Big[\Big|M(t \wedge v) - M(t \wedge u)\Big|\Big] \\ &\stackrel{3}{\leq} K \cdot E\Big[|M(t \wedge v)| + |M(t \wedge u)|\Big] \\ &\stackrel{4}{=} K \cdot \Big(E\Big[|M(t \wedge v)|\Big] + E\Big[|M(t \wedge u)|\Big]\Big) \\ &\stackrel{5}{<} \infty \end{aligned} \tag{3.3}$$

Step-by-step clarification:

1. Follows by equation (3.2).
2. Follows from X being bounded, so there exist a $K \in \mathbb{R}_+ : K > X$.
3. Follows from the triangle inequality.
4. Follows from the linearity of expectation.
5. Follows from the second property in the definition of a martingale.

From $\tilde{H}$, I now construct the class of simple predictable processes H_{simple} as

$$H_{\text{simple}}(t) = \sum_{i=0}^{n} \tilde{H}_i(t) = \sum_{i=0}^{n} X_i \mathbb{1}_{(t_i, t_{i+1}]}(t),$$

where X_i is bounded and $\mathscr{F}(t_i)$-measurable random variable, and the intervals $(t_i, t_{i+1}]$ are a partition of $(0, t]$.

From the simple predictable processes H_{simple}, I can construct the class of predictable processes H. I will need the monotone class theorem to extend the result to processes that satisfies that $Y(t) = \int_0^t H(s)dM(s)$ is a FV martingale.

Theorem 2. ***Monotone class theorem:*** *Let $\mathscr{G} \subset \mathscr{P}([0,\infty) \times \Omega)$ be a $\cap$-stable generator for $\mathscr{B}([0,\infty)) \otimes \mathscr{F}$ and $\mathscr{H}$ be a vector space of functions $X(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ such that:*

1. *$\mathbb{1}_{[0,\infty)\times\Omega}(t,\omega) \in \mathscr{H}$ and $\mathbb{1}_G(t,\omega) \in \mathscr{H}$ for all $G \in \mathscr{G}$.*
2. *For every sequence $X_1,\ X_2,\ \ldots,\ X_n \in \mathscr{H}$, with $X(t,\omega) := \sup_{n\in\mathbb{N}} X_n(t,\omega) < \infty$ for all $\omega \in \Omega$, we have $X \in \mathscr{H}$.*

then the class of functions measurable wrt. $\mathscr{B}([0,\infty)) \otimes \mathscr{F}$ is contained in $\mathscr{H}$.

This concrete formulation of the monotone class theorem is a translation of (Schilling, 2005, p. 68) into the nomenclature of Andersen *et al.* (1993).

Let $\mathscr{H}$ be the class of processes H, where $\int_0^t H(s)dM(s)$ is a FV martingale.

$$\mathscr{H} = \left\{ \begin{matrix} H \text{ is predictable} \\ \text{and bounded.} \end{matrix} \middle| \int_0^t H(s)dM(s) \text{ is a FV martingale.} \right\}.$$

By the arguments above, the processes $\tilde{H}$ are members of $\mathscr{H}$, since $\int_0^t \tilde{H}(s)\, dM(s)$ is a FV martingale.

I first have to show that $\mathscr{H}$ is a vector space.

If $H_1, H_2 \in \mathscr{H}$, then:

$$
\begin{aligned}
&E\Big[\int_0^t \alpha H_1(u) + \beta H_2(u) dM(u) \Big| \mathscr{F}(s)\Big] \\
&\overset{1}{=} E\Big[\int_0^t \alpha H_1(u) dM(u) + \int_0^t \beta H_2(u) dM(u) \Big| \mathscr{F}(s)\Big] \\
&\overset{2}{=} E\Big[\alpha \int_0^t H_1(u) dM(u) + \beta \int_0^t H_2(u) dM(u) \Big| \mathscr{F}(s)\Big] \\
&\overset{3}{=} \alpha \cdot E\Big[\int_0^t H_1(u) dM(u) \Big| \mathscr{F}(s)\Big] + \beta \cdot E\Big[\int_0^t H_2(u) dM(u) \Big| \mathscr{F}(s)\Big] \\
&\overset{4}{=} \alpha \cdot \int_0^s H_1(u) dM(u) + \beta \cdot \int_0^s H_2(u) dM(u).
\end{aligned}
$$

Step-by-step clarification:

1. Follows from linearity of integrals.
2. Follows from linearity of integrals.
3. Follows from linearity of conditional expectation.
4. Follows from the martingale property that is the inclusion criterion for being an element in $\mathscr{H}$.

Then, $\mathscr{H}$ is a vector space, and therefore $H_{\text{simple}} \in \mathscr{H}$.

The first criterion is trivial:

- $\mathbb{1}_{[0,\infty)\times\Omega}(t,\omega) \in \mathscr{H}$ since $\int_0^t \mathbb{1}_{[0,\infty)\times\Omega}(s,\omega) dM(s) = M(t)$, so obviously a FV martingale.

- $\mathbb{1}_G(t,\omega) \in \mathscr{H}$ for all $G \in \mathscr{G}$, since:

$$
\int_0^t \mathbb{1}_G(s,\omega) dM(s) = \begin{cases} M(t) & (t,\omega) \in G \\ 0 & (t,\omega) \notin G \end{cases}.
$$

  M is by definition a martingale and so is the constant zero process.

Next, I show that $\mathscr{H}$ is closed under limits. Let $H_1,\ H_2,\ \dots,\ H_n \in \mathscr{H}$ and $H(t,\omega) = \sup_{n\in\mathbb{N}} H_n(t,\omega) < \infty$, where $H_1,\ H_2,\ \dots,\ H_n$ and H are bounded and predictable.

Then:

$$E\left[\int_0^t H(u)dM(u) \,\middle|\, \mathscr{F}(s)\right] \overset{1}{=} \sup_{n\in\mathbb{N}} E\left[\int_0^t H_n(u)dM(u) \,\middle|\, \mathscr{F}(s)\right]$$
$$\overset{2}{=} \sup_{n\in\mathbb{N}} \int_0^s H_n(u)dM(u)$$
$$\overset{3}{=} \int_0^s H(u)dM(u).$$

Step-by-step clarification:

1. Follows from the convergence property of conditional expectations (Schilling, 2005, pp. 348–9).
2. Follows from the martingale property that is the inclusion criterion for being an element in $\mathscr{H}$.
3. Follows from the dominated convergence theorem (Schilling, 2005, p. 97).

By Theorem 2 since the set of processes $H_{\text{simple}} \in \mathscr{H}$, then all $H \in \mathscr{H}$.

Thus, Y is a FV martingale by equation (3.1).

This completes the proof of Theorem 1 by the monotone class theorem.

3.4 Mathematical Comparison

The two proofs presented in the previous Section 3.2 and Section 3.3 offer insights into the concept of a martingale. This section is a mathematical comparison of the approaches to proving Theorem 1. This contrasts with the epistemic analysis of Chapter 4 and the ontological analysis of Chapter 5.

Andersen *et al.* (1993) presents "formal" and "informal" reasoning throughout *Chapter II*. However, their distinction differs from the notion of "formal" and "informal" proofs as discussed in Section 2.1. In the later, all proofs in Andersen *et al.* (1993) are informal. In Andersen *et al.* (1993), "formal" refers to the standards of measure theory (Andersen *et al.*, 1993, p. 106) and is contrasted with heuristic reasoning (Andersen *et al.*, 1993, p. 45). The informal notation is used in the section, *An Informal Introduction to the Basic Concepts* to introduce dynamics (Andersen *et al.*, 1993, 45–58). Andersen et al. reference this section when using heuristic arguments: “So this informal argument (see Section 11.1 [...])” (Andersen *et al.*, 1993, p. 96).

The dynamic approach employs an informal notation in describing a martingale where the key property is that no increment is expected:

$$E\left[dM(t)\middle|\mathscr{F}(t-)\right] = 0.$$

This local characterization contrasts the global one from Section 3.1:

$$E\left[M(t)\middle|\mathscr{F}(s)\right] = M(s) \text{ for all } s < t.$$

The proof by the dynamic approach demonstrates that Y exhibits the same dynamic behavior in its infinitesimal change as M by arguing via conditional expectations. This approach is direct and intuitive.

In contrast, the proof by the monotone class theorem is a structural argument based on predictable σ-algebras and generating classes using supremum arguments. It focuses on establishing Y as a martingale through approximation, rather than looking at the local dynamic behavior of the process.

While I showed in Section 3.2 that the global and the local characterizations of a martingale are equivalent, this equivalence depends on the acceptance of $E\left[dM(t)\middle|\mathscr{F}(t-)\right]$ as intelligible. This foundational issue will be revisited in Chapter 5 during my discussion of semantic externalism.

In summary, the proof by the dynamic approach is intuitive and examines the behavior of Y locally. The proof by the monotone class theorem is abstract and emphasizes the structural aspect that hides the intuitive idea.

Rigor in Practice

4

In this chapter, I will examine the two proofs presented in Chapter 3. The analysis will not delve into specific interpretations of what constitutes a proof, such as the views in Section 2.1 of proofs as formal derivations and indications of derivations, or the perspective in Section 2.3 of proofs as arguments or recipes. Instead, the primary focus here is on ascription of rigor to each proof. To do this using the conceptual tools of Hamami presented in Section 2.2, I will have to interpret a proof as something that could be translated — which aligns with Azzouni's derivation-indicator view from Section 2.1.

According to Burgess's standard presented in Section 2.1.1, a proof that lacks rigor is not a genuine proof. This conclusion holds regardless of whether one interprets Burgess's standard as standard-as-rigor or rigor-as-standard. However, for the purposes of this thesis, I will adopt the latter interpretation. The former — standard-as-rigor — is insufficient for this analysis because it determines whether a proof is rigorous solely by assessing whether it conforms to a practice's predefined standard. Conversely, I aim to investigate whether these proofs are rigorous, which is necessary for determining whether they qualify as genuine proofs on the account of epistemic value monism.

4.1 Formality of the Proofs

The two proofs are not formal proofs as defined in Section 2.1. The proof by the dynamic approach involves the inference:

$$dY(t) = H(t)dM(t) \vdash E\Big[dY(t)\Big|\mathscr{F}(t-)\Big] = E\Big[H(t)dM(t)\Big|\mathscr{F}(t-)\Big].$$

This inference is valid only if one can conceptualize the conditional expectation of a dynamic, as I will elaborate on in Chapter 5. However, the key issue here is that the inference is not explicitly stated as a premise:

$$dY(t) = H(t)dM(t) \Rightarrow E\Big[dY(t)\Big|\mathscr{F}(t-)\Big] = E\Big[H(t)dM(t)\Big|\mathscr{F}(t-)\Big].$$

Although this step might be seen as an "explanatory gap" (Azzouni, 2009, p. 10) to be filled in the derivation, the proof by the dynamic approach does not conform to the formal standard of a proof presented in Section 2.1.

Similarly, the proof by the monotone class theorem also fails to meet the formal standard. However, the disqualifying factor is not the use of previously proven lemmas, such as the monotone class theorem, monotone convergence for conditional expectations, or the dominated convergence theorem. In contrast to the proof by the dynamic approach, these measure-theoretic results are explicitly referenced as established propositions, aligning with the formal proof's requirement that premises belong to one of three recognized categories: axioms, lemmas, or derivable propositions. Instead, the proof is deemed informal for reasons such as splitting the problem into five distinct cases without explicating the argumentative structure or introducing constructions such as H_{simple} to define the process H. These gaps prevent the proof from being a formal one; instead, it is better described as an indication of derivation.

Azzouni has noted that an indication of a derivation does not necessarily identify which derivation it corresponds to (Azzouni, 2004, p. 94). This raises the question of how to know whether or not the two proofs are indications of the same derivation. If they are, the investigation of their mathematical properties might merely reflect differences on the figurative dimensions from Section 2.4. However, as argued in Section 3.4, there is no clear connection between the proof by the dynamic approach and the proof by the monotone class theorem, since their structures are completely different.

Formalizing the proof by the monotone class theorem appears to be a relatively mechanical task, "albeit a tedious one" (Hamami, 2022, p. 412). Conversely, determining the derivation that the proof by the dynamic approach indicates is more complex but does not conflict with Azzouni's definition of a proof, as it merely requires that the proof indicates the existence of some derivation.

4.2 Rigor of Dynamic Approach

The primary focus of this analysis, as outlined in Section 4.1, is the introduction of conditional expectations in the identity relation of the dynamics. This step is not stated as a lemma, nor could it be, as it is a manipulation that cannot be proven from the global characterization alone. Instead, the step is assumed to be intelligible as a result of the local characterization of a martingale. In Section 3.2, I argued that the local characterization is equivalent to the global one. The requirements for such equivalences will be discussed in Chapter 5.

The key inference at this stage is the transition from:

$$dY(t) = H(t)dM(t),$$

to:

$$E\Big[dY(t)\Big|\mathscr{F}(t-)\Big] = E\Big[H(t)dM(t)\Big|\mathscr{F}(t-)\Big].$$

This step does not change the truth value of the identity relation, as the conditional expectation $E\Big[\cdot\Big|\mathscr{F}(t-)\Big]$ is applied on both sides, based on the same information $\mathscr{F}(t-)$. However, what makes this step non-trivial is the application of conditional expectation to arguments that are not random variables. This involves a *domain expansion*, as conditional expectation is defined only for random variables (Hansen, 2021, p. 341).

Such domain expansions are not unusual in mathematical practice. Some philosophers of mathematics argue that domain expansions are underpinned by conceptual metaphors, a concept from cognitive linguistics (Núñez, 2009, p. 73). *Conceptual metaphors* are not merely linguistic, but involve imaginative idealizations with an inferential structure, allowing insights from a familiar *source domain* to be applied to an unfamiliar *target domain*.

Here, the source domain comprises intuitions developed from the conditional expectations of random variables, while the target domain involves conditional expectations of dynamics. The familiarity with conditional expectations makes mathematicians conceptualize and extend it into the dynamic setting.

The inference:

$$dY(t) = H(t)dM(t) \vdash E\Big[dY(t)\Big|\mathscr{F}(t-)\Big] = E\Big[H(t)dM(t)\Big|\mathscr{F}(t-)\Big],$$

is neither an immediate nor an intermediate inference. It cannot be broken into a series of simple steps. Instead, it relies on a conceptual metaphor from the source to the target domain.

As a result, the proof by the dynamic approach is unrigorous, both in the descriptive and normative part of rigor. In Section 2.2 I presented Harmami's criteria for the normative part of rigor: RT from an informal proof into a formal one. The domain expansion prevents RT. According to Harmami's conformity thesis, the proof is then also unrigorous in its descriptive part. However, our analysis reaches this conclusion independently of the conformity thesis.

The nontrivial step in the domain expansion of conditional expectations is not an enthymematic gap. A competent reader can neither decompose nor verify the inference, since it is a conceptual metaphor. The proof depends on the local characterization of the martingale, formulated in terms of conditional expectations of dynamics. The DV schema is then not feasible and the proof is judged unrigorous by an epistemic agent of mathematical practice.

I will now turn to the second proof of Theorem 1 by the monotone class theorem. In contrast to the proof by the dynamic approach, the second proof is rigorous. This distinction between the two proofs motivates the discussion in Section 4.4 of other noteworthy qualities of the dynamic approach.

4.3 Rigor of Monotone Class Theorem

Compared to the previously examined proof by the dynamic approach, the explanatory gaps in the proof by the monotone class theorem are all enthymematic gaps. The proof relies on lemmas to achieve two objectives:

1. Constructing the vector space $\mathscr{H}$, which contains all predictable process H such that $Y(t) = \int_0^t H(s)dM(s)$ is a finite variation martingale.
2. Construct H through by H_{simple}.

The use of the monotone class theorem, monotone convergence for conditional expectation, and dominated convergence theorem exemplifies higher-level rules of inference. These are rules for which an epistemic agent of mathematical practice is presumed to possess implicit rule certification. This verification aligns with Harmami's DV schema, which characterizes the descriptive rigor.

The proof structure includes the decomposition of H_{simple} and $\mathscr{H}$, which are defined as follows:

$$H_{\text{simple}}(t) = \sum_{i=1}^{n} X_i \mathbb{1}_{(t_i, t_i+1]}(t),$$
$$\mathscr{H} = \left\{ \begin{array}{c} H \text{ is predictable} \\ \text{and bounded.} \end{array} \middle| \int_0^t H(s)dM(s) \text{ is a FV martingale.} \right\}.$$

The ability to decompose the intermediate inferences down to verifiable immediate ones is what gives a proof the descriptive part of rigor. For example, proving that p_m : "$\tilde{Y}$ is a martingale" can be reduced to showing that p_{mp} : "The conditional expectation $E\left[\tilde{Y}(t)\middle|\mathscr{F}(s)\right] = \tilde{Y}(s)$" in five distinct cases: $p_1,\ p_2,\ p_3,\ p_4,$ and p_5. The structure of this argument is as follows:

$$p_{mp} \Rightarrow p_m,$$
$$p_{mp} \Leftrightarrow \bigwedge_{i=1}^{5} p_i,$$
$$p_1,\ p_2,\ p_3,\ p_3,\ p_5,$$
$$\bigwedge_{i=1}^{5} p_i,$$
$$p_{mp},$$
$$p_m.$$

Similarly, the construction of $\mathscr{H}$ can be decomposed into linear combinations and limits of H_{simple}:

$$H_1,\ H_2,\ \ldots,\ H_n \in \mathscr{H},$$
$$\forall \alpha, \beta \in \mathbb{R} : \alpha H_1 + \beta H_2 \in \mathscr{H},$$
$$H \in \mathscr{H} \quad \text{where } H(t,\omega) = \sup_{n \in \mathbb{N}} H_n(t,\omega).$$

These are examples of the decomposition process described by Harmami's DV schema. Together with the verification processes, this defines the descriptive part of rigor. Although a full exposition would be tedious and beyond the scope of this thesis, it suffices to observe that the proof by the monotone class theorem is one that an epistemic agent in mathematical practice would judge as rigorous; in other words, it has the descriptive part of rigor.

Under the conformity thesis, the descriptive part implies the normative part of rigor. Furthermore, through RT the vernacular proof can be converted into a lower-level proof adhering to elementary logical rules. Hence, in Harmami's framework, the proof using the monotone class theorem can be attributed the normative part of rigor without the assumption of the conformity thesis.

If the proof by the monotone class theorem is rigorous, why do epistemic agents of mathematical practice also consider the proof by the dynamic approach? According to Burgess's standard, rigor is the standard for good mathematics, and this is not a case where the proof by the dynamics approach proves a theorem that rigorous proofs cannot prove. If this were the case, the lack of a rigorous proof could be seen as a shortcoming that creates a need for an unrigorous alternative. However, this is not the situation. Why then do epistemic agents of mathematical practice still use unrigorous proofs? This question will be explored further in the following Section 4.4.

4.4 Plurality of Qualities in Practice

Since the proof by the dynamic approach lacks rigor, it should be excluded from practice on Burgess's standard. Nevertheless, one could adopt epistemic value pluralism instead of monism, if it explains the diversity observed in mathematical practice better. As a methodological assumption in the philosophy of mathematical practice, I assume that the practice is justified in its conduct — as stated in Chapter 1, our task is to understand how. So, an exclusion of proofs by the dynamic approach is not an option. There must be a reason for applying it to prove Theorem 1. Using the terminology of Section 2.3.1, this proof is correct even though it is unrigorous.

When is such a proof correct? The answer lies in the introductory context provided in Andersen *et al.* (1993). At this stage, the authors are not focused on grounding results in measure theory but rather on helping the reader build familiarity with the involved concepts. The structural nature of the proof by the monotone class theorem would not fulfill this purpose. Intuition is not fostered by the construction of expanding inclusions. Choosing the rigorous proof by the monotone class theorem would be the wrong action for this epistemic task.

While the proof by the monotone class theorem holds value due to its rigor, it is not suitable for all epistemic objectives. This observation is only accessible if one rejects epistemic value monism. If rigor were the sole standard of good mathematics, it would be impossible to explain why an unrigorous proof might better serve specific epistemic objectives. However, through the lens of epistemic value pluralism, this distinction becomes apparent.

In Section 2.4, I introduced four dimensions for evaluating proofs: aesthetics, intricacy, utility, and precision. The first two were classified as figurative dimensions, while the latter two were classified as epistemic ones. Rigor is strongly associated with precision. Consequently, the unrigorous proof by the dynamic approach is unlikely to be described as "precise". Instead, it is more likely to be evaluated by qualities associated with other dimensions.

If the proof by the dynamic approach is unrigorous but plays an epistemic role in mathematical practice, it must excel in the complementing epistemic dimension of utility. This dimension is associated with adjectives such as "practical" and "informative", which align with building intuition. This illustrates that different proofs can serve different epistemic objectives.

The focus on qualities' association with other dimensions than precision may seem indirect. A more direct approach would analyze the positive and negative correlations between rigor and other qualities. While this cannot be inferred directly from the data in Inglis and Aberdein (2015), it would offer insight into how epistemic agents evaluate rigorous and unrigorous proofs.

Inglis and Aberdein (2015) provide a reduction of the epistemic aspects of proofs to a surface spanned by two dimensions. Although this methodological reduction overlooks some epistemic nuances, it remains an operational tool for analysis. Since "rigor" — and not "precision" — is the addressed standard, I allow "rigor" to serve as a proxy for the epistemic dimension of precision. Unrigorous proofs contribute to mathematical practice by excelling in the dimension of utility: they are informative in establishing familiarity with the mathematical objects involved.

Here, I am only hypothesizing how epistemic agents in mathematical practice might evaluate the proof by the dynamic approach. To draw definitive conclusions, an empirical study like the one of Inglis and Aberdein (2015) would need to be conducted. However, such an empirical task is outside the scope of this thesis, and the application of the four-dimensional framework provides valuable conceptual insights in this case study.

As argued in Section 2.3.5, Tanswell's multi-modal rigor pluralism is still a version of epistemic value monism. It does not address the contrasting strengths of the two proofs presented in Chapter 3. Interpreting Burgess's standard as standard-as-rigor would classify both proofs as rigorous but under different interpretations of "rigor".

Tanswell might argue that his semantic relativism accounts for the different epistemic functions of the two proofs. The proof by the monotone class theorem is preferred for grounding results in measure theory because it is rigorous_1. The proof by the dynamic approach is preferred for building intuition because it is rigorous_2. Although this is a coherent position, it is an unnecessarily obscure one. Why not simply use different predicates such as "rigorous" and "heuristic", while endorsing epistemic value pluralism? The collapse of such distinct concepts into a single term seems only motivated by Tanswell's implicit endorsement of epistemic value monism.

In this chapter, I have analyzed both proofs in Chapter 3. In contrast to how the predicate "formal" is used in Andersen *et al.* (1993), neither proof aligns with the criteria for a formal proof in Section 2.1. Using the standard view of rigor from Section 2.2, I argued that the proof by the monotone class theorem is rigorous, but the proof by the dynamic approach is not. These demarcations were made independently of the conformity thesis by directly addressing the normative and descriptive part of rigor in both cases. Turning to the insights from Section 2.3, I argued that, instead of following the epistemic value monism represented in Burgess's standard, one should acknowledge the informative nature of unrigorous mathematics. Its heuristic is more suited for intuition-building than rigorous proof by the monotone class theorem. Both are important epistemic tasks, each related to different dimensions of utility and precision. Furthermore, I argued that this difference between proofs is concealed in the relativistic language of rigor pluralism that I discussed in Section 2.3.

I will now take an overview of the case into broader discussions in the philosophy of mathematics. With the insights from this chapter, I have the necessary luggage to discuss the ontological implications of using conditional expectations of dynamics. This epistemic step in mathematical reasoning is an example of a broad class of inferences in mathematical practice.

5 Martingale Externalization

The application of unrigorous proofs is not unique to stochastic analysis (McCullough-Benner, 2022, p. 113). In discussing Oliver Heaviside's operational calculus as a method for translating symbolic abstractions into solutions for physical problems, Colin McCullogh-Benner introduces the concept of *the robustly inferential conception* [*RIC*](McCullough-Benner, 2022, p. 117). This concept explains how mathematics can represent diverse target systems (McCullough-Benner, 2022, p. 114).

RIC extends the permissible class of inferences from its original framework to a broader domain. This extension relies on an epistemic agent's ability to infer connections between the original mathematical structure and a new target system (McCullough-Benner, 2022, p. 122). RIC thus provides a justification for engaging in unrigorous mathematics when it leads to productive outcomes. It exemplifies the flexibility of epistemic value pluralism: it highlights that different inferential strategies can satisfy distinct epistemic objectives. However, such extensions are not straightforward; they require that the epistemic agent possesses sufficient conceptual understanding to infer connections from the source domain to the target domain (McCullough-Benner, 2022, p. 122).

McCullogh-Benner argues that reducing Heaviside's operational calculus to a rigorous proof would miss key aspects of its justification (McCullough-Benner, 2022, p. 123). Heaviside leveraged physical interpretations to guide mathematical tools, enabling domain expansions. RIC illustrates how informational content derived from other interpretations can justify new mathematical inferences (McCullough-Benner, 2022, p. 123). Similarly, in stochastic analysis, the heuristic application of the conditional expectation for dynamics mirrors this justification process by introducing new informational content that enriches our intuition of martingales as discussed in Section 4.4.

5.1 Semantic Externalism

The concept of RIC aligns with the broader philosophical framework of semantic externalism (McCullough-Benner, 2022, p. 122). *Semantic externalism* asserts that meanings extend beyond the internal content of a definition. Hilary Putnam's dictum captures this perspective:

> "Cut the pie any way you like, 'meanings' just ain't in the head" (Putnam, 1973, p. 703).

This view opposes *psychologism* — the idea that meanings are purely mental entities — and instead emphasizes that meanings are also shaped by the concepts' *extensions* (Putnam, 1973, p. 700).

The *intension* of a concept refers to its definitional content, while its *extension* refers to the actual set of objects it denotes in the world. Frege's famous example of "the morning star" illustrates this distinction:

> **Intension:** The last bright object visible in the sky at dawn.
> **Extension:** The concrete planet Venus.

Similarly, the *ostensive* action of pointing to water and stating "This is water" grounds the meaning of the concept in its extension, rather than being confined to its intension (Putnam, 1973, p. 699). Reducing a concept's meaning to a purely intensional definition, as in psychologism, therefore, misses a significant aspect of meaning.

Martingales embody this as well. Its meaning is not limited to its definition but extends to their applications and the conceptual work they enable. Consider the definition of a martingale in Section 3.1: For all $t \in [0, \infty)$:

1. $M(t)$ is $\mathscr{F}(t)$-measurable.
2. $E\left(|M(t)|\right) < \infty$.
3. $E\Big[M(t)\Big|\mathscr{F}(s)\Big] = M(s)$ for all $s < t$.

Semantic externalism suggests that this definition presents only *one* intension of the meaning of the concept "martingale". The definition plays an active role in *defining*. Like an ostensive definition: "*This* is a martingale", the definition joins the intension of the concept meaning with its extension.

Addressing the externalization of mathematical objects brings the metaphysical position of *Platonism* to mind. I will not go into this here. Although semantic externalism seems to imply *ontological realism*, it does not need to be the case. A successful argument for *ontological antirealism* in this context would have to address *the Quine-Putnam indispensability argument* in relation to broader discussions of *scientific antirealism*. This endeavor is highly relevant and closely related, but not within the scope of this thesis.

Semantic externalism is obviously in direct conflict with formalism as presented in Section 2.1. If mathematics is merely an abstract manipulation of arbitrary signs, then there is nothing more to a concept than the intension of a definition. Formalism faces, on the other hand, the philosophical challenge of the external success of mathematics — how come mathematics represents anything (McCullough-Benner, 2022, p. 114)? Semantic externalism explains how mathematics can engage and represent external systems by externalizing the object from its definition.

5.2 Conditional Expectation of Dynamics

The global characterization of a martingale in its definition introduces a well-formed concept constructed by conditional expectations. Through deductive reasoning, one can derive additional insights, such as for $u \leq s < t$:

$$E\Big[M(t) - M(s)\Big|\mathscr{F}(u)\Big] = 0.$$

Since

$$\begin{aligned} E\Big[M(t) - M(s)\Big|\mathscr{F}(u)\Big] &\overset{1}{=} E\Big[M(t)\Big|\mathscr{F}(u)\Big] - E\Big[M(s)\Big|\mathscr{F}(u)\Big] \\ &\overset{2}{=} M(u) - M(u) \\ &= 0. \end{aligned}$$

Step-by-step clarification:

1. Follows from the linearity of conditional expectation.
2. Follows from the third property of martingales.

This result may be psychologically illuminating by presenting another property of the martingale. At two different points in time $s,\ t$ we do not expect a difference in value. However, this is not a new characterization beyond that given in the definition. Following Hamami's definition of rigorous proof as proofs routinely translatable into formal ones, the proof of Theorem 1 by the monotone class theorem similarly constitutes an application of definitions. This does not mean that the inference is analytical in the Kantian sense[1], as it involves manipulation of objects within the cognitive faculty of intuition, as discussed in Section 2.1. However, this process does not involve conceptual transformation of the object itself. No RIC through domain expansion.

In contrast, the proof by the dynamic approach is fundamentally different, as discussed in Chapter 4. It requires the application of the conditional expectation to extend beyond its original domain into the target system of dynamics. Whether this constitutes an instance of RIC is not a question for philosophical analysis but a practical concern for epistemic agents of mathematical practice. The fact that this approach is utilized within stochastic analysis indicates its status as RIC. My primary interest in this chapter, however, lies in the implications of this practice.

As noted in Section 2.2, assuming the local characterization of a martingale:

$$E\Big[dM(t)\Big|\mathscr{F}(t-)\Big] = 0,$$

holds for M ensures the global characterization of a martingale holds as well:

$$E\Big[M(t)\Big|\mathscr{F}(s)\Big] = M(s), \textit{ for all } s < t.$$

However, the converse does not hold — achieving this would require a domain expansion of the conditional expectation, which cannot be implied by the definition of a martingale alone.

How, then, can both the local and global characterizations of a martingale be used to derive properties of the same stochastic object? I will now present an answer by *externalization*.

The global characterization, as formulated in the definition, introduces the martingale as a mathematical object for epistemic agents to develop intuitions about. This process involves more than the intension of the definition; the object's extension also plays a role.

[1]See Section 2.1.

Only after this abstract object is externalized beyond its definition can it be characterized differently. If the conditional expectation is limited to stochastic variables and a dynamic is given as an increment:

$$dX(t) = d\left(\int_{[0,t)} dX(s)\right),$$

the dynamic approach remains inaccessible. It is only by externalizing martingales beyond the global characterization that one can heuristically express it by dynamic as:

$$E\Big[dM(t)\Big|\mathscr{F}(t-)\Big] = E\Big[M\Big((t+dt)-\Big) - M(t-)\Big|\mathscr{F}(t-)\Big] = 0.$$

The local characterization of the martingale offers valuable insights into this externalized object. This externalization makes questions about the conditional expectation's value tangible. While this process involves domain expansion through a conceptual metaphor, as discussed in Section 4.2, it does not translate an informal proof into a formal one. Instead, externalization enables the local and global conceptualizations of a mathematical object to have the same extension.

5.3 Externalization and Pluralism

The externalization of a martingale beyond its definition allows epistemic agents to develop intuitions about its behavior. The global characterization introduces the martingale as a well formed object, while the local characterization offers insights into its behavior via conditional expectations of dynamics. This motivates epistemic value pluralism: mathematical objects can be understood through different conceptions, each serving distinct epistemic goals.

In William P. Thurston (1994), he presents seven distinct ways of characterizing the derivative, e.g. geometric (the slope of the tangent to the graph of a function) and logical (via arbitrarily small ε and δ). Thurston notes:

> "This is a list of different ways of *thinking about* or *conceiving* of the derivative, rather than a list of different *logical definitions*" (Thurston, 1994, p. 3).

This triangulation of an object produces intuition through different characterizations. Drawing a graph to confirm the derivative's behavior constitutes an RIC if it takes place within the epistemic context of mathematical practice.

Similarly, the global and local characterizations of a martingale serve different purposes. The dynamic approach uses the local characterization to foster intuition, enabling epistemic agents to visualize the object beyond the intension of the global characterization in its definition. This externalization is not merely a restatement of its properties but a genuine heuristic contribution to mathematical practice.

As established in Chapter 3, the dynamic approach is demonstrably used in practice, confirming its status as a RIC. Its use underscores its epistemic value, even if it does not conform to rigor. This interplay between precision and utility highlights the necessity of epistemic value pluralism to understand the contributions of unrigorous proofs.

The dynamic approach plays a crucial role in developing mathematical intuition. By externalizing the martingale beyond its definition, epistemic agents can explore new insights and conceptions of the object.

Martingale externalization illustrates how heuristic approaches complement rigorous ones. By acknowledging the plurality of epistemic values in mathematical practice, one achieves a more nuanced understanding of the role that unrigorous proofs play in advancing mathematical knowledge. Externalization fosters the development of intuition, enabling mathematicians to engage productively with mathematical objects beyond their definitions.

6 Conclusion

I have argued that stochastic analysis exemplifies the presence of different epistemic values in mathematical practice. The dynamic approach highlights the application of unrigorous proofs and demonstrates the value of such practices.

By engaging with contemporary theories of mathematical rigor, particularly those of Hamami (2022) and Tanswell (2024), I contended that the standard view of rigor is preferable to a multi-modal perspective. The latter emphasizes the diverse epistemic functions of proofs in mathematical practice, a motivation I share. However, I argued that epistemic value pluralism achieves this goal more effectively than rigor pluralism. Crucially, epistemic value pluralism avoids the pitfall of semantic relativism, which risks conflating the distinct qualities of proofs with different meanings of "rigor".

Within the framework of epistemic value pluralism, the empirical findings of Inglis and Aberdein (2015) were introduced that demonstrate independence in the appraisal of proofs along two epistemic dimensions: precision and utility. These dimensions align with the epistemic functions of rigor and intuition-building, respectively, providing a robust basis for analyzing the case of two proofs of the same theorem from stochastic analysis.

I examined two proofs: one by the dynamic approach and the other by the monotone class theorem. Although the proof by the monotone class theorem meets the standard view of rigor, I classified the proof by the dynamic approach as unrigorous. Despite its unrigorous nature, I argued for its epistemic value. Its heuristic qualities make it uniquely suited for intuition-building, an epistemic task that the rigorous proof by the monotone class theorem does not fulfill. The precision of rigorous proofs, while indispensable, can sometimes impede the process of developing intuition.

In addition, I explored the ontological implications of semantic externalism. The need to externalize mathematical objects from their definitions arises from the coexistence of complementary characterizations. Without such externalization, alternative characterizations could only reiterate the intensional content of the definition, rather than providing a deeper engagement with the concept.

Although a discussion of Platonism and its relation to semantic externalism lies beyond the scope of this thesis, I briefly noted that externalism does not necessarily entail ontological realism. Addressing ontological antirealism in this context would require engaging with the Quine-Putnam indispensability argument and broader debates in scientific antirealism, a task reserved for future work.

In conclusion, epistemic value pluralism is both present and desirable in mathematical practice, as it underscores the dual importance of precision and utility in mathematics. I have presented this pluralism in the practice of stochastic analysis, where unrigorous proofs prioritize utility by fostering intuition, albeit at the expense of precision. While explanatory gaps in some unrigorous proofs must be filled to ensure reliability, mathematicians must not shy away from exploratory heuristic thinking. By embracing epistemic value pluralism, mathematical practice can preserve and enhance both the rigor and intuition that drive mathematics.

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