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Inquiry of the Past and Reflection on the Present:

Teaching Rigour and Reasoning in Area Determination through Authentic Historical Sources

Nynne Milthers & Amanda Wedderkopp

Speciale i matematik og didaktik

Vejledere:
Britta Eyrich Jessen & Tinne Hoff Kjeldsen

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Preface

We would like to thank our supervisors, Britta Eyrich Jessen and Tinne Hoff Kjeldsen for the guidance, support, and encouragement throughout the duration of our thesis. We are appreciative of their patience and dedication in helping us navigate the complexities of our project.

We would also like to thank XXX Gymnasium and the class in question for allowing us to impose our fondness of history on them.

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Also, we would like to thank Cille for all the things she helped us with in this process (including this preface).

Abstract

In this thesis we explore how the implementation of authentic mathematical sources in a teaching sequence can be used to teach students in upper secondary school about rigour and reasoning within the subject of area determination. We are utilising works from two past mathematicians, Archimedes and Newton. In order to create an inquiry-reflective learning environment within the framework of The Anthropological Theory of Didactics, Epistemic configurations and the Multiple Perspective Approach, we propose a research methodology. Our proposal is based on Didactical Engineering within the frame of The Anthropological Theory of Didactics and further developed taking inspiration from a methodology by Kjeldsen and Willumsen of how original sources can be implemented in a teaching sequence.

We design an implement a teaching sequence which aims at encouraging students to an inquiry of rigour and reasoning within the chosen historical episodes in order to reflect upon rigour an reasoning in their contemporary context. Our study demonstrates that engaging with historical texts enables some students to identify the evolution of mathematical rigour and reasoning, but discrepancies in the students' expected and actual knowledge made it difficult to implement the intended teaching sequence. Furthermore is sheds light on the students misconceptions of rigour and reasoning. The findings suggest that mathematical historical sources are valuable tools in identifying misconceptions and encouraging students in discussion within mathematics, which is not usual.

Contents

Introduction

We are interested in how we can use authentic historical sources in a teaching sequence in upper secondary school. Therefore we will investigate the possibilities and limitations of using such sources to teach students about rigour and reasoning regarding area determination. Since the 2000's the use of authentic historical sources has been a theme in the research field of mathematics education and nowadays a very well-established theme. As Evelyn Barbin already noticed in 1994: "Studying the history of mathematics allows one to study the construction of mathematical knowledge and to study mathematical activity – (...) to see mathematics (..) as an activity, a human activity". We see a great potential to bring forth how mathematical rigour and reasoning has changed through time and how mathematics is not a timeless field through the students' encounter with selected historical episodes. Through an observation of these historic episodes, we further see a potential to strengthen the students' own mathematical argumentation and reasoning, hence an opportunity for the students to orient themselves in their present. In this way we aim at creating a lesson design that offers opportunities for students to gain insight into authentic mathematical practice, with an environment in which the students can work in ways which are analogous to, or at least similar to, how mathematicians work.

With this interest we want to create an inquiry-reflective learning environment where students gain insight into and reflect explicitly on inquiries that mathematicians engage in when they conduct research from the perspectives of past mathematicians' situated in their workplaces.

The thesis is structured into four parts. The first part establishes the theoretical framework, discussing the conformity of the chosen theories and their application in didactics and historiography. The second part concerns a content analysis, where we define the concepts of rigour and reasoning as used in this thesis and we examine the historical and contemporary approaches to area determination in the light of rigour and reasoning. The third part presents our teaching design and the considerations involved in its development, including lesson plans and the created compendium. Finally, the fourth part analyses the implementation and outcomes of the teaching sequence, providing insights into in the limitations and possibilities of our teaching design.

Part I

Theoretical Framework

Connecting Theory of Didactics with Historiography

Chapter 1

Introduction to Part I

We are designing a teaching sequence in mathematics in which we use authentic mathematical papers, specifically a paper by Archimedes and by Newton. Therefore, we will incorporate literature and theories from the field of didactics and the field of historiography within mathematics. This combination will give us a foundation from which we can design, test, and analyse our teaching design. In this part of the thesis, we will put forth a theoretical framework consisting of *The Anthropological Theory of Didactics* founded by Chevellard, *the Multiple Perspective approach*, and the *ways of using history* founded by Eric Bernard Jensen and adapted by Tinne Hoff Kjeldsen into the field of mathematics, and the theory of *Epistemic Configurations* by Moritz Epple, which is Epple's adaption of the theory of experimental systems within the field of chemistry by Hans-Jörg Rheinberger. Based on this theoretical framework we will put forth our research questions of this thesis.

Afterward, we will account for two different, already established research methodologies, as there is no research methodology within the frame of both the Anthropological Theory of Didactics and literature within historiography of mathematics. Based on Didactical Engineering within the framework of the Anthropological Theory of Didactics and inspiration from the Methodological Triangle for how one can incorporate authentic mathematical sources in the classroom, we will propose a research methodology that accounts for both fields. We will base our thesis on our proposed research methodology.

1.1 The Conformity of the Theories

The literature that forms the theoretical framework of this thesis shares a view on how mathematical knowledge is constructed.

First, we note that the Anthropological Theory of Didactics is based on the assumption that:

> Doing, teaching, learning, diffusing, creating, and transposing mathematics, as well as

> any other kind of knowledge, are considered human activities taking place in institutional settings. (Bosch and Gascón, 2014, p. 68)

Hence, the construction of mathematical knowledge is regarded as a human activity conditioned by the institutional setting in which it arises. Second, a similar perspective on this matter can be found in the Multiple Perspective approach if we look at what mathematics is regarded as, namely:

> (...) a cultural and historical product of knowledge that is produced by human intellectual activities. The knowledge that is produced by a mathematician, or group of mathematicians, at a certain time in history depends on the knowledge and mathematical culture available for these mathematicians and it (might) shape or define guidelines for further developments of mathematical knowledge. (Kjeldsen, 2011, p. 3)

Knowledge is a product of human activities, in this case dependent on available knowledge and culture. If we consider available knowledge and culture as an institutional setting, there is a shared view between the Anthropological Theory of Didactics and the Multiple perspective approach. Third and last, the view that construction of knowledge is considered as human activities can also be found in the theory, of Epistemic Configurations, as we regard social practices as human activities:

> (...) the activities involved in establishing scientific knowledge (...) are of course practices, or rather a whole set of related but different practices. (...) any meaningful social practice, and hence scientific practice, is a fortiori a cultural practice. (Epple, 2010, p. 217)

Here, the construction of scientific knowledge - such as mathematical knowledge - is considered a cultural practice, in line with how the Multiple Perspective Approach sees the construction of mathematical knowledge as dependent on the available mathematical culture.

We see a conformity in the perception of how knowledge is constructed across the literature which forms the theoretical framework of this thesis, namely that knowledge is regarded as human activities dependent on the institutional setting, the available body of knowledge and the current cultural practices. The shared perception of knowledge described above corresponds to an action-oriented approach to knowledge. Thus, we find no immediate discrepancies in our theoretical framework.

Chapter 2

Historiography in Mathematics

In this chapter, we give an account of the literature that constitutes the historiographical part of our theoretical framework, namely *the Multiple Perspective approach and the ways of using history* and *Epistemic configurations*.

2.1 A Multiple Perspective Approach and the Ways of *Using* History of Mathematics

The Danish historian Bernard Eric Jensen acknowledges that developments throughout history are conditioned by the time in which one lives, and that contemporary time is also conditioned by the past and the expectations of the future. These considerations led Jensen to coin the *Multiple perspective approach*, a theory that takes such external factors into account. The underlying premise in his theory can be stated as: "(...) people are understood as being shaped by history and being shapers of history."(Kjeldsen, 2011, p. 3). The notion multiple perspectives refers to the fact that: "History is studied from perspective(s) of the historical actors, paying attention to these actors' intentions and motivations, as well as to intended and unintended consequences of their actions." (Kjeldsen, 2011, p. 3), which again underlines that the theory is an action-oriented conception of history. As such, the production of mathematics should not be separated from the the time and conditions in which it was and is developed because external factors play a significant role.

Kjeldsen who adapted Jensen's theory to the field of mathematics, argues that the history of mathematics in this action-oriented setting can be pursued if we study the history of mathematics from the perspectives of past mathematicians, i.e. in regard to the historical context in which the mathematicians lived, the available knowledge, the circumstances surrounding the new knowledge, and so on. This we could call the mathematician's *workplace*.

Furthermore, the multiple perspective approach is twofold. On the one hand, it can be used to investigate the historical actors perspective. On the other hand, the historians own perspective can and should also be considered, because the historians research is founded in his/her own questions, problems and so on.

2.1.1 A *User* of History of Mathematics

It is evident that one study the past with some use of it in mind. Whether that being to orient oneself in the present or merely to orient oneself ind the past. As Jensen puts it history is: "When a person or a group of people is interested in something from the past and uses their knowledge about it for some reason."(Jensen, 2010, p. 39). Hence use is a central component of Jensen's theory as this is what makes up history. In this thesis we adopt Jensen's view of history. With this in mind Jensen defines four concept pairs of the way one uses history. It should be stressed that these pairs are not mutually exclusive. They can be present in various degrees and are overlapping, so we can think of them as a spectrum. Furthermore the pairs are not developed to single out a correct use of history, only to describe it.

The first concept pair is *Lay history* and *professional history* which is a distinction of the context in which history is used. Lay history refers to an 'everyday' usage, i.e. the non-professional historians. A concern which can arise with this use is the motivation to modernise old mathematical results in order to bring forth a perceived connection with the past or a motivation to simplify a text for a modern reader. Such a modernisation can lead to a distorted view of the developments. (Kjeldsen, 2011, p. 4). The professional usage refers to the academic historians work, people who are trained in the field.

The second pair is *Pragmatic history* and *scholarly history*. The underlying question with these are roughly speaking: what are history a tool for? Is it a tool with which we can understand and orient us in our present or is it something that merely can be used to understand the past on its own terms. Hence the pragmatic history has a utility perspective - a usage with which we can better ourselves. In contrast with this the scholarly approach this is based on a understanding and an investigation of the past on its own terms. Furthermore this concept pair and the aforementioned pair often overlap in such a way that a person whom uses lay history will often conform to a pragmatic use and vice versa(Kjeldsen, 2011, p. 4).

The third pair is *Action history* and *observer history*(Kjeldsen, 2011, p. 5). Action history is characterised by a persons use of past episodes to act in the present, by this one uses history to orient oneself in the present. In opposition to this the observer history is characterised by a wish to understand the past on its own terms. With this pair we can identify and depict the way a person looks at the past - either retrospectively or forward looking.

Lastly, the fourth pair is *Identity concrete* and *neutral history* which are used to examine the way one presents history. Either with an intend to form the recipients perceptions of themselves or not(Kjeldsen, 2011, 5).

Kjeldsen states: "These notions provide a set of glasses—a lens—through which we can identify, articulate and distinguish between different understandings and uses of history. Together with the multiple perspective approach to history of mathematics outlined above, they provide a theoretical framework that can be used to characterise, analyse and criticise uses and practices of history and implementations of history in mathematics classrooms. They can also be used to orient designs and future implementations of history to clarify and target learning goals and teaching intentions."(Kjeldsen, 2011, pp. 5-6)

These concept pairs can allow us to orient our design in such a way that we can 'place' the students in learning environments with different kinds of usages of history that depends on the learning outcome we aim for.

2.2 Epistemic Configurations

The German historian of mathematics Moritz Epple identifies problems with considering mathematics as timeless. According to Epple, mathematical research is, and must be, fluid if innovative research is possible (Epple, 2010, p. 240). Epple states that he sees most accounts of mathematical knowledgeas trying simultaneously to make room for two claims: 1) the objects of mathematics are timeless, and 2) the actual definitions of mathematical object are made at a particular time in a particular historical context. However, it is evident that these claims are at odds regarding the context of discovery vs. the context of justification.(Epple, 2011a, p. 482) To address this problem, Epple adapts the notion of *experimental systems* coined by the chemist Rheinberger into the field of mathematics. Epple calls these systems, in a mathematical context, *epistemic configurations*, which he regards as the smallest productive units of mathematical research(Epple, 2011a, p. 487). Furthermore, Epple defines *epistemic objects* and *epistemic techniques*, which are the constituents of the epistemic configurations. Epple defines epistemic configuration as:

> "An epistemic configuration of mathematical research is the entirety of the intellectual resources that are involved in a particular research episode. It comprises the mathematical language, the skills and techniques at the disposal of the mathematician or the group of mathematicians engaged in this research, the set of research topics and open problems under consideration, the horizon of aims and more general heuristic guidelines followed by the researchers, etc"(Epple, 2011b, p. 148)

Hence, the production of mathematical knowledge occurs in these epistemic configurations, and one cannot discern the mathematician's workplace, as these past scientific researchers were bound and formed by the time and place in which they lived. From this quote, we get a hint of two constituting categories - something that can be applied in the research and something that drives the research. This leads to the definition of the aforementioned epistemic techniques and epistemic objects:

> "Generating new knowledge means posing and answering new, previously unthought questions, or answering old questions in new, previously unimagined ways. Accordingly, there are two kinds of elements involved in the cognitive practice of scientists: elements that induce questions, that open up the future of research [epistemic objects], and elements that generate answers [epistemic techniques], that produce a stable past for ongoing research activity"(Epple, 2011b, pp. 148-9)

We can regard the epistemic objects as being question-generating and the epistemic techniques as being answer-generating. Because of the historical boundedness of the epistemic configurations, it is possible for an epistemic object in a later time to shift status and become an epistemic technique. Thus, the objects of mathematical research arise in the processes of research as well as changes in these processes. In connection to an inquiry-reflective based learning environment where one wants to create an environment that are similar to a researcher's there seems to be an opportunity to bring forth and emphasise how procedures which might seem stable today occurred in the process of past research.

This leads to Epple's claim that: "The dynamics of the epistemic objects of mathematical research are secondary to the dynamics of the epistemic configurations as a whole. To understand the former, it is necessary to understand the latter"(Epple, 2011a, p. 488). Therefore, when one is trying to understand mathematics from the past, one needs to regard the technical framework in which it is placed, i.e. what knowledge was available at the time, what were the objects to be studied, what motivated the investigation, and so on. For example, we can trace back the word *tangent* to Euclid's work, however, to understand what this expression meant at the time, we should investigate what it meant for Euclid himself and what knowledge he had available because definitions are ever changing.

In relation to the Multiple perspective approach and the notion of the mathematician's *workplace*, we argue that these epistemic configurations that Epple defines are analogous. With the use of Epple's terms, epistemic objects and epistemic techniques, we gain greater precision when we consider a historical episode because this will give us a vocabulary and allow us to analyse the mathematical practices at stake. Furthermore, in connection with the concept of Observer history Epple's theory enables us to identify and analyse the historical mathematical papers with respect to the time in which they are placed. In this way, we can get an analytic tool that enables us to orient our teaching sequence in the design phase in regards of observer history.

In order to create a learning environment where the students have the opportunity to reflect upon their own understanding of mathematical rigour trough an inquiry of two historical episodes, it can be fruitful to pose questions about the production of mathematical knowledge. About such questions, Epple writes:

> If such questions are seriously posed, the activities of mathematicians appear in a different light. Issues such as the specifics of the mathematical language used in a particular period and region, the possibilities offered and the limits imposed by particular conceptual frameworks or ways of imagination, the differences in proof strategies and standards of rigor (...) move into focus. (Epple, 2011b, p. 133)

With this in mind, combined with the concept pair of action vs. observer history use, we see an opportunity to bring forth the standard of rigour in the time the historical episodes took place with an observer history use. The shift to an action use of history will foster an opportunity for the students to orient themselves in the present standards of rigour in relation to area determination.

Chapter 3

Didactical Theory

In this thesis we will employ the theory of *the Anthropological Theory of Didactics* (ATD), which will constitute the didactical part of our theoretical framework.

The Anthropological Theory of Didactics was founded by Yves Chevellard in the 1980s as a research program in mathematics education with the assumption that knowledge is a human activity which takes place in an institutional setting (Bosch and Gascón, 2014, 68). ATD is driven by the search for a rationale for any piece of knowledge to be taught, and in this way, it shares a fundamental assumption with The Theory of Didactic Situations, albeit it is different (Barquero and Bosch, 2015, p. 261).

ATD is a research program in constant development, and new elements are therefore continually added. In this thesis, we will make use of some elements, namely; The Didactic transposition, the Didactic Contract, Praxeologies, Study and Research paths and Media-milieu dialectics. We will account for these elements in this section. Throughout this section, we will also point out the most apparent connections with the two historiographical theories already accounted for.

3.1 The Didactic Transposition

It is apparent that the institutional setting in which knowledge is produced, taught, and learned is of great importance in ATD (Bosch and Gascón, 2014, 68). The process of transformation that the body of knowledge undergoes from being produced in a scholarly setting to being taught in a educational institution, i.e the transformations between the different levels of knowledge, is called *the didactic transposition* (Joaquim Barbé and Gascón, 2005). This notion provides a tool for analysing how knowledge transforms between different institutions. The transposing of the body of knowledge takes place in four stages, namely; 1) Scholarly knowledge, 2) Knowledge to be taught, 3) Taught knowledge, and 4) Learned knowledge (Joaquim Barbé and Gascón, 2005). This process can be illustrated as in Figure 3.1, in which we can also note that the transpositions are happening ’two-ways’, as this is not a process of merely degrading and simplifying knowledge in order to be taught. This is a tool with which the body of knowledge in the different institutions can be made accessible for analysing and, therefore, improvement.

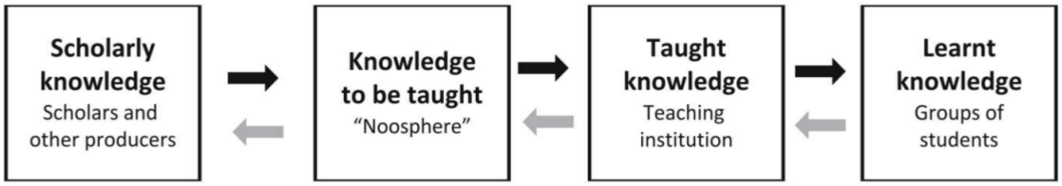

Figure 3.1: The process of the didactic transpositions as illustrated in (Bosch and Gascón, 2014)

In general, scholarly knowledge is produced at scholarly institutions such as universities. According to the theory, the body of knowledge first has to be transposed into *knowledge to be taught*. This happens in the *noosphere*, i.e. it is transposed by actors who "think about teaching" (Chevallard and Bosch, 2014, p. 170), e.g. teachers and producers of knowledge. In the noosphere, different factors need to be considered, such as what part of the knowledge is required to be taught, i.e. what do we want the students to actually learn and be able to do, and how do we best keep the authenticity of the knowledge and not risk distorting it in order to make it easier to understand. This is where knowledge may also be disregarded as irrelevant for the citizens. In general, the knowledge to be taught, and therefore the first transposition results in the official curriculum, textbooks, teaching guides, and so on. This transposition is usually referred to as the *external didactic transposition*, as this happens outside the school.

After the knowledge to be taught has been officially stated, it undergoes a transposition through the teacher in a classroom setting. This transposition is called the *internal didactic transposition* because the knowledge to be taught, handled in the noosphere, turns into actually *taught* knowledge. Due to the transposition, this is different from the scholarly knowledge, but preferably the main elements of the scholarly knowledge is preserved and appears as genuine as possible. Finally, the *learned* knowledge becomes the knowledge that is available to the students.

This thesis investigates the bridge between each of the steps in Figure 3.1. We wish to show how a mathematical historical approach can facilitate the transfer of knowledge from the historical scholarly knowledge about rigour and reasoning to taught knowledge in a classroom setting. Therefore, we note that we regard historical knowledge, which is being used in the design phase, to be on the scholarly level. Because we see a transposition of knowledge from a historical analysis of the subject and sources to historical knowledge to be taught.

3.2 Praxeologies

As already mentioned ATD considers knowledge as human activities, which should be understood as: "(...) any human activity can be decomposed into a succession of tasks of various types." (Chevallard and Bosch, 2020, p. 55). The notion of *praxeologies* is a key tool to describe the knowledge at stake and the learned knowledge. In broad terms, learning is then describes as the formation of coherent praxeologies (Bosch and Gascón, 2014, p. 68).

A praxeology consists of a praxis- and a logos-block. The praxis block is further divided in *type of task (T)* and *technique (τ)*, and similarly, the logos block is divided into *technology (θ)* and *theory (Θ)*. We denote a praxeology as the four-tuple $[T, \tau, \theta, \Theta]$ (Chevallard and Bosch, 2020, p. 55). This can be illustrated as in Figure 3.2. *Type of task* is the starting point of a praxeology and is to be understood in

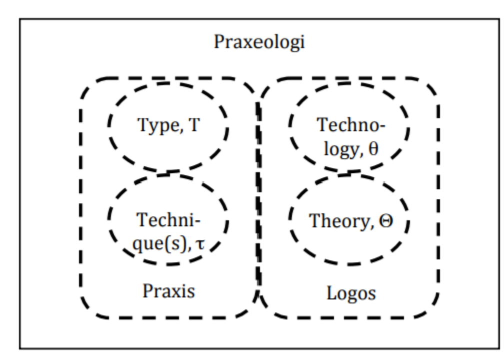

Figure 3.2: Praxeologies schematic representation

the broadest sense possible, form walking down the stairs to solving a quadratic formula (Chevallard and Bosch, 2020, p. 55). In order to solve a task, one needs a *Technique*. When performing a specific task, such as solving a specific equation, there is a specific way of doing it, hence a technique. These two entities constitute the praxis-block of a praxeology. Note that it can be the case that the technique used to solve a type of task actually consists of a sequence of techniques applying to subtasks within the task. Therefore, we adopt the notation that:

> Such techniques are denoted τ_i and the set of techniques solving the task: $\{\tau_1, ..., \tau_n\}$ where $n \in \mathbb{N}$. If we let $P(\tau_i)$ denote the subtask solved by the technique τ_i, then $P(\tau_i) \subseteq T$. p. 13

This idea of viewing a sequence of techniques as one technique is in particular useful in this thesis when we are analysing authentic mathematical sources.

Furthermore, there is a rationale behind the performance of solving a task. When this rationale is put into words, we can regard this as a discourse about how and why the specific techniques can and are used to solve a type of task. This is the definition of the *Technology* part of a praxeology. In this way, the technology relating to a type of task legitimise and justifies the correct way of solving the task(Chevallard and Bosch, 2020, p. 55). *Theory* is the broadest category in a praxeology and the second part of the logos-block. The theory justifies the technology itself, and in this way, it could be thought of as a technology of the technology. This is indeed a very entangled part of a praxeology, and "The distinction between the technological and the theoretical is neither clear-cut nor intrinsic: it is essentially a functional distinction" (Chevallard and Bosch, 2020, p. 56). The theory should be regarded as a bundle of technologies that share an overall theme, for example, probability or integral calculus. With this, it is clear that ATD assumes that no human action can exist without a justification of some kind. The formation of coherent praxeologies constitutes learning, and as Chevellard wrote, we can say that: “praxis thus entails logos which in turn backs up praxis”(Bosch and Gascón, 2014, citted on p. 68)

Praxeologies, i.e. sets of these four described elements form what is called a mathematical organisation, henceforth abbreviated MO (Barbe Farre et al., 2005, p. 237). We can determine praxeologies on three different ’levels’ and thus characterise different mathematical organisations. The smallest level is called the punctual MOs, which consist of a unique type of task solved by a unique technique, i.e an one-to-one correspondence of type of task and technique. On a broader level, a collection of different types of tasks with the shared technological discourse are called local MOs. Finally, a collection of local praxeologies, all drawing on the same theory, are referred to as regional MOs. In conclusion, the notion of praxeologies offers a general epistemological model of mathematical knowledge where mathematics is regarded as a human activity that materialises as the study of types of problems (Joaquim Barbé and Gascón, 2005, p. 5).

With the historiogrpahical theories that we have accounted for in mind, we can now explicitate a connection to the notion of praxeologies. First is the fact that praxeologies are regarded as bounded by the historical time in which they emerge. Because: "Praxeologies do not emerge suddenly and never acquire a final shape. They are the result of ongoing activities, with complex dynamics, that in their turn have to be modelled."(Bosch and Gascón, 2014, p. 69). In this way, praxeologies are closely tied to the historical setting, i.e. praxeologies are defined by and defining of the epistemic configurations in which they arise. Second is that the dynamics of praxeologies can be viewed in relation to the dynamics of epistemic objects and epistemic techniques, because:

> The fact that any piece of knowledge (i.e., any praxeology) can be considered as an answer provided – explicitly or de facto – to a question Q (a problem or a difficulty) arising in an institutional setting (or a “situation”). Question Q then becomes the “raison d’être” of the praxeology constructed, a rationale evolving as the praxeology develops and integrates into other kinds of activities, for instance to provide answers to other kinds of questions. It often occurs that the raisons d’être at the origin of most praxeologies disappear with time, and people end up doing things out of inertia or habit, without questioning their way of doing nor considering the possibility of changing them.(Bosch and Gascón, 2014,

> p. 70)

As such some things can become a mere habit after a period of time. Something that once could be placed in the logos block can with time move and become a part of the practical block. A similar shift as the epistemic things, which once were question generating can through time become an epistemic technique and therefore function as answer generating. Third of is the importance of our inclusion of the multiple perspective approach to the history of mathematics also come forth in the following:

> In the course of their long history, human societies may have arrived independently at similar solutions to some of the issues that beset them. More importantly, human societies are not true isolates, and, consequently, they generally share a part of their "praxeology pool" (Chevallard and Bosch, 2020, pp. 57-8)

In this way praxeologies are tied to the different mathematicians work place in which external factors have an importance of the development of mathematics.

3.3 Study and Research Paths

Study and research paths (henceforth abbreviated SRP) were introduced by Chevallard as a part of the program 'Questioning the World' (Chevallard, 2015, p. 177). An SRP is a design tool intended to foster autonomous inquiry where the conjunction of *studying* material and existing knowledge and *researching* questions is seen as crucial in order to develop knowledge (Winsløw et al., 2013, p. 269).

In a SRP sequence, students' work takes point of departure in a teacher-posed generating question, denoted Q. The generating question fosters an answer process and a question process, because:

> Students should understand the question but not be able to answer it, unless they engage in a study and research process. This process is supposed to be driven by initial hypothesis of an answer, which is incomplete and therefore lead to new, derived questions Q_i. In order to answer the derived questions, the students are supposed to study media to gain new knowledge. Media are the works of others, like textbooks, webpages, podcasts and other materials produced in order to disseminate (mathematical) knowledge (Jessen, 2017, p. 5).

The students' work with the qenerating question in a SRP sequence consists of the three following phases as put forth by Winsløw, Mathreon and Mercier:

1. Identifying "official" knowledge that can help answering Q. This is the activity of "study" of Q, based on the *consultation of resources in media* (books, Internet and so on) and on knowledge previously studied, which can act both as part of the shared praxeological equipment of the students and as media that are (re)consulted.

2. Creating and justifying answers to Q through more or less *pure reasoning*. This is the activity that we designate as "research" on Q (...)
3. Elaborating new questions from Q. These questions may be
   (a) *Subquestions* Q_1, Q_2,... for which answers provide partial answers to Q
   (b) *Derived questions* Q^*, motivated by Q or by answers to Q, but where an answer to Q^* does not itself answer Q wholly or in part (Winsløw et al., 2013, p. 270)

The derived questions that occur in a SRP sequence are naturally connected and are not always student-posed. They can be teacher-posed in order to guide the students or to ensure that the students work on some parts of the intended knowledge to be taught. The intervention of the teacher does not interfere with the natural connection in the sequence. In our teaching sequence, where we will make use of historical sources, it is noted that in the sense of these three phases in a SRP sequence above, these will be regarded as *media*.

A SRP sequence is often used in the design of an inquiry based learning environment and here we tend to use it as a tool for creating an inquiry-reflective learning environment. Together with the multiple perspective approach, that can be used to facilitate such an environment, as argued by Kjeldsen, we see an advantage of choosing and designing our teaching sequence based on SRP because this design tool relies on an open generating question in order to facilitate such an open inquiry.

3.3.1 Representation of Study and Research Paths

In order to simultaneously describe and evaluate the students' inquiry in our teaching design, we will make use of the diagrammatic 'tree-like' representation of how questions and answers are connected in the SRP, as proposed by Winsløw, Matheron and Mercier (Winsløw et al., 2013). Figure 3.3 is an example of such a diagram. The shading in the diagram indicates who raised the question. The black shading denotes the teacher, the white a student, and the grey a collaboration between student and teacher.

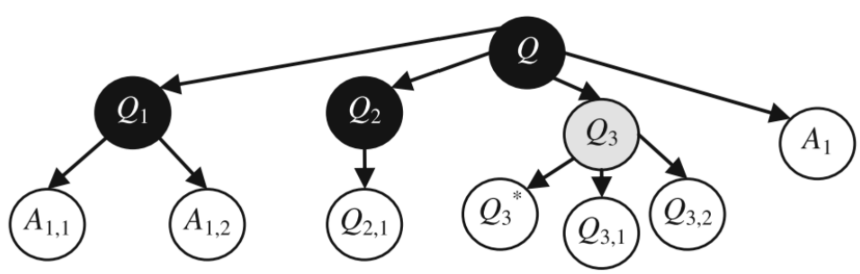

Figure 3.3: Illustration of an SRP tree diagram from Winsløw et al. (2013)

This representation tool is not only useful regarding an analysis of a realised teaching sequence. It is also useful in an a priori analysis of the sequence, as such an analysis will make the possible student outcome clearer and therefore can work as a tool in the design phase in order to orient a teaching design.

3.4 Media-Milieu Dialectics

We can describe the students' autonomous process of study and inquiry using the tool *the Herbartian Schema* if we regard: "Students' knowledge construction (...) as the result of the dialectics between study and research processes" (Jessen, 2017, pp. 5-6). The media-milieu dialectics arises from the students' gestures in the interaction and study of the generating question. It is made up of the continuous interaction and assessment between partial answers given by the media available to the students and their autonomous research through the interaction with an a-didactic milieu (Kidron et al., 2014, p. 158). With this in mind, we can use the Herbartian Schema to describe how partial answers make other questions arise, which consequently will initiate a further study and research process. Jessen (2022) characterises the study process of a SRP sequence as:

> Existing answers, works and data are all media to be studied in SRP's. The study process is characterised as deconstruction of knowledge, where research is considered reconstruction of knowledge as it is when existing knowledge, data and new knowledge are pieced together in terms of partial or more complete answers, $A_i^{\heartsuit}$ to the generating question. (Jessen, 2022, p. 233)

The heart designates that the answer, A, is personal to the student or group who developed it. In short, the Herbartian Schema can be represented as the function:

$$(S(X;Y;Q_0 \Rightarrow M) \hookrightarrow A^{\heartsuit} \tag{3.1}$$

where

$$M = \left\{ A_i^{\diamondsuit}, ...W_j, ..., D_k \right\} \tag{3.2}$$

In this function, S represents the *didactic system* of the SRP, and this consists of X, which is a group of students, Y representing the teacher or other people assisting the students in the SRP sequence, and finally the generating question Q_0, which is studied. In order to produce answers, one is in need of materials, which we say make up the didactic milieu M that is established by $S(X;Y;Q_0) \rightarrow \{A_i \diamondsuit' ...W_j, ..., D_k\}$. Hence in the function above $M = \{A_i \diamondsuit' ...W_j, ..., D_k\}$. In which the A_i's denote the students' previously developed answers, which they often will try to use in order to answer the generating question(Kidron et al., 2014, p. 158). W_j denotes the new knowledge acquired by the study of the various kinds of resources in the SRP sequence, i.e. textbooks, webpages, YouTube videos, or other media regarding mathematical knowledge (Jessen, 2022, p. 233). Finally, D_k denotes the new data consulted in the study process. In relation to our thesis, this will be the historical sources.

3.5 The Evolution of the Didactic Contract

The notion of *Didactic contract* is an interpretation of the commitments, underlying expectations, and beliefs of a *didactical situation* of the involved actors, i.e. students and teacher in the classroom. The objective of this notion is to interpret and account for the actions and reactions of the actors within the didactical situation(Brousseau et al., 2020, p. 197). Such reactions and actions occur due to the fact that the teacher manages the didactical situation, which creates and exploits mathematical situations in which students' mathematical knowledge is developed (Brousseau et al., 2020, p. 197). This notion originates from the older French tradition in didactics of mathematics, namely the Theory of Didactic Situations (henceforth abbreviated TDS) coined by Guy Brousseau in 1997. Recent research has sought to point out points of contact between ATD and TDS as well as some needed theoretical developments (Barquero and Bosch, 2015). In this, they argue that with the implementation of ATD, there is an *evolution of the didactic contract* due to the changed classroom procedures produced by the use of SRPs. In this section, we will account for the evolution of didactic contract (henceforth called didactic contract) as this will be employed in our data analysis.

The traditional sharing of responsibilities in the classroom between teacher and students changes because:

> Implementing an SRP requires students to assume different roles in the inquiry process, such as seeking available answers, validating or rejecting them, raising new questions, deciding which ones to follow or discard, planning the work to do, etc. Teachers also experience essential changes in their tasks: they are no longer the "knowledge holders" nor the sole person bringing new knowledge into the classroom (Barquero et al., 2022, p. 3)

The notion of the didactic contract can therefore be used to identify occurrences in the realised teaching design that could be explained by the new responsibilities in the classroom that both teacher and student have.

Chapter 4

Established Research Methodologies

In this section, we will account for two already established research methodologies for designing teaching sequences and experimentation in classroom. The first is within the framework of ATD, which we intend to evolve by drawing inspiration from the second established research methodology, which concerns the implementation of authentic mathematical sources in the classroom.

4.1 Didactical Engineering

Didactical Engineering (henceforth abbreviated DE) was initially developed in connection with the Theory of Didactical Situations in the early eighties (Artigue, 2014, p. 467). DE was developed as a research methodology that is based on a controlled design and experimentation of a teaching sequence (Artigue, 2020, p. 203). The validation of research carried out within the frame of this methodology is internal and is based on a comparison between an a priori analysis and an a posteriori analysis of the designed teaching sequence (Artigue, 2014, p. 470).

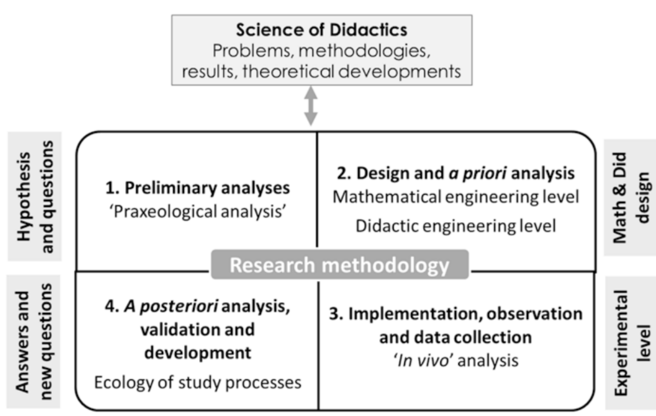

Figure 4.1: Phases of DE within the framework of ATD figure from (Barquero and Bosch, 2015)

DE is structured around four phases, namely; a preliminary analysis, a priori analysis, realisation with the intend of data collection, and a posteriori analysis and validation. This can be illustrated as in Figure 4.1, considering the numbers one through four. Berta Barquero and Marianna Bosch proposed an evolved conception of DE within the framework of ATD, in which they articulated each of the four phases of DE in terms of ATD, which can be stated as in the Figure 4.1(Barquero and Bosch, 2015).

4.2 A Methodological Triangle for using Authentic Mathematical Sources in Mathematics

Mikkel Willum Johansen and Tinne Hoff Kjeldsen have proposed a methodological triangle for using original sources in the classroom that is based on creating an inquiry-reflective learning environment (Johansen and Kjeldsen, 2018), which we intend to draw inspiration from. They regard their model as: "(...) a mediating link between the theoretical analysis of sources from the past and a classroom practice where the students are invited into the workplace of past mathematicians through history"(Johansen and Kjeldsen, 2018, p. 27). They have developed this research methodology based on the view that the possible benefits of using original sources in the classroom do not materialise automatically. Therefore, they aim to accommodate the question of *why* and *how* the use of original sources can support the teaching and learning *of* and *about* mathematics (Johansen and Kjeldsen, 2018, p. 36). They hypothesise that:

> (...) the encounter with the historical artefact will give the students the opportunity to experience something that seems foreign to what they already know, feel familiar with, consider as well-established or take for granted (Barbin 2011). We claim that such experiences may help the students to expand their horizon of understanding and increase their awareness of the function and importance of the cognitive artefacts they normally use. (Johansen and Kjeldsen, 2018, p. 29)

As such, it is clear that their methodology is based on the usefulness of a historical artefact, which they call *cognitive artefact* defined as: "(...) developed with the purpose of partaking in cognitive systems and processes" (Johansen and Kjeldsen, 2018, p. 28). Furthermore, they regard these cognitive artefacts as being culturally and historically situated.

Johansen and Kjeldsen describes three types of considerations which go into the design and implementation:

> (1) Theoretical Analysis of historical Sources (TAS) from the perspective of aspects of the nature of mathematics and historical insights and awareness; (2) Creation and framing of an Iquiry-reflective Learning enviroment *in* Mathematics (ILM); and (3) Instructions for practice promoting Students' situated Relfections (ISR). (Johansen and Kjeldsen, 2018, p. 38)

Along with three processes that combine these considerations:

> (a) designation of which aspects of mathematical research practices the teaching episode should mimic, i.e. which part of a mathematical research 'workplace' should the students be invited into; (b) design of the teaching material that can promote students to reflect upon aspects chosen for inquiry; and (c) evaluation of the development of students' informed conception of the aspects of the nature of mathematics, historical insights and awareness with respect to the results of the theoretical analyses of the sources. (Johansen and Kjeldsen, 2018, p. 38)

These considerations and processed can be illustrated as Figure 4.2.

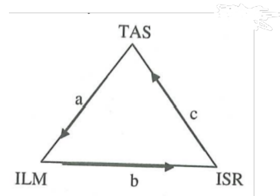

Figure 4.2: Methodological triangle for using original sources in mathematics education from illustrated in (Johansen and Kjeldsen, 2018, p. 38)

In conclusion, this research methodology seeks to put forth the mathematicians' strategies and techniques in the production of mathematical knowledge, which is available to us in the original historical sources "from the perspective of the significance of cognitive artefacts" (Johansen and Kjeldsen, 2018, p. 36)).

Chapter 5

Research Question

Now, we have established the theoretical background necessary to present our research question(s):

> RQ_0: How does the use of historical mathematical sources in an inquiry-reflective environment impact upper secondary school students' comprehension of rigour and reasoning in relation to Area Determination?

In order to answer this question, the following questions have been developed:

- RQ_1 How do we choose original sources, relevant to rigour and reasoning in regards to area determination, to be implemented in an inquiry-reflective learning environment using SRPs?
- RQ_2 How can we identify and accommodate challenges of teaching rigour and reasoning regarding area determination in upper secondary school in the design of a teaching sequence centred around selected historical episodes?
- RQ_3 How can exposing students to a fluctuation between action history and observer history promote reflection on rigour and reasoning in relation to area determination?

We will answer these three questions in order to answer our research question, RQ_0. In our work with RQ_1, we propose a research methodology based on DE within the framework of ATD and further developed with inspiration from the methodological triangle proposed by Johansen and Willumsen (Johansen and Kjeldsen, 2018). Regarding RQ_2, we study the didactic transposition of integral calculus from Scholarly knowledge to Knowledge to be taught. This is done in order to investigate rigour and reasoning within this subject, as it relates to area determination. This will guide us, when we investigate the rigour and reasoning in authentic mathematical sources and how they can be implemented in a teaching sequence. Finally, regarding RQ_3 we design a teaching sequence, which is tested in an upper secondary school class at XXX Gymnasium. Based on gathered data of the realised teaching sequence, we validate by comparing an a priori analysis and an a posteriori analysis of our teaching sequence, thus the validation is internal.

Chapter 6

Our Proposed Research Methodology for Using Original Sources in Upper Secondary School

In order to answer our research question, RQ_1 (*How do we choose original sources, relevant to rigour and reasoning in regards to area determination, to be implemented in an inquiry-reflective learning environment using SRPs?*), we need to establish a research methodology of *how* the implementation of such sources can be done. Here, we see a need for a research methodology that accounts for both ATD and historiography, as we want to create an opportunity for students to reflect upon rigour and reasoning regarding area determination using historical episodes. As such, this chapter relates to the '*how to*' part of our thesis.

We take inspiration from the methodological triangle for using original sources in mathematics education (Figure 4.2) and combine this with the research methodology DE within the framework of ATD (cf. Section 4.1) in such a way that we can create a research model within the framework of ATD and historiography. Further, we propose to use the notions of epistemic configurations (cf. Section 2.2), the concept pair Action history and Observation history (cf. Section 2.1), and expand the notion of praxeologies slightly, with the term historical praxeologies.

Our aim gives rise to three important considerations, which our research methodology needs to accommodate: 1) How do we select authentic sources that can be used in the classroom?, 2) How do we identify concrete passages in the chosen texts that can be used? and 3) How do we create an inquiry-reflective learning environment where students have the opportunity to work with historical mathematical sources and use this to orient themselves in their present?

In this section we will account for how we seek to accommodate each of the these three considerations and sketch out the phases of our research methodology within our frame of ATD and historiography.

6.1 The Choice of Source

The first question that arise when we want to have the students work directly with a historical mathematical source, which is also level appropriate, is: How do we chose among the many sources available? The actual usefulness of the source is also dependent on an in-depth theoretical analysis of the source. We suggest first deciding on a subject of interest within the official common objectives of upper secondary school, followed by an investigation of the historical development of the chosen subject to get a selection of possible sources.

As an example of a subject of interest, we chose in the present thesis to work with integral calculus in upper secondary school. The integral has, through history, been motivated as a tool to determine an area under curves. In upper secondary school, this is done with the Riemann integral, so this became our focus. Based on this choice, we investigated the historical development of the Riemann integral in a very broad perspective. It became clear it was established as a product of a long history of questions regarding area determination. By this, we established a selection of possibles sources through this investigation for our thesis.

One should study the external didactic transposition through a praxeological analysis in order to narrow down selection of sources. This should be studied to identify the institutional conditions under which the subject should be taught in upper secondary school. This study will guide the final choice of sources to implement, as this will make the body of knowledge in different institutions accessible for analysing and make it clear how the scholarly knowledge is made teachable and learnable. It is evident that with a transformation of a body of knowledge, there is a risk of oversimplification of the knowledge in order to fit it to the intended level. The study of the didactic transposition can be used to shed light on such problematic transpositions, which can guide how one sorts through the selection of sources.

6.2 A Historical Theoretical Analysis of the Chosen Source

The second question which arise after one has chosen a historical mathematical source to bring into the classroom is: how do we identify aspects in the source that can be used to develop students' understanding in regard of the chosen learning goals of the teaching sequence? Our hypothesis is that this is done through a theoretical historical analysis of the source (as also proposed by Johansen and Kjeldsen (**?**)). In our theoretical framework, we argued that such an analysis can be based on the notion of epistemic techniques, epistemic objects, and an expansion of the notion of praxeologies.

As epistemic configurations are defined as the entirety of intellectual resources available in a particular research episode, it can be an extensive task to fully comprehend a mathematician's epistemic configuration. Epple argues that in order to understand the dynamics of the epistemic object, one needs to understand the epistemic configuration as a whole (cf. Section 2.2). When implementing a source in a teaching design, which seeks to place students in the workplace of the past mathematicians, we suggest focusing on selected parts of the past mathematicians' workplaces in which the chosen source is situated in order to promote students to reflect upon the chosen aspects for inquiry. The parts one should consider in regard to the mathematician's workplace are: 1) the mathematical objects under investigation in the past research episode (i.e. epistemic objects) and the motivation for the research episode, 2) the available knowledge for the past researcher, e.g. the techniques available as well as the mathematical standards and traditions in which the past mathematician is situated (i.e. epistemic techniques). We argue that these parts constitute a sufficient foundation of analysis in order to implement a historical source in a teaching context, as these parts shed light on the practice of the past mathematician.

After the workplace of the past mathematician has been mapped out in respect to the two described parts above, we suggest to expand the notion of praxeologies slighty, and we wish to use the term *Historical praxeologies* to encapsulate our expansion. As accounted for, knowledge is constructed by human activities, which are dependent on the institutional setting according to ATD, and knowledge can be described as praxeologies, which: "(...) do not emerge suddenly and never acquire a final shape. They are the result of ongoing activities, with complex dynamics, that in their turn have to be modelled" (Bosch and Gascón, 2014, p. 69). We argue that this points to a historical dimension of praxeologies, and if we furthermore regard an epistemic configuration an institutional setting, we can use the notion of historical praxeologies to describe the knowledge at stake in the historical mathematical source. We argue that this can be done because the epistemic configuration is the entirety of intellectual resources involved in the research episode under investigation, thus the epistemic configuration is tied to the practice of the past mathematicians and therefore can be regarded as a kind of historical mathematical institution surrounding the researcher. Based on these consideration, we suggest that the second step of a theoretical historical analysis of the chosen source should be done in terms of historical praxeologies, in which one should seek to point out historical praxeologies to identify passages of the text that can be used in the development of students' knowledge regarding the chosen learning goals.

In this way, we propose that a theoretical historical analysis of the chosen source in order to implement it in a teaching sequence where ATD functions as the didactic framework should be done in two steps; 1) an analysis of the past mathematicians' workplace in light of the two steps described above, and 2) an analysis of the source in terms of historical praxeologies.

6.3 Creation of the Inquiry-Reflective Learning Environment

The third question that arises, is: how do we create an inquiry-reflective learning environment that fosters students' autonomous inquiry in such a way that the knowledge to be taught regarding the source can support an orientation in the students' present mathematical education? As Kjeldsen noted:

> (...) we have to keep in mind that the mere exposure to historical development processes of mathematics is not enough for students to develop informed conceptions about the nature of mathematics - for this to happen, students must be challenged to reflect explicitly and critically upon concrete aspects of the nature of mathematics(...) (Kjeldsen, 2014, p. 43).

For this purpose, we suggest making use of SRPs as a design tool in combination with the concept pair action history and observer history (cf. Section 2.1).

We suggest that the concept pair action history and observer history can be used to identify, articulate, and distinguish between different uses of history. Kjeldsen has already argued that Bernard Erik Jensen's concept pair, when adopted to mathematics, can function to: "(...) orient design and future implementation of history to clarify and target learning goals and teaching intentions" (Kjeldsen, 2011, p. 6). With the incorporation of both the uses of history, we can create a learning environment in which the students fluctuate between observer and action history.

We argue that this fluctuation is crucial in order to create and inquiry-reflective learning environment in which the students are placed as *observers* in the past mathematicians' workplaces, thus are encouraged to understand the past on its own terms through an inquiry of the sources. Hereafter, the students should be promoted to reflect upon present mathematics in the light of their new historical knowledge. By fluctuating between the two uses, and not only conforming to action use of history, we can create an environment where the students can work in way which are analogous to mathematical researchers, as they are encouraged to work as the past mathematicians. Which is exactly one of the obstacles in setting up an inquiry-based learning environment. Therefore, both uses are needed.

6.4 Phases of DE within the Framework of ATD and Historiography

We have now argued how we can accommodate the three considerations that arose when seeking to implement historical sources in the classroom. Based on this, we propose to elaborate on the already established research methodology DE within the framework of ATD by adding the dimension of historiography.

We propose that the first phase in DE should contain both a usual praxeological analysis of the content at stake in order to study the didactic transposition, and a theoretical historical analysis of the chosen sources carried out in terms of epistemic objects and techniques and our suggested notion of historical praxeologies. We also propose, that in the second phase one should seek to create an inquiry-reflective learning environment with the use of SRPs and the fluctuation between an action and an observer history use. Furthermore, we propose, when conducting an a posteriori analysis, that one should both seek to study how the students formed coherent praxeologies and coherent historical praxeologies, as the first is closely linked to an action history use and the latter to observer history use.

Furthermore, when implementing original sources we need to make sure what the validation takes place against. In the more typical modelling context the validation takes place "against the perceived reality, why media representing this must be part of the milieu" (Jessen and Kjeldsen, 2022, p. 110). In terms of the Herbartian Schema, we argue that in a teaching sequence which uses original sources, the validation takes place against the students' perception of rigour and reasoning. Even though this perception is personal to the students, we regard it as part of the students' existing knowledge $A_i^{\diamondsuit}$.

In conclusion, our proposed research methodology incorporates both ATD and historiography. This is an extension of DE which accounts for the use of historical episodes while conforming to the internal validation of DE, based on the confrontation between a priori and a posteriori analyses.

Part II

Content Analysis

Reason and Rigour in Relation to Area Determination

Chapter 7

Introduction to Part II

In this part we will give a brief overview of the history of area determination as this has supported our choice of historical episodes included in the teaching design. Furthermore we will study the didactic transposition of rigour and reasoning in relation to area determination from scholarly knowledge to knowledge to be taught. The didactic transposition will be studied based on rigour and reasoning in relation to the Riemann integral, as this is a common tool for area determination in a scholarly context, which is also taught in Upper secondary school. As such this part concerns RQ_2 (*How can we identify and accommodate challenges of teaching rigour and reasoning regarding area determination in upper secondary school in the design of a teaching sequence centred around selected historical episodes?*).

The analysis of scholarly comes in two parts. First, we analyse the chosen historical episodes from a scholarly viewpoint. Then we will analyse the rigour and reasoning related to the Riemann Integral on a contemporary scholarly level. The mathematics of the historical episodes is related to area determination, and has been a precursor for the development of the Riemann integral. We conclude the chapter with an analysis of the didactic transposition of the Riemann integral as well as rigour and reasoning.

7.1 Defining Rigour and Reasoning

In order to analyse rigour and reasoning in relation to area determination we need to establish what we mean by these terms. Gila Hanna wrote an entry in the *Encyclopedia of Mathematics Education* called *Mathematical proof, Argumentation, and Reasoning*, which she begins by noticing that: "Argumentation, reasoning, and proof are concepts with ill-defined boundaries. More precisely, they are words that different people use in different ways." (Hanna, 2020, 561). From this it is evident that we must establish what we mean when we use these notions.

According to Gila Hanna, argumentation "(...) includes any technique that aims at persuading others that one's reasoning is right."(Hanna, 2020, 563). We adopt this conception of argumentation, and thus we need to establish what we mean by reasoning in a mathematical context. Hanna suggest that we may take reasoning as "to mean the common human ability to make inferences, deductive or otherwise" (Hanna, 2020, 563). Within mathematics we need defined rules of reasoning in order to reach a valid conclusion. For example one form of reasoning is through formal logic with the use of an axiomatic method, in which one would employ formal notation, syntax, and rules of interference in order to permit the validity of the proof to be checked.

In mathematics we could therefore say that we persuade others of the truth of a statement by playing by a set of established rules - this is more or less the role of proofs in mathematics. In the simplest form, a mathematical proof is: "(...) a logical derivation of a given statement from axioms through an explicit chain of inferences obeying accepted rules of deduction." (Hanna, 2020, 562) This is in line with the the modern axiomatic-deductive method in mathematics, and we adopt this definition of a proof in this thesis. It is evident from Hanna's entry that there is no consensus about whether the mathematical proof falls under argumentation. Therefore, we want to clarify that we consider performing formal proofs as argumentation, in accordance with our conception of argumentation, as we find that the aim of a proof is to establish truth, which can be accepted by peers (i.e. 'persuading them that the mathematical assertion is 'right'). However, we note that mathematics can contain other forms of argumentation than the mathematical proof.

There is no direct entry in the *Encyclopedia of Mathematics Education* regarding rigour, so when we use this term throughout our thesis we also need to clarify what we mean by this. For the purpose of this thesis we define mathematical rigour as:

> **Mathematical rigour** pertains to the use of *logical deductions* in order to establish the truth of a stated hypothesis.

This is also in line with how Hanna uses the word, when she states that "contemporary mathematical practice is trending toward the production of proofs much more rigorous and formal than those of a century ago (Wiedijk 2008). In practice, however, one cannot write out in full any formal proof that is not trivial, because it encompasses far too many logical inferences and calculations."

There are many types of proofs in mathematics, some of the most common ones being direct proofs, proof by contradiction and proof by induction. In basic mathematical analysis, the most common type of proof is the direct proof (Eilers et al., 2018, 443). Here, one lists all assumptions made in a theorem, and use already established knowledge, usually previously established definitions, axioms and theorems, to draw the conclusion in question. A direct proof which follow this structure we regard as being rigorous according to our previous definition.

A rigorous mathematical argument leaves no room for ambiguity or doubt and adheres strictly to the rules of logical deduction and mathematical principles in order to prove a hypothesis, and thus establishing said hypothesis as a theorem.

Chapter 8

Scholarly Knowledge

In order to set up an inquiry-reflective learning environment in mathematics where the students have the opportunity to work with mathematical rigour through authentic mathematical sources from Archimedes and Newton, we need to identify in what passages of the sources the rigour and reasoning can be discussed, identified and articulated. For this purpose we proposed, in our established research methodology within the framework of ATD and historiography (cf. Section 6), that this should be done through a theoretical historical analysis of the sources.

First, we give a brief account of parts of the history of area determination, which guides the choice of historical episodes in focus. We will implement two historical sources in our teaching design, namely *Proposition 1* from Archimedes' work *The Method*, and the proof of *Rule 1* from Newton's *On analysis by infinite equations*. Next, in the theoretical historical analysis of the sources, we use the notions of epistemic techniques and objects to establish the workplaces of Archimedes and Newton, and historical praxeologies to analyse the rigour and reasoning of the argumentation given in the authentic sources. Finally, we will investigate the contemporary scholarly knowledge on area determination for rigour and reasoning, which will also be useful for the design phase.

8.1 History of Area Determination

In this section we give a brief overview of the history of area determination. We limit ourselves to the investigation of area determination and disregard the history of rigour and reasoning within this field. As argued in our proposed research methodology, such a historical overview is put forth in order to gather a selection of authentic mathematical sources, which relates to the chosen subject in the official curriculum - for this thesis the integral calculus, which is related to the history of area determination. Thus, this overview only serves to support the question of why the chosen authentic sources is within the field of area determination. In Section 8 we will analyse the rigour and reasoning of the chosen text.

Records show that ancient civilisations such as that of Mesopotamia knew how to calculate simple areas. Methods which still hold true today. The subject of area determination was motivated by a need for humans to measure and divide land for agriculture, trade, construction (Katz, 1998).

The Egyptians had a simple method to estimate the area of the circle, which is evident from the Rhind papyrus, dated to 1650 BCE. In the Rhind papyrus, using modern terminology we are told that for a circle with diameter d the area of the circle is determined by the formula: $\frac{64}{81}d^2$ (Katz, 1998).

In the 5th century Greece BCE it was shown that the area of a disk is proportional to its radius squared, thus the determination of curved bodies entered the scene at this point in the Greek civilisation. The Pythagoreans, a group of mathematicians surrounded the Greek mathematician and philosopher Pythagoras, active in the 6th century B.C.E. believed that numbers (i.e. positive integers) formed the basis of the universe. Part of this believe led to the understanding that all lengths of geometric figures could be *counted*, which in practice meant that the Pythagoreans believed that one could find a common measure for the side and the diagonal of a square. It was however refuted with the discovery that the side and diagonal of a square are in fact incommensurable. This discovery (430 B.C.E.) forced a change of some of the basic mathematical understandings of the Greeks (Katz, 1998, p. 48-51). The incommensurable numbers were not regarded as 'numbers', which led the Greeks to treat area determination in the means of comparing lengths and areas. For example they would compare the area of one plane figure to another which they could calculate the area of in order to determine the area of the first. This ultimately led to Eudoxos of Cnidos (c. 390-337 BCE) to form *The Method of Exhaustion* - a mathematical technique which find the area of a figure by inscribing and circumscribing polygons in- and outside with an increasing number of sides such that the areas of the polygons will 'merge' to be equal to the area of the figure under investigation. Archimedes, another Greek mathematician, also investigated the subject of area determination. He used his own mechanical method *The law of the lever*, which could indicate a result afterwards he would provide a proof of by a geometric method (we will treat this in depth in Section 8, as we have chosen a source from Archimedes). By the time of Archimedes the greeks were able to determine the area of more advanced figures, such as the parabolic segment and ellipses (Katz, 1998).

Now, we make a jump to the 17th century. This does not reflect that nothing happened in 2000 years we skipped - it is merely out of the scope of this thesis to delve into everything. Rather, our focus here is on historical episodes that led to Newton's mathematics as we have already treated the history which led to Archimedes' method. In this time of history there was a search for methods of determining areas which were less complex than The Method of Exhaustion and more straightforward (Katz, 1998).

Descartes, Fermat, commonly known as the fathers of analytic geometry as they in the beginning of the 1600's "both [...] present the same basic techniques of relating algebra and geometry. Both men came to the development of these techniques as part of the effort of rediscovering the "lost" Greek techniques of analysis"(Katz, 1998, p. 432). In short, the development is motivated by trying to replace Greek geometric analysis with a geometric version. The tool which came with the analytic geometry led more or less to the development of infinitesimal calculus, where Newton and Leibniz is credited as the founders - even though their method were very different. Leibniz discovered that differentiation and integration were 'opposites' - with the modern analysis we express this relationship by the main theorem of infinitesimal calculus (Katz, 1998).

The modern way to determine areas under functions is by the Riemann integral which first came to be i 1854 by Bernhard Riemann in 1854 (Katz, 1998).

In conclusion this historical overview guides our choice of historical episodes, namely Archimedes and Newton, as it puts forth possible historical episodes which can be related to The Riemann integral of today.

8.2 Archimedes

Archimedes of Syracuse (c.287 – c.212 BC) was an ancient Greek mathematician and scientist. He did not only contribute to mathematics with the discovery of new results - he also changed the game by disclosing his methods of discovery and exhibiting numerical calculations, in contrast to the mathematicians who came before him such as Euclid (who did not provide details of method of discovery, (Katz, 1998, p. 111)). Euclid mainly provided geometrical proofs of concise theorems, without much explanation of how the results were discovered in the first place (Katz, 1998, p. 103).

We chose the mentioned text from Archimedes because it easily falls into out the subject of area determination, in extension there is a historically clear path from the segment of the parabola and the use of indivisibles to the notion of functions and limits in contemporary mathematics. Furthermore Archimedes provides explicit considerations about the rigour and reasoning regarding his text, which we expect to be accessible to the students without a lot of transposition. Lastly - but definitely not least - the text shows examples of mathematics that is considered rigorous (both then and now), along with arguments from mechanics, of which the rigour is debatable, according to the standards in Archimedes present, and the use of indivisibles, of which the rigour is more than debatable.

8.2.1 Archimedes' Workplace

Archimedes investigated many fields, and mathematics was only one of them. We focus this analysis on his work as a mathematician, in particular on the research episode of developing and publishing *The Method*. With this treatise, Archimedes changed the game by writing a treatise which mainly aimed to investigate methods of discovery. Which is in contrast to Euclid's *Elements* which only contained definitions, axioms, theorems and proofs, and thus never presents a method of discovery.

The Method was discovered in 1899, and was inspected by Heiberg in 1906. Soon after, the Greek text was published. We will base our analysis on the English translation of the Greek text which is made by the English philologist Thomas L. Heath (1861-1940). Therefore one must note that this is a text which we read in translation and some things may be altered or modernised. However it is out of the scope of this thesis to investigate the original Greek text.

In the preface of *The Method*, Archimedes reflects upon his titular method:

> This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge.(Heath, 1912)

It is evident from this quote that the mechanical method is in contrast to the Greek tradition of demonstration by geometry, and Archimedes makes it clear that he as well does not consider an argument an actual demonstration unless it has been demonstrated by geometry. His proposed method is presented as a way to *discover* mathematical results not already known, under the believe that it is easier to prove a result that is already suggested by his method. Archimedes repeats this after he had presented the application of his method, by stating:

> Now the fact here stated is not actually demonstrated by the argument used; (...) we shall have recourse to the geometrical demonstration which I myself discovered and have already published. (Heath, 1912)

From this it is also evident that *Proposition 1* had already been proved rigorously elsewhere by Archimedes (according to Greek standards). In general in the treatise *The method*, Archimedes demonstrates his method on propositions that have already been proved elsewhere with geometry.

Mathematical Objects under Observation

In *The Method*, Archimedes showcases his use of mechanical principles to derive mathematical theorems, particularly those related to areas and volumes. Thus, the mathematical objects under investigation are figures of plane and space, such as the parabolic segment (which is of particular interest to us) and spheres, of which he aims to find the area and volume.

Therefore we need to consider how a parabolic segment were defined in Archimedes' present. According to Heath it is probable that Archimedes adopts basic principles of conics, which he assumes without proof, from some of Euclid's work (Schmarge, 1999). Euclid also deals with solid geometry in Book XI of the *Elements*, where a cone is defined as the trace of rotating a right-angled triangle about one leg:

> **XI.18** When, one leg of a right triangle fixed, the triangle is carried around and restored again at the same position from which it began to be moved, the figure so comprehended is a **cone**. And if the fixed leg is equal to the other leg, the cone will be **right-angled**; if less, **obtuse-angled**; and if greater, **acute-angled** (as cited in (Katz, 1998, p. 91)).

We can think of the hypotenuse of the right triangle as a line generating a cone. In particular, Archimedes is concerned with sections of what Euclid defines as the right-angled cone, and which in modern terms would be called a parabola (Katz, 1998, p. 117). The idea of a conic section is to consider how a plane intersects a cone. For instance, if the intersecting plane is parallel to the base of the cone (i.e. the circle generated by the leg of the triangle that is not fixed), the intersection is a circle. When the intersecting plane is parallel to a generating line of the cone, the intersection is what we in modern terms would call a parabola, and this is exactly what Archimedes considers in *Proposition 1*(Katz, 1998, pp. 111, 117). The idea of cutting a cone with a plane is illustrated in Apollonius' *Conics*, although we note that Apollonius "decided to define the conic sections slightly differently [than Euclid and Archimedes, red.]"(Katz, 1998, p. 117). It is out of the scope of this thesis to delve into details of Apollonius' definition, but we include an illustration that resembles one that appeared in his work *Conics* as a visual aid which is also included in the compendium used in our teaching design. The figure is shown in figure 8.1. It is important for us to note that even though the parabolic segment in

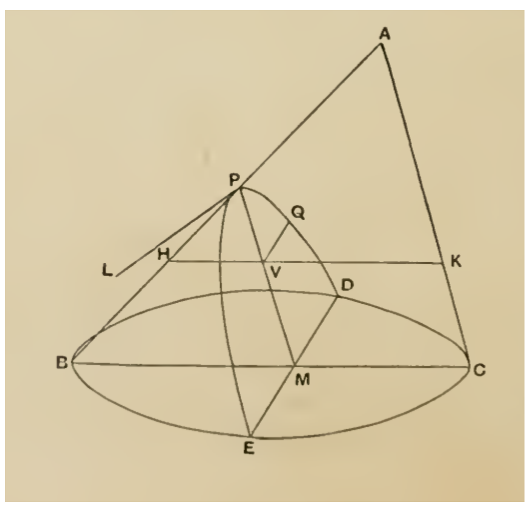

Figure 8.1: Example of conic section from Apollonius' *Conics*

many ways resembles the parabola of modern mathematics, Archimedes understood this figure from a different point of view than we do today. As such, the parabolic segment is also in contrast to the definition of a parabola that students are familiar with from their upper secondary education (usually treated in the first year). As such the epistemic object in Archimedes workplace is very different from today's definition.

Available Knowledge

With the mathematical object clarified we now wish to investigate what knowledge was available to Archimedes' in this research episode. Overall, we identify three 'types' of categories of epistemic techniques; geometric, mechanical and indivisible.

The first type is *established mathematical techniques*, which is a mix of broadly accepted postulates at the time and propositions that had already been established by methods of geometry. We assume that the contents of Euclid's *Elements* are a part of Archimedes' available knowledge. In particular, Euclid has three postulates of geometry that form the basis of construction of geometric figures:

1. To draw a straight line from any point to any point
2. To produce a finite straight line continuously in a straight line
3. To describe a circle with any center and distance(Katz, 1998, p. 61)

The demonstrations in *The Method* require the construction of figures, satisfying certain properties, and in this process, we assume that Archimedes uses straightedge and compass to construct, just as Euclid used. Thus, an important example of an epistemic technique that was available to Archimedes is construction of figures with straightedge and compass, as we will see applied in more detail below (cf. table 8.1). In *The Method* Archimedes also uses results that have been proved in Euclid's *Elements*. For example, *proposition 2* from Book VI of the *Elements* is referred to explicitly in Heath's translation of the text. Finally, Archimedes draws on results he himself has shown in previous treatises. Notable for our purpose is his works *Quadrature of the Parabola*, where Archimedes gives the proof of *Propositon 1* from *The Method*, and *On the Equilibrium of Planes*.

Archimedes work *On the Equilibrium of Planes* brings us to the second type of epistemic techniques we identify in Archimedes' workplace: *The mechanical techniques*. Mechanical techniques are, as the name suggests, techniques that rest on mechanics rather than geometry. While geometry is concerned with the *form* of a body, and not its *matter*, one needs the notion of weight, which relates to the *matter* of a material body when discussing balance and equilibrium. In *The Method* Archimedes is "balancing cross sections of a given figure against corresponding cross sections of a known figure, using the law of the lever"(Katz, 1998, p. 111). As seen Archimedes noted in the preface of *The Method* that a mechanical demonstration do not furnish an actual proof. Therefore it is evident that Archimedes distinguishes between geometrical and mechanical methods, and thus we will do the same here. Methods associated to the law of the lever will also be seen as *mechanical epistemic techniques*, i.e. results proved in *On the Equilibrium of Planes*.

The principle of the law of the lever was well known before Archimedes, but as a physical problem rather than mathematical. From the mathematical model that Archimedes provided, it was possible to derive a mathematical proof, therefore Archimedes is credited for proving the law of the lever (Katz, 1998, p. 103). Of course, in the idealised model, the complicated aspects of the real world are ignored. For instance, Archimedes assumes that the lever was rigid and without a weight, and the fulcrum and endpoints (weights) of the lever were considered as mathematical points, and thus 'has no part'[1]Katz (1998).

Even though we here consider arguments from *On the Equilibrium of Planes* as mechanical rather than geometric, the treatise is actually structured according to the standards of Greek geometry, with postulates which are assumed, followed by propositions which are proved with geometric methods. We wish to note that under the idealised assumptions in *On the Equilibrium of Planes*, the mathematics is actually rigorous according to Greek standards.

Two propositions lead up to the *law of the lever*, which can be concatenated as such:

Proposition 6, 7 (On the Equilibrium of Planes)

> Two magnitudes, whether commensurable [Prop 6] or incommensurable [Prop. 7], balance at distances reciprocally proportional to the magnitudes (Katz, 1998, p. 106)

In the rest of the treatise, the law of the lever is applied to find the centre of gravity of various geometrical figures (Katz, 1998, p. 107).

As seen Archimedes was quite aware that the titular method he provided could not be seen as a rigorous proof. However he does not state explicitly where the issues arise, this discussion is out of the scope of this thesis, and we follow the convention presented in Katz (1998), namely that it is because "neither mechanical principles nor "indivisible" cross sections could appear in a formal mathematical argument"(Katz, 1998, p. 111).

This leads to the third and last type of the identified epistemic techniques: the *method of indivisibles*. Indivisibles are geometric objects, which can be described as the theoretical entities thought to be the smallest possible units that a geometric object is composed of. It is based on the Pythagoeran believe that numbers (i.e. positive integers) formed the basis of the universe. Among others, Aristotle (384–322 B.C.E.) rejected the notion of indivisibles, because he had concerns about the idea of treating continuous quantities as being composed of indivisible units(Katz, 1998). Indivisibles were used in Ancient Greece, but was not accepted as rigorous at the time (Katz, 1998). They appear in *Proposition 1* when Archimedes argues that a geometric object can be 'made up' of objects one dimension lower,

[1]A point is that which has no part Eukl. I definition 1

e.g. a triangle is made up of lines, and a sphere of circles, and so on. This corresponds to accepting the premise that a triangle can be divided into infinitely many indivisible lines, which will make up the whole.

In summary, we find that in Archimedes workplace the epistemic techniques available can be divided into three categories; geometry, mechanics, and indivisibles. The geometric techniques are associated with a high level of rigour, while the mechanical methods are considered less rigorous, and the method of indivisibles is not considered rigorous at all. On a final remark, because Archimedes (as argued) is aware of the lack of rigour, he uses a different method - the method of exhaustion - when proving his findings with "an actual demonstration"(Heath, 1912). However, this method is not relevant in the specific research episode we are investigating, and out of the scope of this thesis to account for.

8.2.2 Historical Praxeologies in *Proposition 1*

Now that we have established Archimedes' workplace in relation to the research episode of developing *The Method*, we wish to identify praxologies in the argument for *Proposition 1*, from the point of view of Archimedes in his workplace.

In Archimedes' text we read the following formulation of *Proposition 1*:

Proposition 1. (The Method)

> Let ABC be a segment of a parabola bounded by the straight line AC and the parabola ABC, and let D be the middle point of AC. Draw the straight line DBE parallel to the axis of the parabola and join AB, BC.
> Then shall the segment ABC be $\frac{4}{3}$ of the triangle ABC.

With the result in *Proposition 1*, Archimedes was able to determine the area of a parabolic segment as $\frac{4}{3}$ of the inscribed area.

The three types of epistemic techniques identified in Section 8.2.1 give rise to three distinct regional MO's. The *established mathemtical techniques* all stem from the theory of *geometry*, Θ_{geom}. The *mechanical techniques* are rooted in the theory of mechanics, Θ_{mech}, and praxeologies that concern the method of indivisibles, we categorise as a theory on its own, Θ_{indi}. We refer to the praxeologies belonging to each of the regional MO's as *geometric*, *mechanical* and *indivisible* praxeologies, depending on the particular MO.

It is debatable whether the use of indivisibles should be categorised as its own theory, but we decided to do so, because it does not really fit with either geometry or mechanics. The 'jump' of dimension that occurs when adding indivisibles to construct a plane figure is not geometrically founded, while

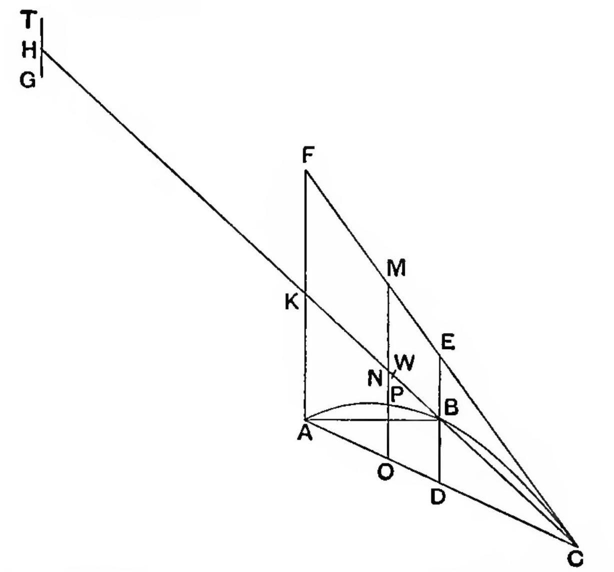

Figure 8.2: An illustration of the construction in *Proposition 1*(Heath, 1912)

lines with no breadth are still 'just' a mathematical concept and does not exist in the real world, and thus it is also at odds with mechanics.

In the argument for *Proposition 1*, we have identified praxeologies from all three regional MO's. In accordance to our proposed way of conducting a theoretical historical analysis with the intend to implement this in a teaching sequence we find it essential to make sure that the students encounter passages from the text related to the three regional MO's, since they all make out the reasoning in the text, and all shed a light on different levels of rigour. Below, we wish to provide some examples from the text.

The first passage in Archimedes' argument for *Proposition 1* is mere construction, which can be illustrated as in figure 8.2. This passage is:

> From *A* draw *AKF* parallel to *DE*, and let the tangent to the parabola at *C* meet *DBE* in *E* and *AKF* in *F*. Produce *CB* to meet *AF* in *K*, and again produce *CK* to *H*, making *KH* equal to *CK*.
> Consider *CH* as the bar of balance, *K* being its middle point.
> Let *MO* be any straight line parallel to *ED*, and let it meat *CF*, *CK*, *AC*, in *M*, *N*, *O* and the curve in *P*.

In this passage, Archimedes' adds lines to the figure, which has already been constructed in the statement of the proposition. Most of the construction consists of geometric praxeologies, with types of task such as "given a line and a point, draw a parallel line", "draw a tangent to meet a line in a certain point", and so on. We will not go into details with all of them, as they appear quite similar, but we will provide one example shown in Table 8.1, note that the Greek letters denoting the lines and points are added by us in order to better explain the associated techniques.

Type of Task	Given a line α and a point β not on α, draw a parallel line γ through β.
Techniques	Locate the given point and line; mark a point on the line; connect the points; draw matching arcs; measure distance with compass; copy distance; draw arcs with compass and mark intersection; connect new points
Technology	Discourse about how geometric objects can be constructed with straightedge and compass
Theory	Geometry

Table 8.1: The geometric praxeology of drawing parallel lines, identifying type of task, techniques and technology

The technique in table 8.1 is actually a set of techniques $P(\tau_i)$. The construction of parallel lines with straightedge and compass is just an example of how Archimedes might have done the construction with Euclidean geometry, as we new he had available (cf. Section 8.2.1). In Table 8.1, the collection of techniques is sketched very loosely, as it is out of the scope of this thesis to delve into the details. It is included to point out the fact that, most of the praxeologies we identify in relation to the construction may have some collection of techniques rooted in the same technology, namely the discourse about constructing geometric objects with straightedge and compass. This technology is of course rooted in Greek geometry. Thus, we identify a local (geometric) MO with regards to construction with straightedge and compass.

Though the praxeologies observed in relation to the passage above are mainly geometric, one line in this passage stands out as rooted in mechanics, i.e.: "Consider CH as the bar of balance, K being its middle point". The imperative word "consider" implies a more passive type of task compared to the imperative word "draw", considered in the previous example. While "draw" promotes action, "consider" simply prompts contemplation. In this way, the associated technique is closely related to the task. The object under consideration is a line, which no doubt is a geometric object, but Archimedes prompts the reader to assign a mechanical property to the line, making it a type of task belonging to a mechanical praxeology. Already in the construction we see mechanical traits that compromises the rigour of the argument, at least in Greek standards; The notion of balance is related to traits of the real world, and one may be concerned about the mathematical foundation that underlies imposing such a property on a line.

The mechanical praxeologies appear throughout the text, mostly related to using the bar of balanced established in the construction as a lever. As an example, consider the following passage from the argument:

> Take a straight line *TG* equal to *OP*, and place it with its centre of gravity at *H*, so that *TH*=*HG*; then, since *N* is the centre of gravity of the straight line *MO*, and
>
> $$MO : TG = HK : KN$$
>
> it follows that *TG* at *H* and *MO* at *N* will be in equilibrium about *K*.
>
> [*On the Equilibrium of Planes*, I. 6,7]

Before this passage, Archimedes establishes proportional relationships of different lines in the figure, which we will not account for here. Instead, we investigate the task of establishing equilibrium between *TG* at *H* and *MO* at *N* about *K*. Which can be presented as in the table 8.2 below.

Type of Task	Establish equilibrium between *TG* at *H* and *MO* at *N* about *K*.
Techniques	Application of the law of the lever
Technology	Discourse from *On the Equilibrium of Planes*
Theory	Mechanics

Table 8.2: The mechanic praxeology of establishing equilibrium

This is a straight-forward use of the law of the lever. When it has been established that the ratio of the line segment *MO* to *TG* is equal to the ratio of the line *HK* to *KN*, it is inferred that *Proposition 6,7* from Archimedes' *On the Equilibrium of Planes* (cf. Section 8.2.1) applies, and in extension that *MO* and *TG* are in equilibrium with *K* at the centre of gravity. Here, we can raise some questions about the rigour. *Proposition 6,7* deals with magnitudes that are proportionally reciprocal to their distances to a point between them. The intuition here is real-world based, and intuitively when we speak of balance, it is highly related to weight. The objects under consideration are not real-world objects, but geometric lines, that one cannot assign a weight within the scope of Greek geometry. Out of respect to Archimedes, we repeat that he never claimed the method was meant to provide rigorous proofs.

Finally, we turn to Archimedes' use of indivisibles in the argument for *Proposition 1*. Above, we discussed how Archimedes uses mechanical notions to establish the relationship between lines. In particular, he establishes the relationship between the line *TG* and the line *MO*. Before this, *MO* was chosen as an arbitrary section of the parabola with the only requirement that it is parallel to the line *ED*, which is parallel to the axis of the parabola.[2] *TG* was chosen to be equal to the part of *MO* on the

[2]"Let *MO* be any straight line parallel to *ED*"

parabolic segment, OP.[3] Thus, we can say that MO could be chosen as any section of the triangle AFC parallel to the axis, and any cross section of the segment ABC parallel to the axis TG corresponds to a choice of MO, and for any such choice of MO, it holds that $MO : TG = HK : KN$. Here, we are comparing lines, which are one-dimensional sections of two-dimensional geometric objects. The next step, however, is where the method of indivisibles is in particular in play:

> And, since the triangle *CFA* is made up of all the parallel lines like *MO*, and the segment *CBA* is made up of all the straight lines like *PO* within the curve,
> it follows that the triangle, placed where it is in the figure, is in equilibrium about *K* with the segment *CBA* placed with its center of gravity at *H*.

What happens here? First, we want to note that Archimedes does not actually use the word *indivisible*, but it is implied when he states that "it follows" from properties of the lines that the same properties hold for triangle AFC and the parabolic segment ABC.

We identify the process of transferring information between dimensions as a type of task in an indivisible praxeology.

Type of Task	Apply properties for one-dimensional objects to two-dimensional objects
Techniques	Consider a two-dimensional object as "made up" of one-dimensional objects
Technology	Intuitive understanding of infinitely small divisions of a two-dimensional figure
Theory	Method of indivisibles

Table 8.3: The indivisible praxeology

As we have already encountered in the first of the mechanical praxeologies we discussed, we here see the imperative word *Consider* again, however this time placed as part of the technique. Once again the distinction between task and technique is not easily drawn, but we aim to point out that the type of task here is to apply the results that have yet been found on the lines to the two-dimensional figure they "make up". Here, the quotation-marks are very deliberate, as the related technology is not so clear. To accept the technique, one must accept that a geometric object of the plane consists of lines, even though lines in Greek geometry (as well as modern geometry) does not have any breadth. As accounted for it was generally accepted that the use of indivisibles was not accepted in rigorous mathematics, and thus the discourse surrounding the technique is highly intuitive.

We have now pointed out different regional MO's in *Proposition 1* and how they reflect different

[3] "Take any straight line TG equal to OP"

levels of rigour in Archimedes' argument. As a final note before we continue to investigate Newton, we want to repeat that Archimedes never actually claimed that the argument was rigorous, but encouraged this more intuitive approach in the process of mathematical discovery.

8.3 Newton

Isaac Newton (1643-1727) was an English scientist whose knowledge spanned a substantial amount of subjects, which among others included mathematics, physics and astronomy. His contribution to the scientific field in the 1600 were of great importance. Nowadays he is often mentioned along with the German mathematician Gottfried Wilhelm Leibniz (1646-1716), because it is generally regarded that they share the credit for developing infinitesimal calculus.

We chose the text *Rule 1* from *De analysi per aeguationes numero terminorum infinitas* (On Analysis by Equations with an infinite number of terms) because we expected it to be accessible for the students, both when it comes to the definitions and the structure, and when it comes to a discussion of possible discrepancies in the argumentation. For example, it can be discussed whether Newton divides by zero, which is breaking a rule we expect the students are familiar with (i.e., it is not possible/allowed do perform zero division).

8.3.1 Newton's Workplace

In Newton's work *On Analysis of Infinite Equations*, which he composed in 1969 based on ideas he developed during the years 1665-1666, he investigates properties of equations of infinitely many terms, including determining the area under the curves which such equations express. However, we focus on the research episode of investigating algorithms for determining the area under curves that can be expressed with a finite number of terms. This is addressed in the paper *On Analysis by Equations Unlimited in the Number of Terms*, as part of *On Analysis of Infinite Equations* and published in *The Mathematical Papers of Isaac Newton* (Newton, 1969).

Objects under Investigation

The objects under investigation in the work in which we find *Rule 1* are areas under curves that can be expressed by a finite number of terms. Newton divides these curves into three types, the simple curves, which are equations of a single term (e.g. $y = x^2$), curves compounded of simple curves (e.g. $y = x^2 + x^{\frac{3}{2}}$), and finally, equations for curves where "the value of y or any of its terms be more compounded than the foregoing"(Katz, 1998, pp. 211-213) (e.g. a hyperbola $y = \frac{a^2}{(b+x)}$).

Newton had a kinematic understanding of a curve, possibly inspired by Isaac Barrow (1630-1677), whose lectures Newton had previously attended (Lund, 2000, 68). Newton considered variables of an equation as a distance dependent on a constant increase of time, without ever giving a definition of time. He thought of curves as the trace left behind by *fluent* quantities changing at a certain rate or *fluxion*, which is dependant on time. Newton adopted the notation established by Descartes, using the last letters of the alphabet, such as $x, y, \ldots$, to denote fluent quantities. Additionally, he independently introduced dot notation for the fluxions of these fluents, represented as $\dot{x}, \dot{y}, \ldots$. It is out of the scope of this thesis to go further into details with the definitions of fluents and fluxions, but the notation introduced here will be useful later. Newton refers to the area under the curve as the *quantity* of the curve.

Available Knowledge

Now, we wish to investigate what knowledge Newton uses in the investigation of the research episode under consideration, i.e. the epistemic techniques he uses in his investigation of the epistemic object, which we clarified in the previous section.

Newton was well studied in Viète, Descartes and WallisLund (2000), but he also developed his own methods. Here, we identify three categories of epistemic techniques; those related to analytic geometry, those related to what we will here call *universal arithmetics*, and finally, the methods that Newton developed himself.

Analytic geometry, pioneered by Descartes and Fermat Katz (1998), is the study of geometry using algebraic methods. It includes describing geometric objects with equations, as we have already seen Newton do with curves. Analytic geometry also involved the study of coordinate systems, although not exactly as we know them today. Newton does not only use analytic geometry in the definition of the object under investigation - he also uses some properties of the curve that follows from the relationship given in the equation in his works. This will be pointed out in the praxeological analysis below (cf. table 8.4)

Now we consider Newton's use of universal arithmetics. The name of the category was actually chosen inspired by the name of his text *Universal Arithmetic*, where he lays forth techniques for algebraic manipulation. We chose to use this text as a reference to the arithmetic techniques in his workplace even though it was published by William Whiston in 1707, thus years after the research episode under consideration. It had been under preparation for around 20 years, so we presume it reflects Newton's arithmetic techniques around the time of *On Analysis by Infinite Equations*. The work *Universal Arithmetic* starts out with simple techniques for summation and multiplication, for example:

> Multiplication: Simple algebraic terms are multiplied by "drawing" numbers into numbers and variables into variables, and then setting the product positive if both factors be positive or both negative, and negative otherwise"(Katz, 1998, 610)

Newton never actually justifies the multiplication rule presented here, or any other of the algorithms presented in the text for that matter. This gives the impression that neither he or his audience speculated much about a rigorous establishment of these techniques for algebraic manipulations(Katz, 1998, 610), and Newton made use of the universal arithmetics in his proofs as techniques without need of justification.

We want to note that the work *Universal Arithmetic* also contains some content on analytic geometry. We decided to distinguish between arithmetics and analytic geometry because this workplace analysis serves as a starting point for our teaching design, and we find that the distinction can be useful in a teaching context for two reasons. Firstly, analytic geometry presents a clear contrast to Greek geometry, whereas the arithmetic of the different time periods is more comparable. Secondly, we assume that the simple algorithms in the arithmetics category are more accessible to the students.

Finally, we want to look at the methods Newton developed himself, which we considers as an independent category of epistemic techniques. In particular, we want to examine how Newton employed infinitesimals in his arguments. This is highly related to Newtons reasoning, since he justifies the rules he established by infinitesimals. As previously mentioned, Newton defined curves using equations, describing fluent quantities as dependent on a constant rate of change over time. In line with his view of change over time, he would define an "infinitely small" (or infinitesimal) period of time as a *moment*, often denoted o. A fluent quantity x with fluxion $\dot{x}$ it becomes $x + \dot{x}o$ after the moment o has passed (Katz, 1998). In this way equations that describe the relationship between two fluent quantities, x and y, will also describe the relationship between $x + \dot{x}o$ and $y + \dot{y}o$, "and so $x + \dot{x}o$ and $y + \dot{y}o$ may be substituted in place of the latter quantities, x and y, in the said equation" (quotation from Newton, (Katz, 1998, 511)).

The rigour of Newtons method is debatable mainly because of his use of infinitesimalsKatz (1998), however he himself did actually find his method to be valid, as he puts in when reflecting upon his method:

> to avoid the tedium of working out lengthy proofs by reductio ad absurdum, in the manner of the ancient geometers. ...I preferred to make the proofs of what follows depend on the ultimate sums and ratios of vanishing quantities [instead of the method of indivisibles]... For the same result is obtained by these as by the method of indivisibles, and we shall be on safer ground using principles that have been proved (Newton in Pourciau (2001)).

It has been discussed extensively over the years that Newton is dividing by zero. As late as 1959 Carl Benjamin Boyer wrote:

> The meanings of the terms ... ”prime and ultimate ratio” had not been clearly explained by Newton, his answers being equivalent to tautologies ... Such an interpretation of Newton’s

> meaning, which of course results in the ... indeterminate ratio $\frac{0}{0}$, is not unjustified (Boyer in Pourciau (2001)).

Regarding *Rule 1* it is exactly the passage: "If we now suppose $B\beta$ to be infinitely small, that is, o to be zero, v and y will be equal and terms multiplied by o will vanish (...)"(Newton, 1969) where Newton seem to be dividing by zero. This leaves us at an interesting crossroad - Newton himself did find his method rigorous, however others raised questions about it. If we go back to Gila Hanna's view of what argumentation is, i.e. the techniques one are using to persuade others that ones reasoning is valid, we find Newton was indeed persuasive. However in his reasoning for *Rule 1* he makes use of rules where the validity can be discussed, thus the validity of the conclusion can be discussed, cf. Section 7.1.

In summary we identify three categories of epistemic techniques within Newtons workplace regarding this research episode, which is: Analytic geometry, Universal arithmetic and the methods developed by Newton, e.g. equations described by fluents and fluxions.

8.3.2 Historical Praxeologies in *Rule 1*

Based on the three categories of epistemic techniques we idenitied above we identify three regional MO's with the theories $\Theta_{\text{an.ge}}$ for analytic geometry, $\Theta_{\text{un.ar}}$ for arithmetics, and finaly Θ_{Newt} for praxeologies that draws on Newtons own developed infinitesimal methods.

We begin by giving a brief account for the constructions Newton made in the beginning of the proof for *Rule 1* and subsequently, we will examine praxeologies within each of the regional MO's.

In *Rule 1* Newton presents an algorithm for determining the area under a *simple* curve, Figure 8.3 shows an illustration of the curve that accompanies *Rule 1*. *Rule 1* is phrased as follows:

> To the base AB of some curve AD let the ordinate BD be perpendicular and let AB be called x and BD y. Let again $a, b, c, \ldots$ be given quantities and m, n integers. Then
> Rule 1. If $ax^{m/n} = y$, then will $\frac{na}{m+n}x^{\frac{m+n}{n}}$ equal the area ABD.

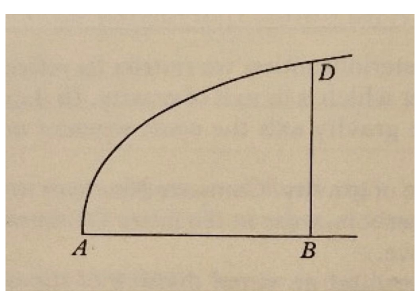

Figure 8.3: Illustration of simple curve from Newton's *Rule 1*(Newton, 1969, p. 206)

After a presentation of *Rule 1*, Newton continues to present similar results on more advanced curves, and provide examples of the applications of his algorithms. This shows that the main motivation of the text is to provide a useful technique for determining areas under curves, rather than to justify their validity. He even includes tables of computations that could assist the reader with interest in trying to apply some of the techniques themselves. Only at the very end, Newton makes considerations on the justification of his techniques: "As I look back, two points stand out above all others as needing proof"**cite243**. The first point he is referring to in this quote is the proof of *Rule 1*, which is given in two parts. First, Newton proofs the Rule for a specific (but arbitrarily chosen) simple curve, namely;

$$\frac{2}{3}x^{\frac{3}{2}} = z,$$

and second, he proof the rule in the general case.

The curve under investigation is illustrated in figure 8.4.

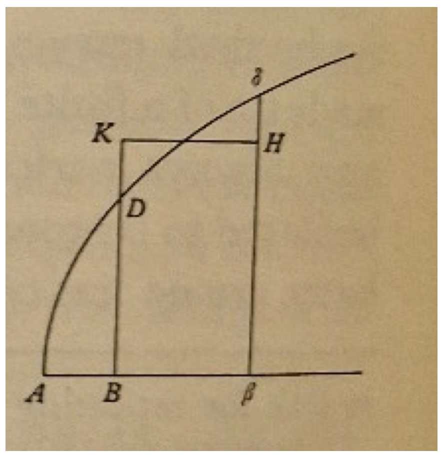

Figure 8.4: Illustration of curve considered in the proof of *Rule 1*(Newton, 1969, p. 242)

Newton renames different parts of the figure as follows:

> Let then any curve $AD\delta$ have base $AB = x$, perpendicular ordinate $BD = y$ and area $ABD = z$, as before. Likewise take $B\beta = o$, $BK = v$ and the rectangle $B\beta HK(ov)$ equal to the space $B\beta\delta D$. It is, therefore, $A\beta = x + o$ and $A\delta\beta = z + 0v$.

By renaming the various components as described, Newton utilises Descartes' symbolism (Katz, 1998, 438). The task of introducing the notation is implied in Newton's phrasing and draws on $\Theta_{\text{an.ge}}$ noting, however, that the use of the letter o actually draws on Θ_{Newt}, but at this point in the text, is not yet considered an infinitesimal increment. We also find praxeologies from the regional MO with theory analytic geometry in the following passage:

> Take at will $\frac{2}{3}x^{\frac{3}{2}} = z$ or $\frac{4}{9}x^3 = z^2$. Then, when $x + o$ is substituted for x and $z + ov$ for z, there arises (by the nature of the curve)

$$\frac{4}{9}(x^3 + 3x^2o + 3xo^2 + o^3) = z^2 + 2zov + o^2v^2$$

In the equality that concludes this passage, we identify two separate types of tasks. First, when Newton refers to *the nature of the curve*, he is referring to the specific algebraic relationship established between x and z. We identify a type of task of translating this relationship to the increments o and ov. This type of task draws on $\Theta_{\text{an.ge}}$. Second, we identify a task of expanding the equation. This task is solved by a set of arithmetic techniques, justified by $\Theta_{\text{un.ar}}$. Table 8.4 and 8.5 provides details of the two praxeologies within two different theories. The implied conclusion that the expanded equation still holds true is again justified by $\Theta_{\text{an.ge}}$.

Type of Task	Translate the relationship between x and z to $x + o$ and $z + ov$
Techniques	Substitute $x + o$ in place of x, and $z + ov$ in place of z.
Technology	Discourse about the nature of curves described by equations
Theory	Analytic geometry

Table 8.4: Analytic Geometry Praxeology

In table 8.4, the discourse about the nature of the curve is rooted in the acceptance that there exists a v and a δ that satisfy that the rectangle $ov = B\beta\delta D$.

Type of Task	Expand the equation $\frac{4}{9}(x + o)^3 = (z + ov)^2$
Techniques	Apply arithmetic algorithms to the left side and the right side of the equation, respectively.
Technology	Discourse about the distributive property of multiplication over addition and raising quantities to 2^{nd} and 3^{rd} powers
Theory	Universal arithmetics

Table 8.5: Universal Arithmetics Praxeology

In table 8.5, the set of techniques could be split into techniques for expanding the left side, and techniques for expanding the right side. A possible technique for expanding the right side could be to apply the rules for the square of a binomial, $(a + b)^2 = a^2 + 2ab + b^2$. In this case, we get:

$$(z + ov)^2 = z^2 + 2zov + o^2v^2 \tag{8.1}$$

by direct application, just as Newton does in the included passage. The arithemtics of the left hand side are similar, though a bit more extensive and we will omit the details here.

Newton continues to simplify the expression in equation 8.1. The relation $\frac{4}{9}x^3 \; =^2$ allows him to remove equal terms, which results in o being a common factor in all terms. After dividing by o, "there remains $\frac{4}{9}(3x^2 + 3xo + o^2) = 2zv + ov^2$"***cite***, and from this expression Newton continues:

> If we now suppose $B\beta$ to be infinitely small, that is, o to be zero, v and y will be equal and terms multiplied by o will vanish and there will consequently remain $\frac{4}{9} \times 3x^2 = 2zv$ or $\frac{2}{3}x^2 (= zy) = \frac{2}{3}x^{\frac{3}{2}}y$, that is, $x^{\frac{1}{2}} (= x^2/x^{\frac{3}{2}}) = y$.***cite***

Here we can identify a praxeology drawing on Θ_{Newt}, which can be schematised as in table 8.6.

Type of Task	Simplify equation with *moments*
Techniques	Remove terms multiplied by o
Technology	Discourse about *moments*, including an understanding of "infinitely small" as to be 0.
Theory	Newton's infinitesimal method

Table 8.6: Universal Arithmetics Praxeology

We made the choice to phrase the type of task of the praxeology presented in table 8.6 as "Simplify equation with *moments*", to reflect that the type of task in its essence resembles the universal arithmetics of simplifying an equation. However, the introduction of moments is not justified with $\Theta_{\text{un.ar}}$, but with Θ_{Newt}. When Newton finds that he can remove terms multiplied by o, it is because he explicitly considers "infinitely small" to be zero. One might now have issues with the comparison, if not for the fact that prior to this, Newton simplified the equation by dividing all terms with o, which was not problematic as o at that point was not considered as equal to zero, but Newton fails to justify that the value of o can be considered in different ways at different places in the argument.

As in the workplace of Archimedes, we have found that Newton has developed praxeologies from the three different regional MO's we identified, and also here they reflect different levels of rigour. However, while Archimedes was explicit about issues with his methods, Newton shows no remorse in the use of infinitesimals in his argument for *Rule 1*, but explicitly states that it is in fact a proof.
When Newton speaks of limits, he does not have the same well-defined conception as has been developed today, and which we will discuss below. However, it seems that Newton did have enough intuition about the concept to at least have convinced himself of the merits of his method.

8.4 Area Determination in Contemporary Mathematics

Now, we move on to contemporary scholarly knowledge on area determination, while still focusing on rigour and reasoning. This is part of a loose praxeological reference model, which will guide the design and a priori analysis of our teaching sequence.

Before we delve into area determination, we remind the reader that we already discussed rigour and reasoning in Section 7.1.

Simply put, the word *area* refers to the measurement of the space inside a two-dimensional shape. As we are working with rigour and reasoning in relation to area determination we have chosen to focus our study of area determination on a scholarly level on the *Riemann* integral. From the previous section it is evident that we can see the development of integral calculus in the 19^{th} century in connection to area determination of a parabolic segment in Ancient Greece and in connection to area determination of simple curves in the 16^{th} century analytic geometry. In modern mathematics the Riemann integral is a common tool for determining the area under a curve, thus it is related to the subject of area determination. In this chapter we will therefore study rigour and reasoning regarding the Riemann integral on a scholarly level, i.e. how the argumentation is build and what makes this rigorous according to modern mathematics.

The background literature for our study of scholarly knowledge is the book *Indledende matematisk analyse* used in the first year course *Analyse 0* at the University of Copenhagen.

When consulting the index of *Indledende matematisk analyse* in search for the word *area*,[4] one is pointed to one of the later chapters (chapter 9), which begins at page 351. This chapter concerns integrals of two and three dimensions. However the formal definition of an area is first stated on page 359. This indicates that on a scholarly level we do not have the privilege to just intuitively talk about an area if we want to do it rigorously. Therefore we will start this section of kind of backwards and consult this definition first in order to shed light on how much is needed to be established in order to talk about an area at a scholarly level. Afterwards we will cover and establish basic definitions and theorems that we need prior to this in order to close of with an discussion of the rigour and reasoning within this field of mathematics on a scholarly level.

The formal definition of the *area* of a *figure* is stated in the textbook as:

Definition 1. *The* area *of a figure* $A \subset \mathbb{R}^2$ *is given by*

$$\operatorname{area}(A) = \left\{ \sum_{i=1}^{n} \operatorname{area}(R_i) \mid R_1, \ldots, R_2 \textit{ are essentially disjoint rectangles contained in } A \right\} \tag{8.2}$$

First, we note that definition 1 is given without any references to integral calculus, even though this has been accounted for in chapter 5 of the textbook. In fact, the connection of the area under a curve and the Riemann integral is only made after the formal definition of the area of a figure - as an example. This is stated as follows:

[4] DA: areal

Example 2. *Let* $f : [a, b] \to \mathbb{R}^2$ *be continuous, and let* $f \geq 0 \forall x \in [a, b]$*. Consider the region under the curve given by*

$$A = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq f(x) \right\} \tag{8.3}$$

Then A *is a figure and*

$$\text{area}(A) = \int_a^b f(x) dx \tag{8.4}$$

Of course, the result in the example is expected, since considerations on determining the area under the curve was what inspired the definition of the Riemann Integral in the first place (Eilers et al., 2018, 360).

In order to fully comprehend Definition 1 and the result in Example 2 on a scholarly level, a comprehensive understanding of the regional MO of real analysis is required. Before giving Definition 1, formal definitions must be given of what a figure and a rectangle are, and what it means for the rectangles to be essentially disjoint. These definitions fall within the regional MO of real analysis, which includes the necessary praxeologies for identifying and working with basic geometric shapes. One must also be familiar with the formal definitions of curves and boundary points, which encompasses praxeologies associated with understanding the properties of curves and the behaviour of boundary points, which also falls within the scope of real analysis. Further, one must have an understanding of basic set theory. The regional MO of real analysis builds on set theory, with praxeologies on operations with sets and set relations, which are associated with Definition 1 and Example 2.

To study rigour and reasoning, we need to consider the definition of an area, how the definition relates to mathematical rigour and the reasoning behind this relation. The argument for the result in Example 2 forms part of the technology associated with determining areas under a curve, and the result provides techniques for solving types of tasks related to determining the area under a curve. A formal proof of Example 2 is not given, but the result is still well argued, and the argument draws on all the praxeologies we point to above, and more. This included praxeologies relating to the notion of functions, limits and continuity, Riemann integrals, properties of $\mathbb{R}$ and more, all in the scope of the regional MO of real analysis. In the following sections we will delve into how some of these praxeologies are established with a high level of rigour and formal argumentation. Regarding basic set theory we loosely define a *set* as a collection of related objects. We view that basic set theory such as complements, unions and intersections, as well as the use of logical quantifiers, have broad interpretations, thus we consider this rigorous from the get go. For this reason we will not delve into detail here, and it is assumed that the reader is familiar with these notions.

Regarding notions from Topology we assume that the reader is familiar with the definition of the interior of A, denoted $int(A)$, a contact point, and if all contact point are contained in A we say that A is closed, boundary points and lastly a bounded set, which is either upward or downward bounded.

8.4.1 Limits and continuity

In order to establish the definition of continuity we need to establish the notion of limits and in order to establish the notion of limits we need to include the definition of a map as it is stated in *Analyse 0*. A *map* is defined as:

Definition 3 (map)**.** *A* map *f is an operation between two sets X and Y, which associates exactly on $y \in Y$ to each $x \in X$. We refer to the set X as the domain of the map f, and to Y as the codomain, and write $f : X \to Y$*

For our purposes later on when we consider the Riemann integral, we define a *real function* to be a map with codomain $\mathbb{R}$. Until then we consider consider more general maps as defined in definition 3.

Limits

One of the most essentiel definitions in modern analysis - and a more recent addition - is that of a *limit point*. Which is defined as:

Definition 4 (limit point)**.** *Let a set $A \subset \mathbb{R}^k$, a map $f : A \to \mathbb{R}^m$ and points $a \in \bar{A} \setminus A$ and $b \in \mathbb{R}^m$ be given. We say that f has b as* limit point *as x approaches a, $x \in A$, if*

$$\forall \varepsilon > 0 \exists \delta > 0 : \quad \|f(x) - b\| < \varepsilon \text{ for all } x \in A \text{ with } \|x - a\| < \delta.$$

We write

$$f(x) \to b \quad \text{for } x \to a, x \in A, \quad \text{or} \quad \lim_{x \to a} = b$$

In modern mathematics, limits are typically introduced with an $\varepsilon - \delta$-definition, as in definition 4. With epsilon-delta arguments, mathematicians have established a solid foundation for mathematical analysis. This methodology is an essential example of rigour in mathematical reasoning on a scholarly level. For this reason, we will now present a few important limit theorems. Here, we also lay out an example of a direct proof to showcase the standars of rigour and reasoning in a contemporary mathematical argument.

Theorem 5 (uniqueness of limit points)**.** *Let a set $A \subset \mathbb{R}^k$, a map $f : A \to \mathbb{R}^m$ and points $a \in \bar{A} \setminus A$ and $b \in \mathbb{R}^m$ be given. Then f has at most one limits point as x approaches a, $x \in A$.*

In order to prove this theorem we need two central results. Thus, we permit ourselves to use the following two results without proof:

Lemma 6. *Let $x \in \mathbb{R}^k$. Then it holds that*

$$\forall \varepsilon > 0 : \|x\| \leq \varepsilon \Rightarrow x = 0 \tag{8.5}$$

Theorem 7 (triangle inequality for vectors)**.** *Let $x, y \in \mathbb{R}^k$. Then*

$$\|x + y\| \leq \|x\| + \|y\|$$

Proof (Theorem 5). Assume $f(x) \to b'$ and $f(x) \to b''$, for $x \to a$, $a \in A$.
We want to show that $b' = b''$.
Let $\varepsilon > 0$ be given. It follows from definition 4, since b', b'' are limit points, $\exists \delta', \delta''$ such that

$$\|f(x) - b'\| \leq \frac{\varepsilon}{2} \quad \forall x \in A \text{ with } \|x - a\| < \delta', \tag{8.6}$$

$$\text{and} \quad \|f(x) - b''\| \leq \frac{\varepsilon}{2} \quad \forall x \in A \text{ with } \|x - a\| < \delta'' \tag{8.7}$$

Now, let $\delta = \min\{\delta', \delta''\}$. Since a is a point of contact for A, there exists $x_0 \in A$ such that $x_0 \in K(a, \delta)$. Now we have $\|x_0 - a\| < \delta \leq \delta'$, hence $\|f(x_0) - b'\| \leq \frac{\varepsilon}{2}$. Likewise, $\|f(x_0) - b''\| \leq \frac{\varepsilon}{2}$. Thus, applying Theorem 7, we have

$$\begin{aligned} \|b' - b''\| &= \|b' - f(x_0) + f(x_0) - b''\| \\ &\leq \|b' - f(x_0)\| + \|f(x_0) - b''\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Now, since ε was chosen arbitrarily, it follows from Lemma 6 that $b' - b'' = 0$, that is, $b' = b''$, which is what we wanted to show.

□

The proof of theorem 5 is a good example of how an $\varepsilon - \delta$-argument usually looks in a direct proof. First, we establish some assumptions we want to make, including that an arbitraty ε has been given. Then one lays out previously established knowledge and definitions. In this, case, definition 4 is useful, because we have assumed that b and b' are limit points, and we want to be able to say something about them. In (8.6) and (8.7), the information we need from the definition is clarified. Then, we are ready to define a δ which will always be smaller than any given ε.
The epsilon-delta argument builds on Newton's methods, by formalising the concept of a limit. Newton's infinitesimals were fundamental to the development of calculus. The epsilon-delta definition, further developed by Cauchy and Weierstrass, provides a precise and rigorous way to describe limits, drawing on Newton's intuitive concepts of continuity and rates of changeKatz (1998). As such, we can think of the contemporary notion of limits as Newton's ideas in a more formalised setting.

Continuity

Often, the limit point of a function as x approaches some point a corresponds to the function value $f(a)$ (this is not always the case). When this is the case, we say that f is continuous. In the following, we will formalize this intuition.

Definition 8 (pointwise continuity)**.** *Let some point $a \in A$, $A \subset \mathbb{R}^k$ be given. We say that a map $f : A \to \mathbb{R}^m$ is continuous in a if*

$$\forall \varepsilon > 0 \exists \delta > 0 : \quad \|f(x) - f(a)\| < \varepsilon \textit{ for all } x \in A \textit{ with } \|x - a\| < \delta$$

In this case, we say that a is a point of continuity *for f. If a is not a point of continuity, we say that a is a* point of discontinuity *for f.*

A map is continuous if it is continuous in all points of its domain:

Definition 9 (continuity)**.** *We say that a map* $f : A \to \mathbb{R}^m$, $A \subset \mathbb{R}^k$ *is* continuous *if it is continuous in every point* $a \in A$.

One might notice that the definitions and theorems above are considered in respect to the real numbers, however this is not a necessary condition for the validity of definitions and theorems. But as stated in the *Indledende matematisk analyse*: "The main theorems about continuous functions, and with them the entirety of differential and integral calculus, break down unless one takes the real numbers as the foundation."[5], hence with the purpose of our study of scholarly knowledge in the present thesis, there are some important properties of the real numbers, which we need to state in order to consider integral calculus the Riemann integral. We will not delve into the construction of $\mathbb{R}$ here as this is out of the scope of this thesis as well as not actually taught in the *Analyse 0* course. Therefore we will account for an important property, namely the least-upper-bound-property.

We start by introducing an important formal definition an *upper bound*:

Definition 10 (upper bound)**.** *Let* $A \subset \mathbb{R}$, $A \neq \emptyset$.
A number $b \in \mathbb{R}$ *is called an* upper bound *for* A *if* $a \leq b$ *for all* $a \in A$.
Further, a number $c \in \mathbb{R}$ *is called a* east upper bound

Of course, if a subset of $\mathbb{R}$ has an upper bound at all, it has infinitely many. In ordered sets such as $\mathbb{R}$, we can talk about some upper bounds being greater or lesser than others. Which gives rise to the definition of the *least upper bound*

Definition 11 (least upper bound)**.** *Let* $A \subset \mathbb{R}$, $A \neq \emptyset$.
A number $b \in \mathbb{R}$ *is called a* least upper bound *or* supremum *for* A *if* b *is an upper bound for* A*, and if any upper bound* c *of* A *satisfies* $b \leq c$.

In fact, what distinguishes $\mathbb{R}$ from other number sets is the least-upper-bound property, which is:

> **Least-upper-bound Property:** Let $A \subset \mathbb{R}$ be a non-empty, upwardly bounded set. Then a least upper bound for A exists.

Similarly, for any subset of $\mathbb{R}$, a greatest lower bound also exists - this is defined similarly as the least upper bound and thus details will be omitted.

The least upper bound property is an fundamental property of $\mathbb{R}$, and is essential, for instance, in

[5]DA: "hovedsætningerne om kontinuerte funktioner og med dem hele differential- og integralralregningen bryder sammen, medmindre man lægger de reelle tal til grund" (Eilers et al., 2018, p. 72)

demonstrating the Riemann integrability of any continuous function. Although the detailed proof is beyond the scope of this thesis, its mention underscores the substantial effort required in rigorous mathematical reasoning at a scholarly level.

8.4.2 Integral Calculus

We have now accounted for limits and continuity, as such we have put forth some of the foundational definitions and theorems we need in order to consider integral calculus. Before we can establish the *Riemann integral*, we need to establish some basic definitions, so that it is very clear what we mean when speaking of notions such as *partitions* and *Riemann sums*.

In the following, we assume that $[a, b] \subset \mathbb{R}$ is a closed and bounded interval. A *partition* is defined as:

Definition 12 (partition). *A finite sequence $x_0, \ldots, x_n$ satisfying*

$$D : a = x_0 < x_1 < \cdots < x_n = b$$

is called a Partition *D of $[a, b]$.*

We can consider the partition of $[a, b]$ as n subintervals of $[a, b]$. A sequence of n intermediate points $\xi_1, ..., \xi_n$ can be chosen such that $\xi_i \in [x_{i-1}, x_i]$. We say that a partition along with a choice of intermediate points $\xi_1, \ldots, \xi_n$ is a **tagged partition** $P(D, \xi)$. Further, we let $\Delta x_i = x_i - x_{i-1}$, the length of the i^{th} interval. Based on this definition of a partition, we can define the *norm* of the partition as:

Definition 13 (norm of partition). *Let D be a partition $[a, b]$, and let $\Delta x_i = x_i - x_{i-1}$, that is, the length of each sub interval in the partition.*
We define the norm *of the partition as*

$$\text{norm}(D) = \max\{\Delta x_i \mid i = 1, \ldots, n\}.$$

For large $\text{norm}(D)$*, we say that the partition is* coarse*, and for small* $\text{norm}(D)$*, we say that the partition is* fine*.*

Furthermore we define a *Riemann sum*:

Definition 14 (Riemann Sum). *Let a function $f : [a, b] \to \mathbb{R}$ be given, and let $P(D, \xi)$ be a tagged partition of $[a, b]$. We then define a* Riemann Sum *for f with respect to $P(D, \xi)$ as the sum*

$$M = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

The Riemann Integral

At this point, we are ready to define the Riemann integral. The Riemann integral works well, especially on bounded functions on closed, bounded intervals. At this point, we briefly want to note that even though the motivation for the integral was to determine the area under a curve, we do not actually address the definition of the area in the definition of the Riemann integral.

Definition 15 (Riemann integral)**.** *Let f be a real-valued function$f : [a, b] \to \mathbb{R}$. If there exists $I \in \mathbb{R}$ with the property that for all $\varepsilon > 0$ there exits $\delta > 0$ such that any partition of $[a, b]$ with $norm(D) < \delta$ and any choice of intermediate points, giving us a tagged partition $P(D, \xi)$, the Riemann Sum $M = \sum_{i=1}^{n} f(\xi_i)\delta x_i$ satisfies that*

$$\left| I - \sum_{i=1}^{n} f(\xi_i \delta x_i) \right| < \varepsilon,$$

we say that f is Riemann Integrable*, and we say that I is the* integral *of f on $[a, b]$, denoted as $\int_a^b f(x)dx$.*

Some theorems follow on applications of the Riemann integral. These are omitted here as the focus is on rigour and reasoning rather than application and computation, but we note that the applicability of Definition 15 is limited due to the fact that a value of I must be guessed or known. This conundrum can be avoided with the *Bolzano Property*, which is also beyond the scope of this thesis.

Fundamental Theorem of Calculus

At this point, we have only discussed the definite Riemann integral. Now, we wish to briefly touch upon the relationship between the definite integral and the anti-derivative because integrals are often motivated and introduced by being the 'opposite' of the anti-derivative. As we will see, the entire subject of integral calculus is introduced through the *Fundamental theorem of calculus* in upper secondary school (cf. Section 9). We find it appropriate to include this theorem in a scholarly context, because this is actually secondary to the establishment of integral calculus. This manner of introducing integral calculus has been highly transposed from scholarly knowledge to knowledge to be taught. We only do a brief account because it would be out of the scope of this thesis to treat the subject with great detail - as we are working with area determination in upper secondary school in a way that it should not be related to differential calculus.

The fundamental theorem of calculus is stated as follows:

Theorem 16 (fundamental Theorem of Calculus)**.** *Let a function $f : I \to \mathbb{R}$ be defined on some interval $I \subset \mathbb{R}$, and let $x_0 \in I$ be arbitrarily chosen.*
If f is continuous on I, then the function defined by

$$\Phi(x) = \int_{x_0}^{x} f(y)dy \quad \textit{for } x \in I$$

is differentiable in every point $x \in I$ with derivative $\Phi'(x) = f(x)$. In this case, we say that $\Phi(x)$ is the antiderivative *of f.*

With Theorem 16, we have finally established the relationship between the definite integral and the derivative.

In conclusion, we have seen that the standards of proofs and mathematical reasoning follow the axiomatic-deductive method. We have found that area determination on a scholarly level is closely related to the Riemann integral, and important theorems within integral calculus rely heavily on properties of $\mathbb{R}$ which were only developed in recent times. The introduction of the integral was initially independent of differential calculus, and is generally treated as such on a scholarly level. In particular, the final point here made is in opposition to how integral calculus is introduced in upper secondary school, as we will see in the next section.

Chapter 9

Knowledge to be Taught

Before we study the didactic transposition from scholarly knowledge, put forth in the previous chapter, to knowledge to be taught, we will give an account of what the official curriculum for STX-A level mathematics states should be taught regarding area determination, mathematical reasoning as well as the history of mathematics. After this we will account for why reasoning and rigour should be taught in upper secondary school mathematics according to our beliefs. We will then study the didactic transposition based on two textbooks in use in upper secondary school, referring back to the previous Section 8.4. We have chosen to base the study on the text book system that the test class used, *Plus STX A*, and on another text book system *Hvad er Matematik?* in order to broaden the perspective of the study.

9.1 Curriculum for STX A-level Mathematics in the Danish Upper Secondary School

Henceforth, we refer to the official common objectives of upper secondary school STX A-level mathematics, 2017, as the *curriculum* (Da: læreplan). The curriculum states that students must work reasoning within mathematics.

> The students should encounter mathematical theory continuously throughout the course of high school course, and they should independently work with various elements of mathematical reasoning[1][p. 22](Børne- og Undervisningsministeriet, 2022).

Because of our definition of rigour in Section 7.1 we argue that the statement above has some annotations to this term as it encourages students to work with mathematical reasoning. It is further noted in the official curriculum that it is important to point out to students exactly when mathematical reasoning is in focus, and that "In this way, students can achieve such familiarity with mathematical thinking that they will immediately distinguish between "what is known," "what is assumed," and "what one

[1] Da: "Eleverne skal møde den matematiske teori løbende gennem hele gymnasieforløbet, og de skal selvstændigt arbejde med forskellige elementer af matematisk ræsonnement"

wants to know" when problem-solving"[2](Børne- og Undervisningsministeriet, 2022, p. 22). Thus gain an understanding of the 'set of rules' one ought to play by when performing a rigorous argument in mathematics (cf. Section 7.1).

Area determination is mainly treated in relation to integral calculus, therefore we will account for this subject. It is encouraged that integral calculus is introduced through the notion of the antiderivative, and "one should draw upon students' experiences from differential calculus and discuss the determination of antiderivatives as the reverse process of differentiation, so that students can initially find reference in previously known material"[3][p. 19](Børne- og Undervisningsministeriet, 2022). Based on this statement in the official curriculum we should expect a strong transposition from scholarly knowledge to knowledge to be taught, because as accounted for the Riemann integral is introduced without reference to differential calculus on a scholarly level, cf. Section 8.4.2.

After the introduction to the anti-derivative, as well as associated rules, integral calculus is related to area determination:

> The relationship between area and antiderivative should be given special attention, and it would be natural to consider this in conjunction with supplementary material on deductive methods and proof[4](Børne- og Undervisningsministeriet, 2022, p. 19).

Regarding history of mathematics the official curriculum states:

> Whenever possible, especially in cohesive teaching sequences, mathematical historical sources should be included, encouraging investigative work that challenges and develops students' curiosity regarding the development, form, and use of mathematics[5](Børne- og Undervisningsministeriet, 2022, p. 23).

We see that using history to investigate form and use can be in line with an understanding of how mathematics is produced, as is also stated in the official curriculum in relation to proofs and reasoning:

> The students should gain knowledge that there is a difference between the way mathematical topics are presented in books and the way the theory associated with the topic

[2]Da: "derved kan eleverne opnå en sådan fortrolighed med matematisk tankegang, at de i en problembehandling umiddelbart vil skelne mellem "hvad man ved", "hvad man antager", og "hvad man ønsker at vide"

[3]Da: "man bør her trække på elevernes erfaringer fra differentialregningen og omtale bestemmelse af stamfunktioner som den omvendte proces a differentatioon, så eleverne som udgangspunkt kan finde reference i allerede kendt stof"

[4]Da: "Sammenhængen mellem areal og stamfunktion skal gives en særlig behandling, og det vil være naturligt at tænke dette sammen med det supplerende stof om deduktive metoder og bevisførelse"

[5]Da: "Der bør om muligt, specielt i sammenhængende forløb, indgå matematikhistoriske kilder, som lægger op til et undersøgende arbejde, der udfordrer og udvikler elevernes nysgerrighed med henblik på matematikkens udvikling, form og brug"

> originally emerged[6](Børne- og Undervisningsministeriet, 2022, p. 22).

Here, the treatment of mathematical rigour and reasoning in the context of history of mathematics is not suggested explicitly, but we see a great opportunity in doing so.

In conclusion students in upper secondary school with A-level mathematics should work with area determination in the light of integral calculus, the history of mathematics and mathematical rigour and reasoning.

9.2 Why Teach Reasoning and Rigour in Upper Secondary School?

From the previous section it is clear that the official curriculum states that the students should work with reasoning within mathematics. As we accounted for in Section 7.1 we regard reasoning and argumentation to be related to rigour, because mathematical argumentation is often associated with a high level of rigour. Thus, even though *rigour* is not used explicitly in the official curriculum, we find that students should acquire a certain level of rigour in their argumentation.

A practical reason why students need to learn about mathematical reasoning is that students are expected to present a mathematical argument in the oral examination in upper secondary school. Of course, this this reflects the incorporation of mathematical reasoning in the official curriculum. We will briefly discuss the merits of this incorporation.

Since ancient Greece, and still today, the discovery of new mathematics has been communicated through formal proofs. In the established didactic research, according to an entry in the *Encyclopedia of Mathematics Education*, there is a division of opinions on the usefulness of encouraging students to engage in more informal mathematical argumentation as a step to learning how a mathematical proof is constructed (Hanna, 2020, p. 564). Gila Hanna put forth two different opinions on this matter:

> Boero (in La lettre de la preuve 1999) and others see a great benefit in having students engage in conjecturing and argumentation as they develop an understanding of mathematical proof. Others take a quite different view, claiming that argumentation, because it aims only to establish plausibility, can never be more than a distraction from the task of teaching proof (e.g., Balacheff 1999; Duval – in La lettre de la preuve 1999)(Hanna, 2020, p. 564).

Even though there is an apparent division of opinions, there is a consensus that students should engage in conducting mathematical proofs. We agree that it can be beneficial for students to engage in formal as well as informal argumentation, and in this, investigate different levels of rigour in different

[6]Da: "Eleverne skal opnå viden om, at der er forskel på den måde, hvorpå matematiske emner fremstilles i bøger, og den måde, hvorpå teorien hørende til emnet oprindeligt er fremkommet"

types of arguments. This view is further supported in that, according to Hanna, most of the recent studies showed how students can benefit from: "(...) argumentation's openness of exploration and flexible validation rules as a prelude to the stricter uses of rules and symbols essential in constructing a mathematical proof" (Hanna, 2020, p. 564).

By placing students in an inquiry-reflective learning environment that makes use of SRP where they study the argumentation of past mathematician in order to discuss the rigour pertaining these texts we can create an environment that can strengthen students' comprehension of proofs in mathematics, as this can function as a prelude to conduct mathematical proofs later on in their mathematics education in upper secondary school.

9.3 Rigour and Reasoning According to the Textbooks in Use

In this section we investigate how *Plus STX A* and *Hvad er Matematik?* describe mathematical argumentation, reasoning and proofs. This is done to shed a light on how the notions of rigour and reasoning is transposed into *knowledge to be taught*. In both text book systems, mathematical reasoning is addressed in independent chapters.

The first book of the *Plus STX A-system*, *Plus A1*, addresses this subject in chapter 8, hence quite late in the book, almost as an after thought. The chapter begins with a brief historical background, to motivate the subject:

> Around 300 BC, the mathematician Euclid lived in ancient Greece. His work "Elements" is an important foundation for modern mathematics. It is, among other things, based on his work that mathematics is structured as a logical system of;
>
> - definitions
> - axioms
> - theorems[7]

As such the chapter takes point of departure in ancient Greece, where the axiomatic-deductive method of mathematics, which was developed, a method that is still common practice in modern mathematics. This is a common use of history in both textbooks used in our analysis.

The textbook "defines" definitions by their use; to "name and delineate new concepts."[8] It uses Euclids definition of a point as an example of a definition (i.e. "A point is that which has no part"[9]). As such,

[7]Da: "Ca. 300 f.Kr. levede matematikeren Euklid i det antikke Grækenland. Hans værk Elementerne er et vigtigt grundlag for den moderne matematik. Det er bl.a. på baggrund af hans arbejde, at matematikken er opbygget som et logisk system af definitioner, aksiomer, sætninger"(Madsen et al., 2024b)

[8]Da: "at sætte navn på og afgrænse nye begreber"(Madsen et al., 2024b)

[9]Da: "et punkt er det, der ikke kan deles"(Madsen et al., 2024b)

definitions in upper secondary school is simplified in a way which we could view as a 'labelling'.

Furthermore, axioms are introduced as rules that are assumed to be true, i.e. a rule that we do not need to prove in order to use them. The way of introducing axioms could lead to the apprehension of them as being merely 'rules for calculation'.

Theorems are defined through their structure and purpose with the statement that: "Theorems build upon definitions and axioms, and they provide us with new knowledge"[10].

Thus, in the text book, mathematics is structured as the axiomatic-deductive system developed in Ancient Greece, which is similar to the structure of mathematics on a scholarly level. We do, however, see a transposition of the historical content at stake. As accounted for in Section 8.1 Euclid (and Aristotle) actually distinguishes between axioms and postulates, which seems to fall under one category in upper secondary school. As such, it is evident that historical knowledge is simplified.

Further, *Plus STX A1* states that "Mathematical theorems must be proved. That is to say, based on axioms, definitions, and other theorems, we must be able to argue that the theorem is true."[11]. This statement hints the importance of proofs in mathematics, and proofs are compared to valid arguments. The validity of proofs is further elaborated by: "For a proof to be accepted as a valid mathematical proof, it must be logically rigorous and cover all conceivable cases."[12] This is in line with our conception of a proof as a formal argument that follows the axiomatic-deductive method.

In *Hvad er Matematik? C*, we see a different approach concerning the placement of the subject in the book. In this book there is also an independent chapter on this subject, but this time placed as chapter 0 - a kind of prelude (Bruun et al., 2017). It points to an understanding of mathematical reasoning as a necessary prerequisite to be able to address the rest of the book, compared to *Plus STX A1*, where the late appearance of the subject makes it seem more independent from the rest of the book system. Of course, it is up to the teacher to decide the order in which the chapters are covered.

Regarding proofs and mathematical reasoning, it is stated that:

> A mathematical statement that expresses that something always holds true - such as the sum of angles in any triangle being 180^o - is called a *mathematical theorem* in most countries. In Denmark, it is called a *matematisk sætning*. [...] A mathematical proposition is always associated with a *mathematical proof* of the proposition. A proof is one (or more) chains of

[10]Da: "Sætninger bygger videre på definitioner og aksiomer, og de giver os ny viden"(Madsen et al., 2024b)

[11]Da: "Matematiske sætninger skal bevises. Det vil sige, at vi med udgangspunkt i aksiomer, definitioner og andre sætninger skal kunne argumentere for, at sætningen er sand"(Madsen et al., 2024b)

[12]Da: "Hvis et bevis skal kunne accepteres som et gyldigt matematisk bevis, skal det være logisk stringent og dække alle tænkelige tilfælde"(Madsen et al., 2024b)

> arguments (reasonings), or series of rewrites of an expression, where we only use logical rules throughout, leading us from something we already know to the new assertion.[13]

Here, the definition of a theorem and the constituents of a mathematical proof for the theorem, such as logical rules and rewriting, are put forth in a slightly different manner. In this there is an inclusion of arguments as being constituents of a proof, and arguments are equated with "ræsonnementer", which is not easily translated to English, but is a countable form of the word 'reasoning'. In our conception of the notions, we would probably say that a 'ræsonnement' is a type of argument, e.g. a deductive argument. Overall, the two books convey the same ideas and both are based on axiomatic-deductive methods, but the terminology in *Plus STX A1* is more in line with ours.

Chapter 8 of *Plus STX A1* also introduces fives types of proofs: Proof by inspection (concerning finite sets), the pigeonhole principle, the direct proof, proof by contradiction, and proof by induction (Madsen et al., 2024b). In relation to analysis, the direct proof is the most relevant one, as we have also accounted for in Section 8.4. For this reason, we will put forth an example of a direct proof in the textbook *Plus STX A1* later on, in order to shed light on the rigour and reasoning in upper secondary school.

9.4 Area Determination in Upper Secondary School

In this section, we aim to follow the progression from Section 8.4. We start by considering how the notion of an area is treated in the text books used in upper secondary school. Afterwards we will account for the notion of limits and continuity as treated in the *Plus STX A*-system. Henceforth, this text book system will be our primary reference.

In Section 8.4 we assumed that the reader were familiar with notions from topology. These notions are not treated in upper secondary school. We also assumed familiarity with basic set theory, which is, in fact, a taught subject in upper secondary school, however highly transposed. As an example the real numbers are defined in *Plus STX A1* as follows:

Definition A (The real numbers)**.** *The real numbers consist of all numbers on the number line. They are denoted by the letter R. We will use the notation R_+ for the positive real numbers, i.e., all numbers greater than 0.*[14]

[13]Da: "En matematisk påstand, der udtrykker, at et eller andet altid gælder - som fx at vinkelsummen i enhver trekant er 180^o - kaldes i de fleste lande for et *matematisk teorem*. I Danmark kaldes det en *matematisk sætning*. [...] En matematisk sætning er altid knyttet sammen med et *matematisk bevis* for sætningen. Et bevis er en (eller flere) kæder af argumenter (ræsonnementer), eller rækker af omskrivninger af et udtryk, hvor vi hele vejen kun anvender logiske regler, og som fører os fra noget, vi ved i forvejen, til den nye påstand" (Bruun et al., 2017, p. 12)

[14]Da: "De reelle tal består af alle tal på tallinjen. De betegnes med bogstavet R. Vi vil bruge notationen R_+ om de positive reelle tal, dvs. alle tal større end 0"

The treatment of real numbers in upper secondary school is very limited and restricted to an intuitive definition. On a scholarly knowledge, we argued that without properties of the real numbers, in particular the least-upper-bound property, the foundation of integral calculus 'breaks down'. It is evident that the scholarly knowledge has been transposed in such a way in definition A that there is no real substance regarding the logos block. It is not clear from this definition how the real numbers differ from the rational numbers, which are defined as all fractions of integers. This is reflected in the way limits are introduced without fully justifying their existence (cf. Section 9.4.1). In regards to integral calculus there is no assumptions of the real numbers in order to apply the theorems in the text book.

When investigating how areas are treated in *Plus STX A3*, one is led to the definite integral, and part 2 of what the book calls *the main theorem of integral calculus*(Da: Integralregningens hovedsætning) - a theorem for calculating the area under a function:

Theorem B (Main theorem of integral calculus, part 2)**.** *Let f be a non-negative and continuous function in an interval $[a;b]$, and let F be an arbitrary antiderivative of f in this interval.*
The area of the region between the graph of f and the x-axis in the interval $[a,b]$ is now given by:

$$\text{Area} = F(b) - F(a).$$[15]

Type of Task	Calculate the area under a curve formed by a non-negative continuous function f over a closed interval $[a,b]$
Techniques	Find the anti-derivative F, insert the endpoints of the interval in F, calculate the difference $F(b) - F(a)$.
Technology	The discourse regarding the use of anti-derivatives to determine areas
Theory	Calculus

Table 9.1: Praxeology of determining areas under curves

A common type of task that employs Theorem B is described in table 9.1. The theorem supports the technology regarding the task, in that anti-derivatives can be used to determine the area, and also provides a technique for determining the area, assuming the students knows the anti-derivative. In order to employ the theorem, one needs to ensure the the function is non-negative and continuous in the given interval and determine the anti-derivative F of the function f. Determining the anti-derivative comes with its own set of techniques, depending on the function equation, which we will

[15]Da: "Lad f være en ikke-negativ og kontinuert funktion i et interval $[a;b]$, og lad F være en vilkårlig stamfunktion til f i dette interval. Arealet af punktmængden mellem grafen for f og x-aksen i intervallet $[a,b]$ er nu givet ved: Areal $= F(b) - F(a)$"(Madsen et al., 2024a)

address later. In general, integral calculus and the anti-derivative is introduced in the text books in relation to differential calculus, utilising the fact the F is anti-derivative if $F'(x) = f(x)$, and differential calculus is introduced through the notion of limits. In this way, the technology in table 9.1 is justified on a foundation of other technologies that need to be established.

9.4.1 Limits and Continuity

As the technology of the praxeology of determining areas is finding justification in the notion of limits, we investigate how limits are introduced in upper secondary school.
After providing two thorough examples of limits of functions, the notion of a limit value is given in *Plus STX A2*:

Definition C (limit value)**.** *A function f is said to have a limit a as x approaches x_0 if $f(x)$ can get as close to a as we want, provided we choose x close enough to x_0.*
Symbolically, this is written as:

$$f(x) \to a \quad \textit{for} \quad x \to x_0 \qquad \textit{or} \qquad \lim_{x\to x_0} f(x) = a$$

The existence of the limit is equivalent to the existence of the left and right limits, which are equal to each other:

$$\lim_{x\to x_0} f(x) = a \quad \Leftrightarrow \quad \lim_{x\to x_{0-}} f(x) = \lim_{x\to x_{0+}} f(x) = a$$ [16]

The notation of $\lim_{x\to x_{0-}} f(x)$ and $\lim_{x\to x_{0+}} f(x)$ has previously been established through one of the examples, but it has not been given a formal definition, it has only been written as x "moving towards" a certain value from left and right, respectively. In this way, the definition relies a lot on intuition.

We also find that definition 4 of a limit point is a transposed $\varepsilon - \delta$-definiton, boiled down to "if $f(x)$ can get as close to a as we want, provided we choose x close enough to x_0" in the didactic transposition. The definition also establishes the conditions under which the limit exists, by an equivalence between the existence of the limit and the existence and equality of left and right limits. However, the considerations of how to determine if the right and left limits exists are purely intuitive, and formal proofs are omitted from the text book. As such the logos block regarding limits has been simplified to an extend that the notion of limits are only introduced as an intuitive means to define continuity. The notion if limits is used in Definition D of pointwise continuity:

Definition D (continuity in x_0)**.** *A function f is continuous at some x_0 from its domain if $f(x)$ approaches $f(x_0)$ as x approaches x_0:*

[16]Da: "En funktion f siges, at have en grænseværdi a for x gående mod x_0, hvis $f(x)$ kan komme så tæt på a, som vi ønsker, bare vi vælger x tilstrækkelig tæt på x_0.Symbolsk skrives så: $f(x) \to a$for$x \to x_0$ eller $\lim_{x\to x_0} f(x) = a$. At grænseværdien eksisterer, er ensbetydende med, at grænseværdierne fra venstre og højre eksisterer og er lig med hinanden: $\lim_{x\to x_0} f(x) = a \Leftrightarrow \lim_{x\to x_{0-}} f(x) = \lim_{x\to x_{0+}} f(x) = a$ "(Madsen et al., 2024c)

$$f(x) \to f(x_0) \quad \textit{for} \quad x \to x_0 \qquad \textit{or} \qquad \lim_{x \to x_0} f(x) = f(x_0)$$

If x is replaced by $x_0 + h$, we can write:

$$f(x_0 + h) \to f(x_0) \quad \textit{for} \quad h \to 0 \qquad \textit{or} \qquad \lim_{h \to 0} f(x_0 + h) = f(x_0)$$[17]

This book does not have an explicit definition of when we say a function in its entirety is continuous, it just notes after Definition D that: 'a function which is continuous in every point is considered a continuous function'. In contrast to the *Plus STX A* system, *Hvad er matematik? 2* actually provides a formal ε-δ-definition of limits, a long with two more intuitive definitions written out in words. This shows a lack of consensus in the two text book systems on whether the ε-δ-definition is relevant for upper secondary school students. However, they both provide intuitive definitions of the concept, and it is likely that many students will rely mostly on the intuition in the praxeologies they create around limits.

9.4.2 Integral Calculus

The third book in the *Plus A STX-system*, *Plus STX A3*, begins with integral calculus and the first chapter begins with the following introduction:

> In differential calculus, we showed how the derivative function can be used to determine the monotonicity of a function and to optimise solutions to mathematical problems. In this chapter, we demonstrate how the reverse process, integral calculus, can be used to determine areas in a remarkable way. [18]

As we have seen in Section 8.4, the development of integral calculus is motivated as an answer by questions regarding area determination on a scholarly level, and only after the fact it has become clear that the integral and the anti-derivative is in fact related to the derivative.[19] However, as we see here, and as is also suggested by the official curriculum, integral calculus is introduced directly as the opposite of differential calculus in upper secondary school. This in particular highlights how the logos block concerning integral calculus is very different in upper secondary school from the scholarly level. As a result, the first definition given in the chapter is that of the anti-derivative:

[17]Da: "En funktion f er kontinuert i et x_0 fra definitionsmængden, hvis $f(x)$ går mod$f(x_0)$ for x gående mod x_0:

$$f(x) \to f(x_0) \text{ for } x \to x_0 \text{ eller } \lim_{x \to x_0} f(x) = f(x_0)$$

Hvis x erstattes med x_0 + h, kan man i stedet skrive:

$$f(x_0 + h) \to f(x_0) \quad \text{for } h \to 0 \qquad \text{eller } \lim_{h \to 0} f(x_0 + h) = f(x_0)$$

"(Madsen et al., 2024c)

[18]Da: "I differentialregningen viste vi, hvordan den afledede funktion kan anvendes til at finde en funktions monotoniforhold og til at optimere løsninger til matematiske problemstillinger. I dette kapitel viser vi, hvordan den modsatte proces, integralregningen, på forunderlig vis kan anvendes til at bestemme arealer."(Madsen et al., 2024a)

[19]Of course, this is implied in the choice of word 'anti-derivative', but this is not as evident in Danish.

Definition E. *A function F is the anti-derivative of a function f, if $F'(x) = f(x)$*[20].

On a scholarly level, this was a very fundamental result, stated in Theorem 16, but in the didactic transposition, this result has transposed into a definition.

Another definition, given shortly after, is that of the indefinite integral - that is, the indefinite integral is presented *separately* to the anti-derivative, though the two are in fact the same thing. It seems the distinction is made in the book to introduce the notation of the integral. Henceforth, we will use the term anti-derivative.

With the definition of the anti-derivative, we are ready to return to the main theorem of integral calculus. In part 1, this theorem states that the relationship between the derivative of a non-zero continuous function defined on an interval, and the area underneath it. That brings us to Theorem B, which we stated in the beginning of Section 9.4:

Theorem B (Main theorem of integral calculus, part 2). *Let f be a non-negative and continuous function in an interval $[a;b]$, and let F be an arbitrary anti-derivative of f in this interval.*
The area of the region between the graph of f and the x-axis in the interval $[a, b]$ is now given by:

$$\text{Area} = F(b) - F(a).$$[21]

Thus, in upper secondary school calculus, we can determine the area under a function with the notion of anti-derivatives, and without even discussion the notion of the definite integral.
The definite integral, defined in terms of the anti-derivative, finally yields the conclusion that the area can be found with the definite integral:

Definition F (Definite integral)**.** *Let f be a continuous function on an interval with an antiderivative F. Furthermore, let a and b be two numbers in the interval. The definite integral of f from a to b is written as $\int_a^b f(x), dx$ and is defined by*

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

a is called the lower limit of the integral and b the upper limit of the integral. [22]

[20]Da: "En funktion F er stamfunktion til en funktion f, hvis $F'(x) = f(x)$" (Madsen et al., 2024a).

[21]Da: "Lad f være en ikke-negativ og kontinuert funktion i et interval $[a;b]$, og lad F være en vilkårlig stamfunktion til f i dette interval. Arealet af punktmængden mellem grafen for f og x-aksen i intervallet $[a,b]$ er nu givet ved: Areal $= F(b) - F(a)$"(Madsen et al., 2024a)

[22]Da: "Lad f være en kontinuert funktion i et interval med en stamfunktion F. Lad endvidere a og b være to tal i intervallet. Det bestemte integral af f fra a til b skrives $\int_a^b f(x)\,dx$ og er defineret ved

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

a kaldes integralets nedre grænse og b integralets øvre grænse."(Madsen et al., 2024a)

This definition in is great contrast to the corresponding scholarly definition of the Riemann integral, Definition 15, where the same mathematical objects $\int_a^b f(x)$ was introduced without any references to differential calculus or the antiderivative. This shows how the associated logos block has transposed. It is also evident from Definition F that there is an emphasis on associates techniques, since the relation $\int_a^b f(x) = F(b) - F(a)$ is included in the definition.

In order to investigate the rigour of mathematical arguments in relation to area determination in the knowledge to be taught, we examine a direct proof given in the text book. First, we present how the proof of Theorem B is given in the text book:

Proof (Theorem B) Given that *A(x)* is the area function, the sketched area has the area *A(b)*. Thus, we need to show that $F(b) - F(a) = A(b)$. Since both F and A are antiderivatives of *f*, they differ by a constant *k*:[23]

$$F(x) = A(x) + k$$

In this equation, substitute b and then a for x:

$$F(b) = A(b) + k$$

$$F(a) = A(a) + k$$

Now we obtain:

$$\begin{aligned} F(b) - F(a) &= (A(b) + k) - (A(a) + k) \\ &= A(b) - A(a) \\ &= A(b) \end{aligned}$$

where we in the last equality used that; *A(a) = 0*.

Thus the desired result is proved.[24]

□

The theorem draws on part 1 of the Main Theorem of Integral Calculus, which includes the area function (cf. Theorem G).

[23]Da: "Idet *A(x)* er arealfunktionen, har det skitserede område arealet *A(b)*. Vi skal altså vise, at $F(b) - F(a) = A(b)$. Da både *F* og *A* er stamfunktioner til *f*, er de ens på nær en konstant *k*(...)"(Madsen et al., 2024a)

[24]Da: "Idet A(x) er arealfunktionen, har det skitserede område arealet $A(b)$. Vi skal altså vise, at $F(b) - F(a) = A(b)$. Da både F og A er stamfunktioner til f, er de ens på nær en konstant k: $F(x) = A(x) + k$ I denne ligning indsættes b og derefter a på x's plads: $F(b) = A(b) + k, F(a) = A(a) + k$ Nu fås: $F(b) - F(a) = (A(b) + k) - (A(a) + k) = A(b) - A(a) = A(b)$ hvor vi ved sidste lighedstegn benyttede, at $A(a) = 0$. Hermed er det ønskede bevist." (Madsen et al., 2024a)

Theorem G. *Let f be a non-negative and continuous function in the interval $[a; b]$.*

Furthermore, let the area function A be defined such that $A(x)$ denotes the area of the region between the x-axis and the graph of f over the interval $[a; x]$, where x belongs to $[a; b]$.
It now holds that:

$$A'(x) = f(x)$$

That is, A is an antiderivative of f.[25]

The structure of the proof is direct. Definitions are called and applied. In the statement "Since both F and A are anti-derivatives of f, they differ by a constant k", the first part of the statement draws on Theorem G, the second part draws on a previously stated theorem which is omitted here. The substitution of x by b is not justified. It appears that students are expected to be able to justify deductions from previously stated theorems, and be able to evaluate a function at a certain value. The rest of the proof is done with basic algebraic manipulations. The proof never explicates the theory it draws on, leaving it up to the student to either be well-versed in related theory, or trust the statements without justification. Students who just "trust the process" will memorise the proof without any connection to the related technology. We also note that the proof is dependant on the theory of differential calculus being established, as this is used in the definition of the anti-derivative.

In conclusion, the main theorems covered on integral calculus has a similar content to the scholarly knowledge we accounted for in Section 8, but it is introduced almost in the backwards order of the scholarly knowledge, as differential calculus is a convenient consequence of the Riemann Integral on a scholarly (and historical) level, and a necessary prerequisite in the knowledge to be taught.

[25] Da: "Lad f være en ikke-negativ og kontinuert funktion i et interval $[a; b]$. Lad endvidere arealfunktionen, A, være fastlagt ved, at $A(x)$ angiver arealet af punktmængden mellem x-aksen og grafen for f i intervallet $[a; x]$, hvor x tilhører $[a; b]$. Der gælder nu: $A'(x) = f(x)$. Dvs. A er en stamfunktion til f." (Madsen et al., 2024a)

Part III

The Teaching Sequence

Design and a Priori Analysis

Chapter 10

Introduction to part III

In this part we will first present the conditions for our teaching sequence, which includes the context of the test class and the students' expected prior knowledge. Then we will present the generating question where we will discuss its possibilities and limitations in regards to answering our research question. This leads to a presentation of our teaching design, before ending the chapter with an a priori analysis of the design. The a priori analysis serves as the foundation for the a posteriori analysis in part IV. Therefore this part and the following constitutes our research of RQ_3 (*How can exposing students to a fluctuation between action history and observer history promote reflection on rigour and reasoning in relation to area determination?*)

Chapter 11

Presentation of our Design and the Generating Question

The overall learning objective for our teaching sequence is to create an inquiry-reflective learning environment, that gives students an opportunity to reflect upon mathematical rigour and reasoning in the context of area determination. we want to investigate how this learning objective can be met through an investigation of rigour and reasoning in two authentic mathematical papers by Archimedes and Newton through a sequence of SRP's. By implementing authentic mathematical sources in the classroom, we aim to place students as observers of history, and enable them to use this experience to orient themselves in a contemporary context, reflecting upon rigour and reasoning in relation to integral calculus, thus confine in an action use of history.

11.1 Conditions for the Test of the Teaching Sequence

In this section we account for the context of the test class and their prior knowledge, in order to argue that the curriculum accounted for in the Section 9 is relevant to the level of the class. In the process of designing the teaching sequence we had already arranged where and when the teaching it should be tested. In cooperation with the usual teacher of the class, we therefore had the opportunity to discuss what subjects that needed to be covered prior to our teaching sequence, which we wish to elaborate on in Section 11.1.2.

11.1.1 The Test Class

The teaching sequence was designed to be tested in a 2^{nd} year STX class at XXX Gymnasium with A-level mathematics as part of their study programme. At this gymnasium the duration of a lesson is 70 min. Our teaching sequence consists of four lessons, hence we have a total of 280 min. Furthermore, the students were given an assignment set to be answered within the frame of two hours. This assignment were scheduled to be handed in two weeks after the last lesson had been conducted. The assignment were an individual hand-in, however the students' work in class during the four lessons were in groups which their usual teacher had composed.

11.1.2 Prior Knowledge

We agreed with the usual teacher of the class that he would cover the curriculum on differential calculus and parts of integral calculus prior to the implementation of our design, and that they would put off proofs from integral calculus until after our implementation. This was due to the fact that we were unsure at that point how big a role proofs of integral calculus would play in our design, and wanted the opportunity to be the ones introducing them to the students. The condition to have concluded differential calculus, including proofs, was due to two considerations; a) integral calculus is introduced in upper secondary school as the 'opposite' of differential calculus (cf. Section 9.1, and b) we wanted the students to be familiar with proofs and rigour in analysis.

According to their teacher, the students had not been taught about rigour and reasoning in mathematics independently. However, the students had had practice conducting proofs. Further, the text book follows an axiomatic-deductive method in its structure. Thus, we expect the students are familiar with mathematical reasoning and the structure of mathematical argumentation. This is an essential part of the milieu the validation takes place against.

The students' prior experience with area determination outside the subject of integral calculus is limited to primary school, where we assume that the 'formulas' of calculating the areas of circles, triangles, squares and so on have been covered.

11.2 The Generating Question and the Process Choosing this

In the process of designing an inquiry-reflective learning environment with the use of SRP's as a design tool, the generating question needs to be chosen carefully, such that it can foster an autonomous inquiry in a oriented way. In our design we have chosen one guiding generating question from which we pose two sub-questions that are almost identical in order to accommodate the two historical episodes.

In this section we wish to present our chosen generating question as well as the two sub-questions and we will discuss the possibilities and limitations associated with these questions. Furthermore, we will discuss the choice of including two generating questions and we will close of with an a priori analysis of just one of these because the sequence of SRP's are very similar.

11.2.1 Considerations of Choosing the Generating Question

A good generating question is one that: "(...) should be strong enough to guide an exploration of a knowledge domain. Students should understand the question but not be able to answer it, unless they engage in a study and research process." (Jessen, 2017, p. 5) In this way it should be a question which fosters students' investigations which can lead to various SRP's. We found choosing a question that was broad enough to foster various explorations and narrow enough to accommodate our aim with our teaching design challenging.

As stated in the introduction, the learning objective of our design was to create an inquiry-reflective teaching environment, which could foster student-discussions about mathematical rigour and reasoning in the context of area determination. This led us to the question:

Q_0*: How would you determine the area under a curve, and how do you argue for your method?*

This question seemed appropriate in consideration of our research question. However, we noticed that it did not have a lot of possibilities regarding an autonomous exploration as the students were already familiar with integral calculus, and we feared that an SRP generated from the first part of the question would be quite limited if students quickly concluded that it could be answered with integral calculus. Regarding the second part of the question, about how one would argue for the chosen method, we found a great possibility for students to engage in an inquiry-reflective process, exploring what it actually means to argue in mathematics, leading them to engage in a discussion of reasoning and rigour. However, we were concerned that students would not be motivated to explore this direction without further guidance.

In particular, three things stood out to us regarding Q_0. First of all, the word *rigour* is not stated explicitly in the question, but instead we use the word *argue*. This choice was made, because the curriculum in Danish upper secondary school never in fact uses the word 'rigour'[1], and as we have discussed in Section 7.1, a clear definition of rigour is not so straight-forward. However, it is related to argumentation and reasoning, and we see that one way to attack a question of how arguments are made is through an exploration of mathematical rigour. By guiding students to work with argumentation relating to the specific problem of area determination, rather than posing a broader question (such as 'what does it mean to argue in mathematics?), we manage to narrow down the focus of the teaching sequence. In order to make sure that students actually will engage in an SRP regarding rigour, we will need to create a teaching design with sub-questions leading in the right direction.

Secondly, we need to reconsider the use of the word *curve*. From a historical point of view, this phrasing stands out, as the definition of a curve we know from contemporary mathematics is different from the curve that was known to Archimedes and Newton. We are concerned that students

[1] altså, stringens

will not see the curve as an epistemic object that has had a different meaning in different epistemic configurations. For example, the way Archimedes considered a curve is not necessarily comparable to how Newton considered it. If students are not aware that the meaning has changed over time, they may rely on their own contemporary conception of a curve, and thus it might not be a motivating factor for an autonomous historical inquiry.

Our third observation concerns the lack of historical dimension in the question. We found it challenging to compose one single question which relates to two different historical episodes while not being too narrow and guided, which would lead to a too restricted inquiry process.

Based on these considerations we found regarding Q_0, we decided to rephrase it into two generating sub-questions and not pose Q_0 explicitly.

11.2.2 The Generating Sub-Questions

The considerations in the previous section guided our choice of two generating sub-questions. In order to avoid using the word 'curve', the sub-questions relied on a visual representation of an area. We decided on the following two generating sub-questions:

> Q_1: How would you determine the *yellow* area in figure 11.1, and how would you argue for the method you have used?
>
> Q_2: How would you determine the *blue* area in figure 11.2, and how would you argue for the method you have used?

The only difference between the two questions are the figures they refer to. This is chosen in order to motivate an exploration of the sources in the compendium by Archimedes and Newton, respectively. The figures resemble illustrations from the original works by Archimedes and Newton, which we find motivational for the students to address the compendium in search for answers in the authentic historical texts.

Q_1 and Q_2 also represent how the teaching sequence is divided into two parts, each with a different historical focus, on the works of Archimedes and Newton, respectively. The historical dimension is not achieved in the phrasing, but will be more teacher guiding through work with the compendium. In the first encounter with the two figures, we expect the students will be unable determine the area precisely with their prior knowledge. Therefore we see a potential to spark interest in regards to investigating the historical sources, as this will enable them to actually determine the areas of these figures.

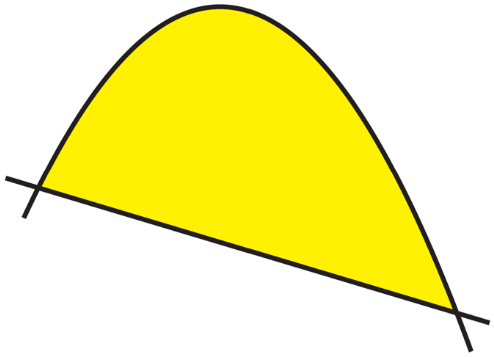

Figure 11.1: The first area shown

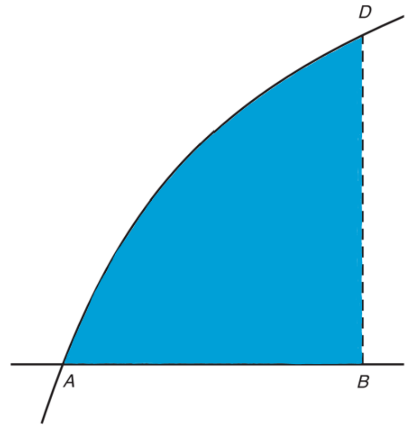

Figure 11.2: The second area shown

11.2.3 A Priori Analysis of the Generating Question Q_1

Before we present the design of the teaching sequence surrounding the chosen generating sub-questions, we present an a priori analysis of Q_1 without any limitations other than our own creativeness. As the questions are quite similar we only include an a priori analysis of Q_1 because the paths would be somewhat similar apart from the point that when the students work with Newton, they also have knowledge of Archimedes' method. However they should quickly realise that this can not be used to determine the area in figure 11.2.
The a priori analysis also sheds light on how the chosen sub-questions could not be answered properly without an historical investigation of Archimedes' text (and Newton's, with regards to Q_2).

In order to read the diagram, we want to make a note on the notation. In order to support a reading of the subscripts, questions derived from Q_1 will be subscripted to the left with a 1, e.g. ${}_1Q_1$ is the first expected question derived from Q_1, and likewise, questions derived from Q_1 will be subscripted as ${}_2Q_1$, which will be relevant in the a priori analysis of our design. The notation will be elaborated later on. We find two derived question from Q_1 when our teaching material are not available, namely ${}_1Q_1$ (*Which methods can I use that I am already familiar with?*) and ${}_1Q_5$ (*When have i argued enough for my method and result?*) (here, the numbering of the questions follow the numbering in our full SRP diagram in Section 12)
This is due to the fact that not much information is given in the question, and thus the student does not have many options but to search for answers regarding their prior knowledge. The SRP for Q_1 is represented in the diagram in figure 11.3 along with a list of the questions it represents.

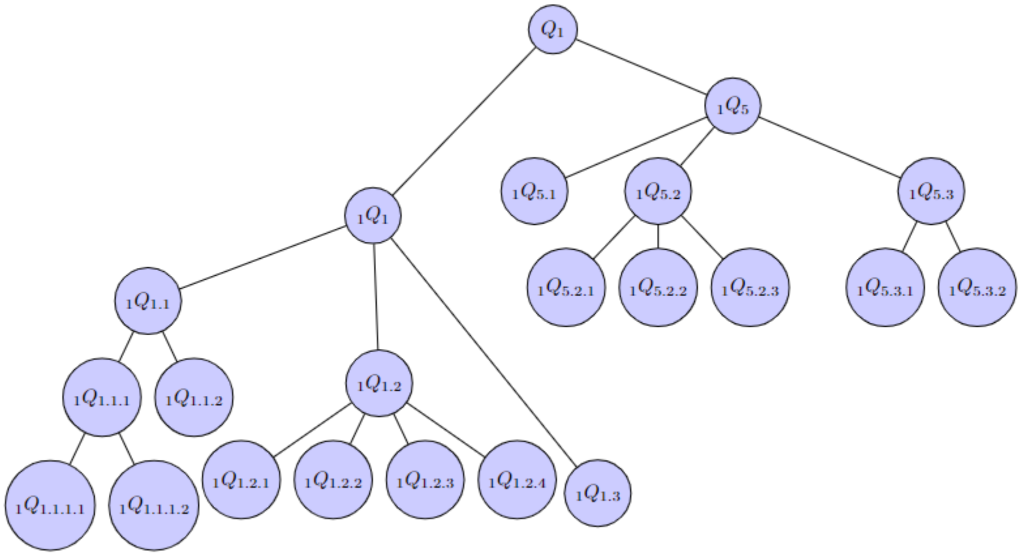

Figure 11.3: Questions derived from Q_1

Q_1	How would you determine the yellow area in figure 11.1, and how would you argue for the method you have used?
$_1Q_1$	Which methods can I use that I am already familiar with?
$_1Q_{1.1}$	Can I use integral calculus to determine the area?
$_1Q_{1.1.1}$	Can I use the definite integral?
$_1Q_{1.1.1.1}$	Is it possible to determine the borders of the figure as functions?
$_1Q_{1.1.1.2}$	Can I view the figure as a space between functions?
$_1Q_{1.1.2}$	Can I use the indefinite integral?
$_1Q_{1.2}$	Can I partition the figure into figures that I can determine the area of?
$_1Q_{1.2.1}$	How can I use grid paper (i.e. squares)?
$_1Q_{1.2.2}$	Can I approximate the figure with triangles?
$_1Q_{1.2.3}$	Can I arrive at an exact area by counting?
$_1Q_{1.2.4}$	Can I use straightedge and compass construction to measure the area?
$_1Q_{1.3}$	How can I plot the figure with digital tools?
$_1Q_5$	When have I argued enough for my method and my result?
$_1Q_{5.1}$	What does it mean to argue for my method?
$_1Q_{5.2}$	What is an argument in mathematics?
$_1Q_{5.2.1}$	What are my assumptions?
$_1Q_{5.2.2}$	What is a theorem?
$_1Q_{5.2.3}$	What is a proof?
$_1Q_{5.3}$	What is mathematical rigour?
$_1Q_{5.3.1}$	When is an approximation good enough?

${}_1Q_{5.3.2}$ When is an empirical verification good enough?

At this point, we do not provide the possible answers to the derived questions, as we simply wish to shed light on the possibilities inherent in our choice of generative sub-questions. An elaboration of possible answers will be dealt with in the a priori analysis of the design in Section 12.

The path above is quite short and limited, however we see many possibilities when students are informed that Archimedes had a way to determine the area precisely, and are guided towards the treatment of Archimedes' *Proposition 1* in the compendium. This will open a new path of exploration, namely the one tied to how Archimedes could determine the area, and in particular, how he argued for his method. As the students are working in an unknown territory, we also see a motivation to verify that the method in fact works, and thus prompt a discussion of rigour and reasoning in relation to Archimedes.

In conclusion, our chosen generating sub-questions do not immediately foster a large sequence of SRP's, but it is constructed in such a way that the students are not able to answer the question based on the expected prior knowledge which motivates the investigation of Archimedes text. This construction of a generating question is atypical, however, we regard it as beneficial for our purposes.

11.3 Overview of the Teaching Sequence

The intention of this section is to give an overview of the teaching sequence regarding the compendium, the lesson plans and the assignment in order to support the reader in our a priori analysis.

The teaching sequence has two main objectives. The first one is to create a necessity for the students to consult the historical sources in search of how they would determine the two areas and how they would argue for their answer, which relates to an observation oriented use of history. The second one is to orient themselves in their present, i.e. to bring the discussion of rigour and reasoning pertaining Archimedes' and Newton text into their present.

11.3.1 Lesson Plans

The first two lessons were focused on Archimedes and the last two around Newton. Therefore the first lesson takes point of departure in question Q_1, which is teacher posed and at the start of lesson three question Q_2 is teacher posed. In the lesson plans we have included a fair share of conferences, in which the students are asked to share their thoughts and ideas throughout the process of investigation. This is done to avoid students getting stuck, and to encourage students to 'steal' ideas from each other. The four lesson plans are presented in tables 1 to 4 in appendix A.

11.3.2 The Compendium: *Historical Expeditions*

We have created the compendium based on our theoretical historical analysis of Archimedes' and Newtons text, carried out in Section 8.2 and 8.3. And it is constructed in such a way that it should foster the students autonomous investigation from the perspectives of the past mathematicians and minimise the teachers interference with the students inquiry.

The analysis guided us to collect the 'tools' which is necessary to be familiar with in order to discuss the reasoning and rigour in the historical sources. This collection can be regarded as the epistemic techniques and epistemic objects in the different workplaces of Archimedes and Newton. One needs to be familiar with these tools in order to discuss the reasoning and rigour in the historical sources as observers. In the compendium, we included these tools in two *tool boxes* belonging to Archimedes and Newton, respectively. For example, Archimedes is concerned with the parabola, and thus we have accounted for how he understood a parabola, which is different from the students' prior understanding. Newton on the other hand considers a curve, which we expect that the students relate to the notion of a function - however, Newton did not have the notion of a function available in his tool box, as it was not yet defined, which is pointed out in the compendium. In this way our analysis supported the creation of the compendium.

Furthermore, as our analysis hinted the authentic historical sources can be difficult to read we chose to include them in the compendium with comments guiding the students to read some of the difficult passages as well left out some passages which have been paraphrased in order to keep the text coherent. The included passages is chosen based on the the analysis, which made it clear what parts could foster a discussion of rigour and reasoning which is fit to the students' level.

We have also included perspectives from other people, who wrote about the works of Archimedes and Newton (e.g. Klines statement from 1953 regarding Newtons limit concept: "[Newton] ... succeeded in doing [nothing] more with the limit concept than confusing himself..."), as well as a substantial amount of historical context in order to emphasise that Archimedes and Newton did not only shape the history but were also shaped by history, hence foster an inquiry of the mathematics of the past from the perspective of the past mathematicians, by which we seek to take on a multiple perspective approach to history of mathematics (see Section 2.1).

11.3.3 The Assignment

We have made the assignment in such a way that a student who have attended the four lessons should be able to complete it without the need for any knowledge which have not been available at class or in the compendium. In this way we constructed the questions in the assignment with the intend for it to function as a reflection paper on the teaching sequence.

Chapter 12

A Priori Analysis of our Design

In this section we will make an a priori analysis of our design, in which we will present possible answers to selected questions and connect them to our account of knowledge to be taught (cf. Section 9). The questions are chosen based on that they are likely to be posed, and/or because we want to demonstrate how potential answers may contribute to the learning goals. For the latter case, the presentation of answers serve as an argument for some of the questions to be included in the lesson plan as teacher posed questions.

The two generating sub-questions that lays the foundation for the teaching sequence are presented in Section 11.2.2. As the first two lessons were centred around Q_1 and the second to around Q_2 we will present the first two lessons together, and the second two lessons together in the next section.

12.1 The Five Main Derived Questions

Overall, we expect five questions to arise from Q_1 and Q_2. Some of the five questions are expected to arise from both Q_1 and Q_2, and some are only expected in relation to one of them.
In the following, we expand the notation introduced in Section 11.2.3. As already established, the left side index refers to whether the question is derived from Q_1 or Q_2. Questions that are expected to be derived from *both* Q_1 and Q_2 are subscripted to the left with a 0 here, and in cases when we discuss them in relation to the specific cases of Q_1 or Q_2, we will subscript them accordingly. In this way, ${}_0Q_1$, ${}_1Q_1$, and ${}_2Q_1$ are all the same question. This is done because even though the phrasing of the questions are the same, the expected answers and derived questions are dependant on the context. Further, possible answers are noted with an A and the same indexing as the questions, e.g. possible answers to ${}_1Q_{1.1}$ are noted ${}_1A_{1.1}$.

The five main derived questions we expect are presented below and are numerated to the left as to give an overview of how they relate to the two generating questions.

$_0Q_1$: Which methods can I use that I am already familiar with?

$_1Q_2$: How did Archimedes determine the area, and how did he argue for it?

$_2Q_3$: How did Newton determine the area, and how did he argue for it?

$_2Q_4$: What could Newton do with his method that Archimedes could not?

$_1Q_5$: When have I argued enough for my method and my result?

Question $_0Q_1$ draws on the students' prior knowledge regarding area determination. $_1Q_2$, $_2Q_3$ and $_2Q_4$ only arise through the students' work with the compendium, thus are related to a observation use of history. $_1Q_5$ will be posed continuously throughout the whole teaching sequence. This is done in order to create an explicit relation between the historical sources and the modern methods that the students are taught in upper secondary school. In this way we foster an action use of history, as the students are encouraged to orient themselves in their contemporary context.

12.2 Lesson 1 and 2: *Archimedes' Workplace*

In the beginning of lesson 1, Q_1 is posed without any introduction to the compendium. We remind the reader that Q_1 is:

> Q_1: How would you determine the yellow area in figure 11.1, and how would you argue for the method you have used?[1]

Since this question will be posed prior to distributing the compendium, we do not expect any changes from the expected SRP already presented in Section 11.2.3.

$_1A_{1.1.}$ Can I use the definite integral?

We expect that the students will be challenged by the lack of information in the figure, there is no numerical values, functions nor coordinates. Since the students are familiar with integral calculus, we expect $_1Q_{1.1}$ to be posed quickly because this is a common tool used in upper secondary for determining area. This will prompt students to consider the assumptions required if one were to use the definite integral and we expect them to conclude that they can not use this tool as they do not have any of the required information.

$_1A_{1.1.2}$ Can I view the figure as a space between functions?

We expect students to investigate possible function equations to represent the figure. Hence we expect $_1Q_{1.1.2}$ to be student posed. We expect a possible answer $_1A_{1.1.2}$ to be that it is not possible to determine the equation of the functions as they do not have any coordinates from which they could

[1]The question was posed in danish: *Hvordan ville I bestemme arealet af det farvede område, og hvordan argumenterer I for jeres metode?*

determine a function.

The two questions above sets the stage for a debate about how the tools they have today haven't always existed and how the coordinate system is not a given entity.

${}_1A_{1.2.1}$ **How can I use grid paper (i.e. squares)?**
A common method taught in primary school is to use grid paper to count the squares of a figure in order to determine the area. Therefore we expect that some students consider this method and reach the conclusion that this can only lead them to an approximation of the area.

In conference 1.1, the students present their answers to the posed question, which will foster a discussion of what assumptions one need to know in order to use theorems in mathematics and reach a valid conclusion. After this discussion, the compendium is provided to the students. Archimedes is briefly introduced, and the students are informed that he in fact could determine this area. Therefore ${}_1Q_2$ (*How did Archimedes determine the area, and how did he argue for it?*) is teacher posed. Through working with the compendium, we expect the following questions to be derived from ${}_1Q_2$.

${}_1Q_2$	How did Archimedes determine the area, and how did he argue for it?
${}_1Q_{2.1}$	Which methods does Archimedes use in the demonstration of Proposition 1?
${}_1Q_{2.1.1}$	Which method does Archimedes refer to in the text?
${}_1Q_{2.1.1.1}$	What is the law of the lever?
${}_1Q_{2.1.1.1.1}$	What advantages are there to using the law of the lever?
${}_1Q_{2.1.1.1.2}$	What are the disadvantages of using the law of the lever?
${}_1Q_{2.1.1.1.3}$	Why did Archimedes find it meaningful to use the law of the lever?
${}_1Q_{2.1.1.1.4}$	Why can't the law of the lever be included in proofs, according to Archimedes?
${}_1Q_{2.1.1.1.5}$	When can the law of the lever be used?
${}_1Q_{2.1.1.1.6}$	Are we familiar with the principle of equilibrium from elsewhere?
${}_1Q_{2.2}$	When did Archimedes believe something to be proved?
${}_1Q_{2.2.1}$	What is a mechanical method?
${}_1Q_{2.2.1.1}$	What was the attitude towards mechanical arguments at the time?
${}_1Q_{2.2.2}$	Why did Archimedes believe that a proof was only valid if it was geometric?
${}_1Q_{2.2.2.1}$	What does it mean to demonstrate something by geometry
${}_1Q_{2.2.3}$	What did Archimedes think of his demonstration in Proposition 1?
${}_1Q_{2.2.3.1}$	Why does Archimedes state that the result was only indicated?
${}_1Q_{2.2.3.1.1}$	Can a two-dimensional figure be made up of one-dimensional lines?
${}_1Q_{2.2.3.1.1.1}$	What is the definition of a line?
${}_1Q_{2.3}$	When can I use Archimedes' method?
${}_1Q_{2.3.1}$	Can I use Archimedes' method if I want to determine the area under a function?

${}_1Q_{2.3.1.2}$	If so, does it work to determine the area under *any* function?
${}_1Q_{2.3.2}$	Is the law of the lever still used to discover new mathematics today?
${}_1Q_{2.3.2.1}$	Why/why not?
${}_1Q_{2.4}$	Where did Archimedes acquire his knowledge from?
${}_1Q_{2.4.1}$	How did Archimedes access mathematics that had already been discovered?
${}_1Q_{2.4.2}$	Which prominent mathematicians preceded Archimedes?
${}_1Q_{2.4.3}$	In what ways were mathematics different for Archimedes compared to contemporary mathematics?
${}_1Q_{2.4.3.1}$	Did Archimedes understand mathematical objects in the same way as we do today?
${}_1Q_{2.4.3.2}$	Was Archimedes familiar with the coordinate system?
${}_1Q_{2.4.3.3}$	Was Archimedes familiar with the notion of functions?

In conference 1.2, a plenum discussion of the groups' initial encounter with the text is initiated and ${}_1Q_{2.2}$ (*When did Archimedes believe something to be proved?*) is teacher posed if students have not posed it themselves because this is an important question to treat for our learning goal. We see a possibility that the students will pose this question themselves, as Archimedes addresses this matter in the preface of the text, and when the students investigate the compendium, this is the first text they will encounter of Archimedes' own writings. We expect the students to be able to identify that Archimedes refers to a mechanical method and a geometrical method, which we expect will lead to the students posing ${}_1Q_{2.2.1}$ *(what is a mechanical method)* and ${}_1Q_{2.2.2.1}$ *(what does it mean to demonstrate something by geometry?)*. In this way they get 'in touch' with two of the three historical praxeologies we identified in Section **??**. Since answers ${}_1A_{2.2.1}$ and ${}_1A_{2.2.2.1}$ both feed into the answer of ${}_1Q_{2.2}$, they are included below in a possible answer ${}_1A_{2.2}$

${}_1A_{2.2}$ When did Archimedes believe something to be proved?

If we assume that the students only consult the preface of *Proposition 1* in the compendium, we expect them to be able to give an answer such as: "Archimedes believed that something was only proved, if it was shown with geometry, rather then mechanical methods", because Archimedes wrote: "*for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration*". However, this answer is also based on the assumption that the students think of "*an actual demonstration*" as a proof in mathematics. They may be satisfied and not seek a more elaborated answer, or they may go a little further to gain insight into what that in fact means.

The meaning of mechanics and geometry is not elaborated in the compendium, but we expect that students, based on their prior knowledge, have a conception of geometry as something related to shapes and figures, and mechanics as something more real-world oriented, and thus will be able to distinguish the concepts to some extend. Using a search engine, students may find a definition of 'mechanics' such as "a path of physics, deals with the motion or rest of bodies[2]", and 'geometry' as an "area within mathematics, originally dealing with the description and measurement of shapes[3]. If students gain this insight, a more elaborated answer is available, providing details of the distinction between mathematical geometric methods as an idealisation of the real world, where mechanics concern the world as it actually is, and is thus less idealised. The insight into this distinction will give rise to a more nuanced answer, concerning why Archimedes distinguished between the two methods, and why he thought that mechanics could not provide an actual proof.

Conference 1.3 yields a plenum discussion of the students' answers ${}_1A_{2.2}$. Here, the focus is shifted from the past to the present, in order to lead them from an observation usage of history to an action usage. Here, we intend for the students to make some considerations of their own conception of rigour and reasoning in contemporary mathematics, hence relating to ${}_1Q_5$ (*What is your own conception of rigour and reasoning in contemporary mathematics?*). As the students have now worked with the distinction between two methods, i.e. mechanical and geometric, they should be able to reflect upon whether they have used or is able to use either of these methods in a proof today. Based on our content analysis we expect that they are able to state that a geometrical method can still be used.

From ${}_1Q_5$ we expect the following derived questions:

${}_1Q_5$	When have I argued enough for my method and my result?
${}_1Q_{5.1}$	What does it mean for something to be rigorous?
${}_1Q_{5.1.1}$	How does rigour relate to intuition?
${}_1Q_{5.1.2}$	Is it important to be rigorous in mathematics? (And why?)
${}_1Q_{5.1.3}$	What role does examples and special cases play in relation to rigour?
${}_1Q_{5.2}$	What is a mathematical proof?
${}_1Q_{5.2.1}$	Which methods do I use today when proving a mathematical theorem
${}_1Q_{5.2.2}$	When would I say that something has been proved?
${}_1Q_{5.3}$	What is mathematical reasoning?
${}_1Q_{5.3.1}$	What significance does mathematical language have for mathematical reasoning?

We find ${}_1Q_{5.2.2}$ (*When would you say that something has been proved?*)[4] to be a crucial question regarding the learning goal of strengthening the students comprehension of proofs in mathematics, and to make the connection to the present even clearer. Thus, the question is included in the lesson plan as a teacher posed question.

${}_1A_{5.2.2}$ When would you say that something has been proved?

[2]Mekanik, gren af fysikken, som behandler legemers bevægelse eller hvile, `https://denstoredanske.lex.dk/mekanik`

[3]"[...] område inden for matematikken, som oprindelig omhandler beskrivelse og måling af figurer." `https://denstoredanske.lex.dk/geometri`

[4]Phrased in the lesson as: "Hvad mener I? Hvornår er noget bevist?"

We have identified four types of distinct answers that we expect are some guiding categories for the students answer, namely that the student conception of when something is proved is:

1. based on understanding of standards
2. based on convictions
3. relying on authority
4. based on formality

We regard the first and fourth to be tied to a strong comprehension of proofs in mathematics, and the second and third to a weak comprehension.

We regard the first comprehension likely to occur, as theorems are introduced as something that should be proved based on axioms, definitions and other theorems, cf. Section 9.3. Therefore we expect some students to be able to identify these as standard within a proof.
For the second type, we expect that some students might think that something has been proved merely because they feel confident that the fact stated is true. In this case, the conception of a proof is based on intuition and/or previous experience.
The third type of conception of proofs is authority based. This arises when a student equates 'proof' with authority, i.e. a statement by a teacher (or textbook provided by a teacher), and accepts something as true because this authority has asserted it. This weak comprehension of when something has been proved can occur if students have experienced that the theorems and proofs which they encounter during their education are always valid, and thus they see no reason to doubt them. As we do not expect students to have encountered invalid statements in their mathematics, we expect some students will rely on their teacher regarding validity.
Lastly, regarding the fourth type, some students may in fact view something as proved when it has been demonstrated with logical deductive principles, however we do not expect students to be able to point to which principles we categorise as such, thus they are not expected to have a strong discourse about this. This is the conception we would expect from scholars, and thus we find it unlikely to appear among upper secondary school students, and if so, only from the more advanced students. As we have seen in Section 9, students have not been introduced to mathematical formalism on a scholarly level, which is why this is unlikely.

Conference 1.4 concludes lesson 1. This is a plenum discussion of the students' personal comprehension of proofs, and is continued in the beginning of lesson 2, where the students (after a brief recap) are asked what methods they use when proving a mathematical theorem, thus lesson two takes point of departure in ${}_1Q_{5.2.1}$ (*Which methods do I use today when proving a mathematical theorem?*) in order to promote a reflection of the things that constitutes a proof. Students who provided answers ${}_1A_{5.2.2}$ of type 1 (and 4, for that matter) might already have made considerations of this as these types are closely tied to the theory of what we are able to use when proving things in mathematics. Students who provided answers of type 2 or 3 may be forced to reassess their conception when working on this question. By this we 'force' students who have a weak understanding of proofs to work explicitly on what makes a proof valid. We expect the students to be able to give the following answer to ${}_1Q_{5.2.1}$:

${}_1A_{5.2.1}$ Which methods do I use today when proving a mathematical theorem?
In order to answer this question some students might restrict themselves to consult their prior knowledge. If this is the case we expect advanced students to answer this question by listing types of proofs, e.g. direct proofs, proofs by contraposition or proofs by induction, as these are the types listet in thier text books, cf. Section 9.3. On the other hand, if students are not able to identify different types of proofs, they may instead list tools often used in proofs, such as algebraic manipulations or illustrations, and some may even have the misunderstanding that a theorem can be proved by example, or even using digital tools.
We also see a possibility that some students will consult their prior knowledge in light of their new knowledge from lesson 1 regarding what Archimedes thought of as methods used in proofs. If this is the case we expect that the students are able to elaborate on geometrical methods.

After a brief plenum discussion of ${}_1A_{5.2.1}$ we make a switch back again to Archimedes, by posing ${}_1Q_{2.1}$ (*Which methods does Archimedes use in the demonstration of Proposition 1?*). This switch is made in order to guide students back to conforming to observer history. In the following Conference 2.1, after the groups have shared their findings, the teacher will initiate a discussion of the contrast between geometric and mechanical methods, as this is an important part of Archimedes text regarding the discussion of rigour and reasoning (cf. Section **??**). This has already been discussed in the previous lesson in relation to the preface, but we expect that the actual use in Archimedes text will make the distinction more clear. In particular, we expect that the emphasis on mechanical methods in relation to the demonstration of *Proposition 1* will prompt students to pose ${}_1Q_{2.1.1.1}$ (*What is the law of the lever?*), and if not, it will be teacher posed.

${}_1A_{2.1.1.1}$ What is the law of the lever?
We expect students to consult the material in the compendium regarding the law of the lever, where a link to a visual representation is included. We also expect students to draw on their own experience e.g. from a seesaw or similar, which leads to an intuitive understanding, because the law of the lever has been connected to the mechanical method in the previous lesson. Thus, by now, it should be clear that this method can be connected to a physical conception, and is highly real-world related. This answer can support (and is supported by) the students' understanding of the distinction between geometry and mechanics - in fact, we expect it is possible to have a strong intuition about the law of the lever without consulting mathematics at all.
In extension, we don't expect the answer to be provided in rigorous mathematical terms, which aligns

well with our focus being more on rigour and reasoning than on the conducting of mathematical manipulations. Examples of expected answers are along the lines of "something you use to investigate the relationship between two entities" or "weighing stuff".
With this first encounter with the law of the lever, we do not have any expectation that the students will point out precise problems with using the law of the lever in a mathematical demonstration. However we see an opportunity that they will find it somewhat problematic, as it is a mechanical method, which by lesson 1, is not regarded to furnish an mathematical demonstration according to Archimedes.

In Conference 2.2, students share their understanding of the law of the lever with the class, followed by an institutionalisation of the concept by the teacher. Which leads to question ${}_1Q_{2.1.1.1.2}$ (*What are the disadvantages of using the law of the lever?*)[5] to be teacher posed.

${}_1A_{2.1.1.1.2}$: What are the disadvantages of using the law of the lever?
We do not expect students to find many issues with the law of the lever initially. We expect them to merely state that it is a disadvantage to use the law of the lever because it is a mechanical method. Therefore the teacher will quickly guide students to investigate how a line was defined (in the historical context). The compendium contains a few line-related definitions, e.g. "A line is breadthless length[6]". Based on this investigation we expect students to be able to conclude that lines have no breadth, and thus no weight, which makes it complicated to weigh them on a lever. If students do not come to this conclusion themselves, the teacher will guide them by posing questions related to lines, e.g. "how would you weigh a line?" or "what dimension is a line?". In this way we expect the students to be able to argue that there are issues with the law of the lever, when a mechanical method is employed in a geometrical setting.

In conference 2.3, the groups' answers are discussed in plenum to ensure that all students have worked with the passage in which Archimedes may have considered his demonstration as non-rigorous. This will initiate a discussion of indivisibles. The teacher makes sure that the notion of indivisibles is included, if it is not brought to attention by the students. To foster a discussion of the question, students are asked to consider a quote from Democritus included in the compendium, where the notion of indivisibles is criticised.

${}_1A_{2.2.3.1.1}$: Can a two-dimensional figure be made up of one-dimensional lines?
We expect the students to read the statement of Democritus in the compendium (cf. Appendix C) by which we expect the them to be able to connect the conic section Democritus is treating to the parabolic segment, which Archimedes is investigating. We expect them to arrive at something like

[5]Posed as: "Hvilke problemer kan der være med at bruge vægtstangsprincippet?"
[6]Included in Danish: *En linie er en længde uden en bredde*

the parabolic segment would not have a smooth surface if we regard it as being made up from indivisibles. As indivisibles are a difficult entity the compendium also includes a brief description of the notion, in which we have written:

> Disse skiver har ikke nogen højde, og er således to-dimensionelle, mens selve keglen er tre-dimensionel. I eksemplet her er det skiverne, vi kalder de indivisible, men på samme måde kunne man forestille sig, at en flad to-dimensionel figur består af en-dimensionelle linjer, der ikke har nogen bredde.

This enables the students to connect the indivisibles with a shift in dimension, as dimensions are a taught subject in upper secondary school. And this is not meaningful in Archimedes argument.

Since the concept of indivisible is highly abstract, we only expect students to get the slightest understanding. For the aim of our teaching sequence, it is in fact not important that students understand the notion of indivisibles completely, but rather that they gain insight into the fact that the establishment of rigorous mathematics is complicated, and has been discussed for thousands of years. And if an argument makes use of a tool of which the validity is up for discussion, e.g. indivisibles, we can criticise the the state of rigour in the entire argument.

After the conclusion of lesson 1 and 2 we expect students to have formed some answers in relation to the rigour of Archimedes method, as well as how his argument is composed. In relation to these formed answers, we expect that they will have started reflecting on the rigour and reasoning of contemporary mathematics due to the teacher posed question regarding the students own comprehension of proofs (${}_1Q_{5.2.2}$ and ${}_1Q_{5.2.1}$).

12.3 Lesson 3 and 4: *Newtons Workplace*

The historical focus in lesson 3 and 4 is shifted from Archimedes to Newton. Therefore lesson 3 begins with a brief recap of the previous lessons, where the distinction between the mechanical method and geometrical methods is emphasised along with how we can discuss the rigour pertaining Archimedes demonstration, even though his result in fact is valid. Afterwards the students' investigation of Newtons 'workplace' will be motivated with the teacher posed Q_2:

> Q_2: How would you determine the in blue area figure 11.2, and how would you argue for the method you have used?

The students are asked to discuss Q_2 in their assigned groups. When Q_1 were posed in lesson 1, we mainly expected ${}_1Q_1$ and its derived questions to be posed, cf. Section 12.2. The phrasing of Q_2 i slightly different - as it is now the blue area in figure 11.2 that are under investigation. However ${}_1Q_1$ is in fact the same question as ${}_2Q_1$, but we have changed the left hand indexing to underline that now we are looking at ${}_0Q_1$ as derived from Q_2. This is because we expect the derived questions to unfold a bit differently, since students have already gained some new insight into area determination through their work with Archimedes, and some are expected to employ Archimedes result. In extension of this, we want to make a note that the right hand indexing is kept intact if the phrasing of the question has not changed.

We expected students to reject the option of using integral calculus when we posed Q_1 in the first lesson, because it is not possible from a picture to determine accurate functions. Thus, we do not expect ${}_2Q_{1.1}$ (and all its sub-questions) to be derived from Q_2. If it is posed we expect the students to quickly state that they can't use this tool, based on their work with Q_1. We do however see the possibility that all other sub-questions of Q_1 can be derived from Q_2 to some extend. In addition, we suspect that the students work with Archimedes may lead to more possible sub-questions, as presented below.

${}_2Q_1$	Which methods can I use that I am already familiar with?
${}_2Q_{1.5}$	Can I use Archimedes method to determine the area?
${}_2Q_{1.5.1}$	How is the blue figure different from the figure in Proposition 1?
${}_2Q_{1.5.1.1}$	Is the figure a segment of a parabola?
${}_2Q_{1.5.1.2}$	Can I inscribe a triangle in the figure as described in Proposition 1?
${}_2Q_{1.6}$	Can I use the law of the lever to determine the area?

${}_2A_{1.5}$: Can I use Archimedes method to determine the area?

We expect that some students will try to use Archimedes' method to determine the area. In order to do so, we expect that they will have to compare the blue figure to the figure in *Proposition 1* (${}_2Q_{1.5.1}$) in order to try and inscribe a triangle (${}_2Q_{1.5.1.2}$), which is a necessity to use the statement in the proposition. We expect that students will discover that it is not possible because based on lesson 1 and 2 they have knowledge about that we need the right assumptions in order to employ a method correctly. Archimedes' method can not be employed due to that fact that the parabola has been cut *twice*, leaving a figure with two straight lines at a 90° angle, rather than just one straight line meeting the parabola. Some students may try to extend the parabola in order to create a figure which satisfies the construction in *Proposition 1*. From this we expect some students to divide Archimedes' result by 2, i.e. $\frac{4/3}{2}$.

In conference 3.1, students are encouraged to share their ideas on how to determine this area. We expect them to be able to conclude that they can not determine the blue area in the figure presented. This leads to, that the teacher informs the students that Newton actually had a method for determining the area of this figure without. Therefore a brief introduction to Newton is made. The teacher then poses ${}_2Q_3$[7] (How did Newton determine the area, and how did he argue for it?), the students

[7] Phrased as: "Hvordan bestemte Newton arealet? Hvordan argumenterede han for det?"

work on the question in groups where they are told to investigate the part of the compendium which concerns Newtons *Rule 1*. Below, we present the questions we expect to be derived from ${}_2Q_3$:

${}_2Q_3$ How did Newton determine the area, and how did he argue for it?
${}_2Q_{3.1}$ Which methods did Newton use in his proof of Rule 1?
${}_2Q_{3.1.1}$ What is analytical geometry?
${}_2Q_{3.1.2}$ What does Newton mean by 'Curve'?
${}_2Q_{3.1.2.1}$ What type of curve does Newton present in Rule 1?
${}_2Q_{3.1.3}$ Was Newton familiar with the notion of functions?
${}_2Q_{3.1.4}$ How did Newton perform algebraic manipulations?
${}_2Q_{3.1.4.1}$ Does Newton divide by zero?
${}_2Q_{3.1.5}$ What are 'infinitely small' increments?
${}_2Q_{3.1.5.1}$ What are infinitesimals?
${}_2Q_{3.1.5.2}$ What does Newton mean when he states that 'terms multiplied by o will vanish'?
${}_2Q_{3.2}$ How did Newton argue for his method?
${}_2Q_{3.2.1}$ How does Newton proof that his method works in general?
${}_2Q_{3.2.2}$ How rigorous is Newton's proof?
${}_2Q_{3.2.2.1}$ Did Newton question the rigour of his mathematics?
${}_2Q_{3.2.2.2}$ What would a contemporary mathematician think of Newtons mathematics?
${}_2Q_{3.2.2.3}$ Does Newton divide by 0?
${}_2Q_{3.2.3}$ Why does Newton begin by proving a specific example rather than the general case?
${}_2Q_{3.3}$ When Can I use Newton's method?
${}_2Q_{3.3.1}$ How generalisable is Newton's method?
${}_2Q_{3.4}$ Where did Newton acquire his knowledge from?
${}_2Q_{3.4.1}$ What was algebra to Newton?
${}_2Q_{3.4.2}$ What did Newton mean by 'quadrature'?
${}_2Q_{3.4.3}$ Was Newton the first to work with infinitely small quantities?
${}_2Q_{3.4.3.1}$ Who worked with infinitely small quantities before Newton?
${}_2Q_{3.4.3.2}$ What similarities can we find between Newton's infinitesimals and Archimedes' indivisibles?

In conference 3.2, the students initial thoughts on ${}_2Q_3$ will be discussed in plenum. In particular, the teacher will ask if students found anything strange or noteworthy. As we can see from the SRP for ${}_2Q_3$, it has potential to go in a lot of different directions and this part of the teaching sequence will be more guided than the first part. Below, we put forth some possible answers to selected questions.

${}_2A_{3.2}$ How did Newton argue for his method?
When consulting Newton's way of argumentation, students may answer that Newton performs a direct proof with algebraic manipulations. In a discussion of Newton's argumentation, the question of

whether his argument had problems should be derived - either posed by the students or the teacher ${}_2Q_{3.2.2}$ (*How rigorous is Newtons proof?*) in order to discuss rigour pertaining Newton's *Rule 1*. In order to get closer to an answer to this questions, students may look in Newton's text to see whether he himself pointed out issues, i.e ${}_2Q_{3.2.2.1}$, (*Did Newton question the rigour of his mathematics?*) and find that he does not, as Newton states that the argument is a 'proof' without posing any concerns. In the first two lessons the students have discussed rigour regarding Archimedes, where it was evident that Archimedes was thorough in his self-criticism. In this respect, Newton stands in contrast to Archimedes. The mathematics lessons the students usually participate in are based on stating and proving facts, the didactic contract may result in the students concluding that the argument is rigorous enough when not exposed to criticism. This is in particular true for students who have a proof conception of type 3, (*relying on authority)*, as they may think that a theorem is proved simply because they are told by an authority that it holds. Other students may trust that *Rule 1* has been proved after it has been confirmed by example, which is what Newton does in the first part of his argument. This reflects a proof conception of type 2 (*based on convictions)*.

${}_2A_{3.1}$ Which methods did Newton use in his proof of Rule 1?

At this point in the teaching sequence, students have gained some familiarity with our design, and thus may tend to try and identify methods Newton might use in his argument. This we expect based upon the students work with Archimedes' text, by which the students are assumed by now to be able to state e.g. 'that the law of the lever is a method'. In this way we expect the students to search for if Newton uses the word *method* in his text, as Archimedes did. Newton uses this word in the first quote we have incorporated in the compendium in the Newton part - in which the students are expected to notice the following sentence:

> And whatever the common Analysis [that is, algebra] performs by Means of Equations of a finite number of Terms (provided that can be done) this new method can always perform the same by Means of infinite Equations.

From this quote the students should be able to identify which methods Newton states he does not use and which he does, namely a method which is based on infinite equation. We do not expect the students to be able to say anything about what this method is by now. But they should be able to identify the 'name'.

The investigation of this question is further encouraged by the information provided in the compendium. If students consult the compendium, they will also find descriptions of analytical geometry, curves and Newtons notion of functions, which gives rise to ${}_2Q_{3.1.1}$, ${}_2Q_{3.1.2}$, ${}_2Q_{3.1.3}$ being student posed. In particular, we expect that ${}_2Q_{3.1.3}$ will promote students to reflect upon the concept of a function as it is introduced in upper secondary school, and thus use history in an action oriented way. Here we expect them to be able to derive that the notation and formalism regarding functions did not just come to be by chance, but is the product of a long history of developing analytical geometry.

After conference 3.3, the teacher will pose ${}_2Q_{3.2.2}$ (*How rigorous was Newton's proof)*. In the group discussion, if the students are struggling and/or have not yet brought attention to the issue with zero division, the teacher will have pointed to the following passage during conference 3.3:

> On taking away equal quantities ($\frac{4}{9}x^3$ and z^2) and dividing the rest by o, there remains $\frac{4}{9}(3x^2 + 3xo + o^2) = 2zv + ov^2$. If we now suppose $B\beta$ to be infinitely small, that is, o to be zero, v and y will be equal and terms multiplied by o will vanish

and the groups are asked to specifically consider the mathematical rigour in this passage. In conference 3.4, we expect that some students will point to the fact that Newton divides by zero, and if not, the teacher will point to this issue. Thus, we will force ${}_2Q_{3.2.2.3}$ (*Does Newton divide by 0?)*.

${}_2A_{3.2.2.3}$ Does Newton divide by 0?

As can be seen in the SRP for ${}_2Q_3$, we expect this question can be derived from more than one path. We deliberately included the question in several places, since the lack of rigour from a scholarly point of view is a good example of how mathematics have changed over time. Dividing by 0 is a clear violation of the rules of algebraic manipulations, which the students are expected to be familiar with - they are expected to know that 'zero division is not allowed'. We expect that student will derive from the text that, yes is some way Newton divides by zero

Lesson 3 concludes with Conference 3.4 including the discussion of zero division. We start lesson 4 with a continuation of the discussion of zero division, followed by a plenum discussion of this in conference 4.1. In the following devolution, the teacher poses ${}_2Q_{3.1.5.1}$.

${}_2A_{3.1.5.1}$ What are 'infinitely small' increments?

We expect the students answer to be somewhat intuitive, understood at they probably read this section and imagine something being so small, that it in all practicality is equal to zero (but not quite). In the light of the first two lessons, the students may compare them to indivisibles, or consult the compendium to find a description of infinitesimals as infinitely small quantities. In this case, they would be able to state that: *infinitesimals are infinitely small, however they are not 0* (cf. Appendix C). Further, since students are expected to be familiar with the notion of limits, they may think of the 'infinitely small' as variables tending toward zero. Students may be able to answer that infinitesimals has functioned as a tool in mathematics before the notion of limits was properly established. Such an understanding will give rise to a discussion of how mathematics have changed over time, because there has been a need for a more rigorous foundation.

In Conference 4.2, groups share their investigations of infinitesimals. We expect it to be a difficult concept to grasp, and thus there will be an institutionalisation by the teacher. Even if the concept does not settle easily with the students, we expect that they will be able to compare their size to zero and thus raise doubts about Newton's method. The remaining part of the lesson will be centred around placing the content of the teaching sequence in a broader perspective, i.e. comparing the two historical workplaces, which they have encountered and using this to reflect upon their present tool. Thus, we now present questions derived from ${}_0Q_4$.

${}_0Q_4$	What differences and similarities are there between the methods of Archimedes and Newton?
${}_0Q_{4.1}$	Is Archimedes' method evident in the work of Newton?
${}_0Q_{4.1.2}$	Could Newton use Archimedes method?
${}_0Q_{4.1.2.1}$	Why/why not?
${}_0Q_{4.1.3}$	What could Newton do with his method that Archimedes could not?
${}_0Q_{4.1.4}$	Was Newtons method an improvement?
${}_0Q_{4.1.4.1}$	Was Newtons method more generalisable than Archimedes'?
${}_0Q_{4.2}$	Could Archimedes and Newton determine the same types of areas?
${}_0Q_{4.2.1}$	How did Archimedes describe the figures, he determined the area of?
${}_0Q_{4.2.2}$	How did Newton describe the figures, he determined the area of?
${}_0Q_{4.3}$	Is Newton more or less rigorous than Archimedes?
${}_0Q_{4.3.1}$	What is the difference between Archimedes' indivisibles and Newton's infinitesimals?

Note that we do not expect many of the derived question to arise before the written assignment. After Conference 4.2, students are asked to discuss the relation between Archimedes' indivisibles and Newton's infinitesimals, i.e. ${}_0Q_{4.3.1}$. Their answers will not be shared with the class, but are expected to form part of answers given in the assignment. Afterwards, the groups are given different assignments: All groups are asked to discuss a different view on how to determine an area, and how this is argued - Two groups are asked to answer the question in the context of Archimedes, two groups in the context of Newton, and three groups in a contemporary context. This will be discussed by the students in groups. The answers given here are not expected to be new, but this exercise is included to give the students the opportunity to make a recap of the four lessons.

The groups discussion of contemporary mathematics may go more in depth with integral calculus here. We have not discussed contemporary mathematics except for when posing Q_1 and Q_2, but we expect similar answers here, since we expected those questions - especially Q_1 - to be answered with the students' existing knowledge, of which integral calculus is a big part. For this reason, we don't present any new expected answers here.

After the groups have worked, the teaching sequence is concluded with a plenum discussion of their findings. We expect students to take notes from the other groups' presentation about the differences between the rigour and reasoning as well as methods of area determination of different times. In the end, we present the assignment that is to be handed in two weeks later.

12.4 The Assignment

We have made the assignment in such a way that a student who have attended the four lessons should be able to complete it without the need for any knowledge which have not been available at class or in the compendium. We chose to have the students answer the assignment individually in order to gather a broader data set and information about the students' individual reflections. We constructed the questions in the assignment with the intend for it to function as a reflection paper on the teaching sequence.

The focus in the in-class teaching sequence was mainly on direct work with the authentic mathematical sources and the workplaces surrounding them. In the assignment, the contemporary perspective is brought in, when the students are using history in an action oriented way. The assignment is created to function as a ladder from observer history to action history. The question of the assignment are:

- $_*Q_1$ How did Newton and Archimedes determine areas? What could they determine the area of? Here you should comment on the rigor in their arguments.[8]
- $_*Q_2$ What are the differences and similarities between Archimedes' and Newton's methods? What are the advantages and disadvantages of the different methods?[9]
- $_*Q_3$ How do Archimedes' and Newton's methods relate to the usual method you have learned in school for determining areas under curves? Is that method rigorous? (*Hint: Think of the notion of limits!)*[10]
  (Hint: Tænk på grænseværdibegrebet!) (cf. Appendix E.1)

$_*Q_1$ is intended to encourage the students to use history as observers, as it is limited to the respective historical workplaces. $_*Q_2$ is bringing the students closer to an action history use as they are asked to compare the two methods. Even though they are not actually orienting themselves in their present, the comparison prompts an action use of history. $_*Q_3$ encourages the students to actually orient themselves in their present through action history. In this way the assignment is built as a ladder with three steps of the two uses of history and a conjunction of these.

[8]Hvordan bestemte Newton og Archimedes arealer? Hvad kunne de bestemme arealet af? I skal her kommentere på stringensen i deres argumenter.

[9]Hvilke forskelle og ligheder er der mellem Archimedes' og Newtons metoder? Hvilke fordele og ulemper er der ved de forskellige metoder?

[10]Hvordan relaterer Archimedes' og Newtons metode sig til den metode, du har lært i den normale undervisning, om at bestemme arealer under kurver? Er den metode stringent?

For question ${}_*Q_1$ we expect the students to draw on the answers which they have formed in class regarding the path derived from ${}_1Q_2$ (*How did Archimedes determine the area, and how did he argue for it?*) and ${}_2Q_3$ (*How did Newton determine the area, and how did he argue for it?*). Regarding question ${}_*Q_2$ it is the path of derived question from ${}_2Q_4$ (*What could Newton do with his method that Archimedes could not?*). Though it is only briefly worked with in class, we expect the students to be able to compare the two methods independently based on the in-class work. In this question the students are explicitly comparing two different mathematicians' workplaces, which will launch their comparison of Archimedes' and Newton's methods to their own, integral calculus. In this question we expect students to employ developed answer relating to the path of ${}_0Q_1$ (*When have I argued enough for my method?*). This question can in fact shed light on how our teaching design has fostered an orientation in the rigour and reasoning regarding contemporary area determination.

Part IV

Implementation of Design

A Posteriori Analysis

Chapter 13

Introduction to Part IV

Our hypothesis is that if we can place students as observers of mathematics of the past by working directly with authentic mathematical sources, we can create an environment which fosters a discussion of mathematics of the present. We suggest that by displaying the rigour and reasoning of past mathematicians' arguments of methods in the field of area determination, we can create a milieu where the students are able to discuss rigour and reasoning in the field of area determination in their contemporary context. Thus we proposed a fluctuation between action oriented use of history and observation use of history.

In order to analyse how the realised teaching sequence has or has not supported our hypothesis, we need a theoretical tool which allows us to tune in on the role played by the authentic mathematical sources. For this purpose we will make use of the media-milieu dialectics from ATD, cf. Section 3.4. This will allow us to identify how the incorporation of both uses of history has furthered or delimited the students' reflection upon rigour and reasoning in their contemporary mathematics. For example when working with questions that relate directly to the authentic mathematical sources, which at first functions a *media*, students form personal answers $A^\heartsuit$. An example of a question which fosters observer-history use is: *When did Archimedes believe that something was proved?*. The students' answers $A^\heartsuit$ will then be available as existing answers of the *milieu* later, when the students answer action-history related questions, such as: *When do you believe that something is proved*? As such this tool can enable us to describe the interplay between media and milieu.

We will first analyse the students' work as observers of history, in order to investigate how the knowledge the students gained in this process affects their action history use. We will refer to the main teacher implementing the design as "teacher" or "visiting teacher", and to the usual teacher of the class as the "regular teacher".

Chapter 14

Data and the Realised Lesson Plans

Before we conduct the analysis of our teaching sequence we will present the data we have gathered and the realised lesson plans.

14.1 Gathered Data

The class consisted of 28 students who worked in seven groups during the teaching sequence. Our data consists of 26/28 assignments, 6/7 screen casts from lesson one, 7/7 from lesson two, 6/7 from lesson three, 3/7 from lesson four. Only two screen casts from each day have sound, and one of them only partly, as the regular teacher instructs the group on how to turn off the microphone. It appears that he thought it was not important for our research. We expect he might have instructed more groups to do the same. Therefore most of our data is based on visuals. This will be a limitation for our analysis, as we are not able to hear the students' group discussion. Furthermore we have obtained 4/7 documents with notes taken in groups during the teaching sequence. These documents will be referred to as *group notes*. Lastly we have recordings of all plenums talks during the teaching sequence, however the enumeration of the students in the transcription is reset every time we start a new transcription, thus "Student 1" in one transcription is not necessarily "Student 1" in another.

A presentation of the full realised SRP diagram will not be presented, as our limited data with sound does not establish a solid foundation and it would therefore not be representative for our study. However, we will present smaller segments of sequences related to some paths. Furthermore, because the assignment functioned as a reflection paper on the four lessons, we will incorporate students' answers throughout the analysis, rather than treat it separately.

14.2 The Realised Lesson Plans

During implementation of our teaching sequence the lesson plans changed slightly, and some things were switched around due to time constraints and the students' way of interacting with the milieu.

For example we did not foresee that including the authentic sources in the compendium in English would cause issues, as the curriculum for upper secondary school requires that students work with mathematical texts in a foreign language, preferably English (Børne- og Undervisningsministeriet, 2022), but during the implementation it became evident that the students had trouble with the language barrier. This stalled the process as the students were focused on translating the texts, some students word by word. Table 5 through 8 in Appendix B shows the realised lesson plans.

Chapter 15

Data Analysis

In Section 8.2.2, we identified and argued for that the three different regional MO's Θ_{geom} (for geometric arguments), Θ_{mech} (for mechanical arguments) and Θ_{indi} (for the use of indivisibles), of Archimedes' workplace are associated with different levels of rigour. Further, regarding Newton's workplace we identified three regional MO's in Section 8.3.2 with the theories $\Theta_{\text{an.ge}}$ (for analytic geometry), $\Theta_{\text{un.ar}}$ (for arithmetics), and finally Θ_{Newt} (for praxeologies that draws on Newtons own developed infinitesimal methods), where the rigour of the latter one is questionable. Following our proposed methodology, we designed our teaching sequence such that the students would engage tasks related to all six of these regional MO's and fluctuate between an observer history use and an actions use of history.

15.1 Students Initial Encounter with Q_1 Before Handing out the Compendium

The first lesson took point of departure in the yellow area of figure 11.1, with the teacher posed question; Q_1*: How would you determine the yellow area in figure 11.1, and how would you argue for the method you have used?*

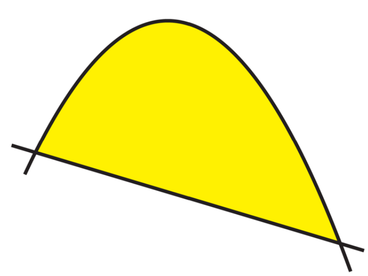

Figure 11.1: Showed to the class when posing the question: "How would you determine the area of the yellow figure, and how would you argue for it?"

The didactical system describing the initial discussion consists of

$$S(X_1, X_2, ..., X_7, Y_1, Y_2, Q_1) \rightarrow A_1^{\heartsuit}$$

where X_i represents each of the 7 groups, and Q_1, which is being studied by the groups guided by Y_1 the visiting teacher, and Y_2 the class' regular teacher who participates in group discussions.
At this point, the compendium had not been distributed to the students. Thus, the milieu brought together can be described as

$$M = \left\{A_1^{\diamond}, A_2^{\diamond}, ..., Q_1, {}_1Q_i\right\}$$

where $A_i^{\diamond}$ represents the students' existing knowledge, and ${}_1Q_i$ the questions derived from Q_1.

The students immediately posed question ${}_1Q_1$ (*Which methods can I use that I am already familiar with?*), which led to the path:

${}_1Q_{1.1}$: Can I use integral calculus to determine the area?
${}_1Q_{1.1.1}$: Can I use the definite integral?
${}_1Q_{1.1.2}$: Can I use the indefinite integral?
${}_1Q_{1.2.1}$: How can I use grid paper (i.e. squares)?
${}_1Q_{1.3}$: How can I plot the figure with digital tools?
${}_1Q_a$: How can I use ChatGpt to determine the area?
${}_1Q_b$: Can I draw a circle and divide it by two?

From this path it is evident that our generating question did not foster the branch of questions related to ${}_1Q_5$ (*When have I argued enough for my method and my result?*). Therefore it seems that the students do not draw on any existing knowledge regarding rigour and reasoning, even though it is treated in a separate chapter in their text book, cf. Section 9.3. It is also evident that the students mostly relied on their existing knowledge of integral calculus, as we expected, as they drew on existing answers such as how to calculate the area between two functions. We did not expect ${}_1Q_a$ to occur, but it is evident that some students regard AI as a tool that can be used in mathematics.

As expected, Q_1 led to a dead end, because the students did not have any existing answers $A_i^{\diamond}$ available to develop an answer, $A_1^{\heartsuit}$ to Q_1. Instead, the question fostered a plenum discussion of why the students could not use the methods they had learned from integral calculus, including what assumptions one needs to establish before using the theory from integral calculus and why they are essential. Here, we present a transcription of part of the plenum discussion from conference 1.1 (cf. Appendix D.1):

Student 1: The definite integral

Visiting Teacher: The definite integral. And what do we need to know to be able to compute the definite integral? [silence from student 1] Others can answer as well. What do we need if we want to find the definite integral here? What information do we need?

Student 2: We need to know the limits.

Visiting Teacher: yes, and what are the limits here?

Student 2: It's where those sides meet.

Visiting Teacher: mmh, Yes, and what do you think? [a third student raises their hand]

Student 3: well, we are not told that, and we don't know the function equations either.

Visiting Teacher: no [affirmative]

Student 3: ... so we actually can't.

Visiting Teacher: yes, it's a bit hard to know exactly what the expression is here, right? So... are there any other ideas on how we can do it? Yes, there in the back [a fourth student raises their hand]

Student 4: An indefinite integral then?

Visiting Teacher: But we still need to know what the function is, right? (...) But if we don't know what this function is and if we don't know how long or what the limits are with respect to the coordinate system. What can we do then? What is your best guess?

Student 5: Draw it.

The students first proposed to use the definite integral and needed a lot of guidance to realise why it could not be used. Even after it was established in plenum that the definite integral cannot be used when we don't have any information on boundaries, a student proposed to use the indefinite integral. This discussion reflects how the students regard theorems, such as the main theorem of integral calculus they know from *Plus STX A3* (cf. theorem B), as mere 'tools' they can use to solve a problem without any connection to the discourse about why. The students exhibit a 'trial-and-error' approach to mathematical problem solving in this transcription, which points to a lack of an active justification of why the theorem can be used, thus the technology tied to the task of determining areas with integral calculus is weak.

However, since the students eventually realise that the techniques from integral calculus cannot be applied, they do come to the conclusion that the use of integral calculus is not fruitful. They continue their shotgun approach by suggesting different methods, which also fall short in answering the question.

15.2 In-Class Work

After the initial investigation the students were introduced to the compendium, and it was revealed that Archimedes had found a way to determine the area. Because the teacher states that it is in not possible to determine the area with integral calculus, it is assumed that the fact is integrated in the students' milieu. The students seek other paths than integral calculus to answer Q_1, and turn to investigate the compendium which also becomes part of the milieu brought in by the teacher.

In the a priori analysis we identified four main branches derived from ${}_1Q_2$ (*How did Archimedes determine the area, and how did he argue for it?*):

- ${}_1Q_{2.1}$ Which methods does Archimedes use in the demonstration of *Proposition 1*?
- ${}_1Q_{2.2}$ When did Archimedes believe something to be proved?
- ${}_1Q_{2.3}$ When can I use Archimedes' method?
- ${}_1Q_{2.4}$ Where did Archimedes acquire his knowledge from?

Question ${}_1Q_{2.3}$ was only posed as a derived question from Q_2 regarding Newtons method. Question ${}_1Q_{2.4}$ was dealt with explicitly when students were asked to investigate Archimedes' tool box in the compendium.

Furthermore we also identified four main branches derived from question ${}_2Q_3$ (*How did Newton determine the area, and how did he argue for it?*):

- ${}_2Q_{3.1}$ Which methods did Newton use in his proof of *Rule 1*?
- ${}_2Q_{3.2}$ How did Newton argue for his method?
- ${}_2Q_{3.3}$ When Can I use Newton's method?
- ${}_2Q_{3.4}$ Where did Newton acquire his knowledge from?

Question ${}_2Q_{3.3}$ and ${}_2Q_{3.4}$ were not dealt with due to Newton's text being to difficult for the students.

The analysis of the students' in class work is structured such that we first investigate the students' observer history use and subsequently their action history use.

15.2.1 Observer History Use

In this section, we analyse how the students engage with the milieu through the lens of observer history.

Archimedes' View of Mechanical Arguments

The question ${}_1Q_{2.1.1}$ (*What is the law of the lever?*) was posed earlier than expected in the design of our lesson plan. It was planned to be content of lesson 2, but ended up being a significant point of discussion in lesson 1, when students were asked how Archimedes argued for his method. With the compendium brought in by the teacher, and the students' free use of all sources available on their computers, the milieu becomes:

$$M = \left\{A_1^{\diamondsuit}, A_2^{\diamondsuit}, ..., W_1, W_2, ..., D_1, D_2, ..., {}_1Q_{2.1.1}\right\}$$

Here, W_i describes works provided in the compendium as well as other sources students might investigate by using search engines or similar sources, and D_i describes data collected by the students, e.g. through experimenting with balancing a pencil on their finger to find equilibrium. The didactical system describing the initial investigation of ${}_1Q_{2.1.1}$ can be described as:

$$S(X_i, Y_i, {}_1Q_{2.1.1} \Rightarrow M) \hookrightarrow {}_1A_{2.1.1}^{\heartsuit}$$

${}_1A_{2.1.1}$

From the screen recordings we see that all the groups approach the question of how Archimedes argues by consulting the compendium, and most groups are quick to 'google' *law of the lever*. It seems that they believed they needed to answer *what is the law of the lever* in order to even consider answering *why can it be used*. Due to the lack of sound on the screen recordings, we are not sure what prompted the students to look into this particular notion. In one of the recordings, it seems that the group looks op the notion after reading the passage (in Danish in the compendium) "After Archimedes used the lever principle to understand how the figures relate to each other, he resorted to some already known results.[1]" This phrasing is accessible to the students, and it implies the *use* of a specific tool. It suggests that the students understand an argument by the tools that are used.

Based on our data it seems that most of the students gained an intuitive understanding of the principles of the law of the lever. The screen cast from a group showed that the students consulted Google as well as the compendium, which eventually ended up with the following group note:

> The law of the lever, is about finding the point of balance, if there is a lot of weight on one side and less weight on the other, the balance point will always be closer to the side with the heavier weight. [2]

Another group answers the question of how Archimedes argues for his method in the notes with the sentence "he talks about gravity",[3] referring to the law of the lever. This prompts the group to pose ${}_1Q_{2.1.1}$. In their investigation of this they consult the compendium before they give the answer in their group notes:

[1]Da: "Efter Archimedes har brugt vægtstangsprincippet til at indse, hvordan figurerne forholder sig til hinanden ovenfor, griber han til nogle allerede kendte resultater"

[2]Da: "Vægtstangprincippet, handler om at finde det punkt hvor der er balance, hvis der er meget vægt i den ene side og mindre vægt i den anden vil balancepunket altid være tættere på den side hvor den tungeste side er".

[3]han snakker om tyngdekraften

> Based on the idea of when something is in equilibrium.
>
> Things that are in equilibrium, at equal distances, are of equal size.
>
> There should be a smaller mass on the left side and a larger mass on the right side. The distance between "centre of gravity" and the mass A is length a. The distance between the centre of gravity and mass B is length b.(Group 7)[4]

The answer from this group is a bit more technical, but at this point, the group does not actually use the denoting of mass and lengths to anything. This is the most technical answer we find in our data in the group notes. Some of the handed in assignments do provide a more formal mathematical description. As an example, one student writes:

> Archimedes utilized the law of the lever. The law of the lever: based on when something is in equilibrium and can be described by the formula $\frac{B}{A} = \frac{a}{b}$[5]

Even though a formula is actually provided here, it does not actually come with an explanation of the meaning of A, B, a and b.

We have no data from students who both manage to give an intuitive explanation of the principle and a more formal technical description. Thus, we do not expect the students would be able to follow the deductions Archimedes makes when he solves tasks of the type presented in table REF***. However, our goal with the focus on mechanical praxeologies was not for them to understand direct application of the law of the lever, but rather gain insight into the place it has in a mechanical argument. Therefore the media brought into the milieu by us intentionally did not include a technical representation of the law of the lever, as we only intended for the students to gain an intuitive understanding of the law of the lever, in order to discuss how it affects the level of rigour in Archimedes' text. However the technical representation was brought into the milieu by the students (and in part spontaneously by the teacher as an answer to the students' questions). The desire to understand the notion as a "set of rules to follow" may be a result of the didactic contract, as they are used to mathematics being straight-forward applicable formulas.

In realising that the students were focusing more than intended on the technicalities, the teacher tried to change the focus of the discussion to what mechanical arguments are, and what their place is in relation to rigour of mathematical arguments. Therefore ${}_1Q_{2.2.1.1}$ (*What was the attitude towards mechanical arguments at the time?*) was teacher posed along with ${}_1Q_{2.2.1}$ (*What is a mechanical method?*)

${}_1A_{2.2.1}$ and ${}_1A_{2.2.1.1}$

[4] Da: "Bygger på ideen om hvornår noget er i ligevægt. Ting som er i ligevægt, i lige lang afstand, er lige store. Der skal være en mindre masse på venstre side, og en større masse på højre side. Afstanden mellem center of gravity og massen A er længden a. Afstanden mellem center of gravity og massen B er længden b".

[5] Da: 2Archimedes benyttede sig af hans vægtstangsprincip. Vægtstangsprincippet: bygger på hvornår noget er i ligevægt og kan beskrives med formlen $\frac{B}{A} = \frac{a}{b}$"

First, we briefly investigate how the students answer ${}_1Q_{2.2.1}$. A good example of this occurs in a plenum discussion (cf. Appendix D.2):

Visiting Teacher: What is mechanics? I mean, what do you think of when you think of mechanics?

Student 2: Well, it's based on a relation you can create in the real world.

Here, we see that the student can provide an intuitive answer ${}_1A_{2.2.1}^{\heartsuit}$ with their already existing knowledge of what mechanics are. The teacher confirmed the intuition an provided a few examples of mechanical methods, before asking why one might use mechanical methods in a mathematical argument, even if it is not rigorous, to which a student answers:

Student 3: Well, if you use the mechanical method, it can give an idea about if, like, if it can actually be proven.

This student shows an understanding that the mechanical method is something that can give an idea on whether or not it can be proved, indicating that they understand that proving the theorem is a process separate from the mechanical method. In general, we see from the screen casts and the group notes that some students formed a comprehension that mechanical methods did not have a place in a mathematical proof according to Greek standards. One group copy-pastes part of the preface of *The Method* into the group notes and follows it up with their own conclusion that the mechanical method is not sufficient:

> *"(...)by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration."*
>
> He tests something mechanically and then demonstrates it geometrically. The mechanical method is not sufficient to prove/demonstrate anything. [6]

This is one of several examples from the data indicating that the preface of *The Method* by Archimedes was used as media by the students to form answers ${}_1A_{2.2.1.1}^{\heartsuit}$ of the type "Archimedes did not regard a mechanical method valid in a mathematical proof," but they are not concerned with *why*. Thus the students did not develop a strong technology in relation to Archimedes' view of the mechanical method.

[6]Han afprøver noget mekanisk og demonstrerer det derefter geometrisk. Den mekaniske metode er ikke nok til at bevise/demonstrer noget.

The Discussion of Archimedes' use of Indivisibles

In order to make sure the students investigate methods in Archimedes' text related to Θ_{geom} the teacher guided the students to investigate the pages of the compendium which contain *Archimedes' tool box*. This is done to establish a foundation with which to discuss the rigour of Archimedes' method, in particular, discuss tasks related to Θ_{indi}. The groups were told to pick different geometric notions and were asked to explain how they understood them in their own words. This fostered an active investigation, and the students were very engaged with the compendium. Because of the nature of this particular task, the students posed different questions ${}_1Q_{2.4.i}$ derived from ${}_1Q_{2.4}$ (*Where did Archimedes acquire his knowledge from?*). They decomposed their existing knowledge $A_i^{\diamondsuit}$ and the new knowledge they had gained from the compendium, and then reconstructed answers ${}_1A_{2.4.i}^{\heartsuit}$. An example of this from Conference 2.1 is how a group considered a *common notion* from Euclid's elements "The whole is greater than the part,[7]" and expressed it in terms of fractions, which they are familiar with (cf. Appendix D.4):

Visiting Teacher: It sounds like you've started to select some things and have written a bit down. I'd just like to start by having one of you tell me something you've chosen.

Student 1: The whole is greater than the part.

Visiting Teacher: What does that mean?

Student 1: Uh, we were talking about fractions, something about having a whole fraction and then having a whole number, and then fractions are smaller than that.

From this transcription it is clear that the students explicitly tried to relate Euclid's definition to to their existing knowledge about fractions.

After the students worked with their chosen definitions, the teacher posed ${}_1Q_{2.2.2}$ (*Why did Archimedes believe that a proof was only valid if it was geometric?*) as it was assumed that their research and answers, $A_{2.4.i}^{\heartsuit}$, relating to selected geometric notions from Archimedes' tool box, was now available for them to use as media in their enriched milieu.

${}_1A_{2.2.2}$

Below, we present a transcription of the plenum talk of this question (cf. Appendix D.5):

Visiting Teacher: (...) Does Archimedes himself think that this is a good proof?

Student 1: But is it actually a proof then in terms of him saying that he uses like *mechanism*. Or *mechanics*?

[7] "det hele er større end en del af det"

Visiting Teacher: Yes, he uses mechanical arguments.

Student 1: to solve problems.

Visiting Teacher: Does Archimedes think that... So the question is, does Archimedes think that one can prove something with mechanical arguments?

Student 1: uhhh, yes

Visiting Teacher: Yes [skeptical], no, was my tone very leading, no not really

Student 1: no, okay, then you can't

Visiting Teacher: [another student raises their hand] what do you think?

Student 2: well he says that he first uses that mechanical method and then demonstrates it with geometry. So the mechanical method cannot really prove it. So it's not an actual demonstration (...)

Visiting Teacher: exactly. He says, he has this mechanical method and, that's correct, then he says but afterwards we shall demonstrate it with geometry. (...)

From this, it appears that the students were able to recognise the fact that Archimedes thought that proofs should be done by geometry, but only with a lot of guidance from the teacher, and their comprehensions seemed to only be based on the fact that they could read in Archimedes' preface that he had this believe. The phrases from the students like "det er ikke en faktisk demonstration" are very closed to the phrasing used by Archimedes, "the said method did not furnish an actual demonstration". They are basically just stating the facts of the text, but at this point, it still seems like some of the students struggle to connect this fact to technology.

In order to guide the students to a discussion of indivisibles, they were asked to consult the notion of a line in Archimedes tool box. The discussion was guided even further, to encouraging the students to investigate how Archimedes *uses* the law of the lever in his text, because at this point the notion of the law of the lever seemed to be somewhat familiarised for the students. ${}_1Q_{2.2.3.1.1}$ (*Can a two-dimensional figure be made up of one dimensional lines?*) was teacher posed.

${}_1A_{2.2.3.1.1}$

This investigation took point of departure in the teacher posing the question: $_1\mathbf{Q}_{\mathbf{2.2.3.1.1.}a}$: *How many lines make up the triangle for Archimedes?*, which was not planned. A transcription of a group discussion on this question is presented below:

Student 1: if we agreed that a line has a width of 0, then it must be infinite

Student 2: yes

Student 1: [calling on their regular teacher] can I ask something? isn't it just infinite? if we agreed that a line had a width of 0?

Regular Teacher: yes

This exchange shows that the students were able to apply Euclid's definition of a line - translated to their own words as 'a line has breadth 0' - and deduce the implications it has for the idea of 'adding lines'. The concept of infinity seemed to be available in some form to the students but they did not make considerations of what infinity means or how the notion was conceived by the Greeks. In addition, they quickly seek their regular teacher for confirmation, in search for validation. The regular teacher confirms that their idea is correct, which limits the possibility of further developing their answer.

From a plenum discussion in Conference 2.3 it seems that students did not have much trouble recognising issues with the notion of indivisibles (cf. Appendix D.6):

Visiting Teacher: (...) what is he [Archimedes] weighing? What are we hanging on the lever? He has something that he puts out here, yes

Student 1: Yes, in that figure he is using, figure 3, he has something THG

Visiting Teacher: Yes, and what is THG?

Student 1: it's some kind of, well, equilibrium with the triangle

Visiting Teacher: Yeees... THG itself, what kind of thing is it?

Student 2: A line

Visiting Teacher: A line, so they are lines! That's actually just it, yes, now I was hinting at something, ha, so he has a point here, where he hangs lines, right, what do we know about lines? What does a line weigh? Yes? What does a line weigh?

Student 3: It doesn't weigh anything

The conclusion from *Student 3* in this transcription is also reflected in some of the screen casts and group notes. The students could recognise that lines do not have any mass and therefore cannot be weighed. The group notes have descriptions of indivisibles of differing quality, for example: "*Noget der ikke kan opdeles. Det består måske af ordene in og devide. Så det er undividable*". Thus it appears from the in-class data that, through the compendium and teacher institutionalisation, the notion of *indivisibles* was part of the media the groups used in constructing answers ${}_1A^{\heartsuit}_{2.2.3.1.1}$. However, as we will see, this was not evident from the handed in assignments.

Newton's Method

As mentioned above, the students had difficulties reading Newton's proof of *Rule 1*. This resulted in a minimal amount of autonomous inquiry from the students, most of the text was reviewed in plenum sentence by sentence, guided by the teacher. A note from one of the groups, figure 15.1, sums up most of the students' approach to Newtons text[8].

Rule 1
Vi forstår ikke noget ☺

- Beviset forløber på følgende måde: områderne får bogstaver, og han laver en ubehageligt svær beregning lige pludselig, fordi han dividerer med 0, som man ikke kan.

Figure 15.1: Group notes

Obviously, this group is particularly discouraged, but this group note generally reflects the discouragement of the class. In a screen cast, we see that a group includes 'ChatGPT' as a media in the milieu, by copying the proof of *Rule 1* along with the prompt *translate to Danish and something understandable*[9], reflecting that they did not find answers in the compendium alone. Here, ChatGPT functions as a milieu which provides new media to be studied. Through the screen cast we see the cursor move along with the lines in the "Danish, more understandable" AI-version of the text, but the students still seem to struggle with following the argument. Therefore they ask their regular teacher to explain the AI-generated text, bringing him into the milieu.

We did not expect how difficult the students would find it to read *Rule 1*, as we expected the language to be more familiar than the language of Archimedes. Due to the general difficulties, the visiting teacher guided the students to approach the text in small steps and identify arithmetic manipulations they were familiar with (cf. Appendix D.9):

Visiting Teacher: [...] Are there any of the things he does that you can understand? Does he add something... multiply something... What does he do?

Student 1: he takes something and then divides the rest by o

[8] Da: Rule 1. We don't understand anything. The proof proceeds as follows: The areas are labeled with letters, and he suddenly makes an uncomfortably difficult calculation because he divides by 0, which one is not allowed to do.

[9] Oversæt til en dansk og noget der kan forstås

Visiting Teacher: yes, exactly. So this first sentence, I can understand, it is a bit strange. He says: 'taking away equal quantities'. What do you think 'taking away' can be translated to?

Student 2: minus

Here, a student is able to provide a translation of an operation in Newton's text to a language they understand, but only when asked directly by the teacher. We had expected the arithmetic operations to be more easily accessible to the students. Instead it turned out the milieu around Newton's text was dependant on the teacher to guide the students in constructing techniques to phrases from Newton's universal arithmetic to the modern arithmetic they are familiar with. The effort to simply comprehend the text ended up constituting a large part of the lesson plan, where the proof of *Rule 1* was discussed in plenum step by step. After explicitly pointing out in plenum that o is made infinitely small, the teacher shifted to look at potential issues with the rigour of Newtons argument, by posing ${}_2Q_{3.2.2}$ (*How rigorous is Newton's proof?*).

${}_2A_{3.2.2}$

In the initial group work, a student marked with a hand that they wanted help. Since their regular teacher was unavailable, the visiting teacher provided assistance. Reluctantly, the student asked if she was correct that Newton divided by zero, which is not allowed. The students existing knowledge (that zero division is not allowed) and the knowledge she had acquired from reading the proof of *Rule 1* was decomposed and reconstructed into an answer ${}_2A_{3.2.2.3}^{\heartsuit}$ to the derived question ${}_2Q_{3.2.2.3}$ (*Does Newton divide by 0?*), which she sought to validate against the regular teacher.

In the following plenum discussion in Conference 3.4, a student is also quick to provide the similar answer (cf. D.9):

Visiting Teacher: okay, great, let me hear what you think here. Are there any thoughts about some problems?

Student 5: he divides by zero

Visiting Teacher: does he?

Student 5: you can't do that

Visiting Teacher: and where does he do that? He says that he divides by 'o', right?

Student 5: yes, but 'o' was zero. He sets 'o' to be zero

In general, the students had no doubt about the validity of the statement "it is not allowed to divide by zero". It was a rule of mathematics that no one questioned, a rule so familiar that they were easily able to identify where Newton broke the rule explicitly.

After it has been discussed in plenum whether or not Newton were dividing by zero, the students are told that *'o'* is an infinitesimal, therefore ${}_2Q_{3.1.5.1}$ (*What are infinitesimals?*) is teacher posed.

${}_2A_{3.1.5.1}$

Below, we present a plenum talk where the students shares their thoughts on infinitesimal after a short inquiry (cf. Appendix D.10):

Visiting Teacher: okay, let me hear if you have a guess on what we are looking at with these infinitesimals? (...) Where have you tried to find answers to this question?

Student 1: Google

Visiting Teacher: Google, yes, and what shows up when you look on Google?

Student 1: that they are so small that you cannot measure or see them.

Visiting Teacher: yes, exactly, does it remind you of, yes, what do you say?

Student 2: well, it is just that you consider them as 0 but they are not quite that (...)

Visiting Teacher: yes, exactly, so you can kind of see that this is what Newton is doing here, right? Does it remind you of something? We have worked with? Something that is so small that it, yeah?

Student 3: Those indivisibles [mispronounced]

Visiting Teacher: yes, indivisibles. I think that is what you meant. Exactly, so are they the same as the indivisibles? [pause]

Student 4: No, because that was something that could not be divided. It was something like lines and such in width .... [unclear at the end]

The description of the difference between indivisibles and infinitesimals given by Student 4 is vague, but the group notes show that some groups have a more clear distinction of the notions, for example:

> Newton's infinitesimals are values that are very close to 0, but will still get larger if you add them together.
> Archimedes' indivisibles are equal to 0, so if you add them, they will not get larger, but remain 0.
> Infinitesimals are one dimension larger than indivisibles.[10].

[10]Newtons infinitesimaler er værdier som er meget tæt på 0, men stadig vil blive større hvis man plusser dem med hinanden. Archimedes indivisible er lig med 0, så hvis man plusser dem, vil det ikke blive større, men stadig 0.
Infinitesimaler er en dimension større end indivisible.

This group note shows an understanding that in their belief, a figure cannot be made up of indivisibles, but it can be made up of infinitesimals, demonstrating an intuitive understanding of the difference. However, the statement in the last line appears to be a misconception, since it is not clear how the dimensions of the two notions are comparable to each other. Rather, this is probably a fallacy derived from a plenum discussion about how indivisibles are a dimension less than the objects they "make up", where infinitesimals match the dimensions of the objects, cf. ${}_1Q_{2.2.3.1.1}$ (*Can a two-dimensional figure be made up of one dimenstional lines?*).

Ultimately, the difficulties the students had with comprehending the proof of *Rule 1* made it difficult for the them to engage in discussions about the rigour of arguments related to Θ_{Newt}. Instead we reoriented the focus by asking about the structure of the argument ${}_2Q_{3.2.a}$ (*What is the structure of Newton Rule 1?*).

${}_2A_{3.2.a}$

The question was posed to the students after group work with the part of the proof concerning the general case, and prompted the following plenum discussion (cf. Appendix D.8):

Visiting Teacher: We talked earlier about him making a rule. But is there more in the structure we can say something about? (...)

Student 1: He provides a proof for the rule with an example

Visiting Teacher: Yes, he provides a proof with an example. Exactly. So at the beginning of this session, we talked about the difference between when something is an example and when something is a proof. But what is this then? Do you have any thoughts on whether it's a proof or an example he's presenting? [long pause] when you prove something with an example.. [student raises hand]

Student 2: Then it's a demonstration. About how he uses the rule [short working pause on what happens after the example]

Visiting Teacher: He presents a rule and then he says, here is an example, I prove it with an example, and what does he do at the end?

Student 4: Provides the proof without an example

Visiting Teacher: Yes, exactly and what is it then, when it is not an example, then it is... [long pause] When I asked you to talk with a partner before. I heard at least three or four of you say it. If it's not an example, then it's more?

Student 4: General

Here, one student suggests that proving something by example is "a demonstration". This shows that the previous work with Archimedes' *Proposition 1* had become part of the milieu for this student, as the word 'demonstration' did not appear to be a frequent word in their mathematical vocabulary beforehand. It also seemed that the student expected that we were looking for an answer similar to the answers formed in lesson 1 and 2. As the teacher elicited here, several groups in the group discussion had mentioned the word 'general' when discussing the general case, but they seemed very uncertain that it was the answer "we were looking for". This is due to the didactic contract, which made it difficult to create a space where the students felt that there was not necessarily one right answer, and they were reluctant to provide an answer that they had not been able to validate. However, several students showed some understanding of how a proof could be constructed, as well as the difference between a general proof and proof of an example.

In conclusion the students did engage with tasks related to Θ_{Newt}. However they did not engage with task relating to $\Theta_{\text{an.ge}}$. Techniques relating to $\Theta_{\text{un.ar}}$ seemed to be available, but did not bring much to a discussion of rigour and reasoning in a historical context, since the rules of the arithmetics have not changed significantly since the historical episode under investigation.

15.2.2 Action History Use

The fluctuation between action and observer history use were made continuously throughout the teaching sequence. Until now, we have looked at the answers $A_i^{\heartsuit}$ the students formed in an observer history context, through working with the provided media and answering questions in context of a specific historical episode. In this section we will analyse the students' in class action history use within two episodes.

When is it Proved?

In lesson 1, after the discussion of mechanical methods and the law of the lever, the teacher shifted the focus of the class with an action use oriented question ${}_1Q_{2.2.1.1}$ (*When would I say that something has been proved?*) which was posed at the end of lesson 1. The following plenum talk from Conference 1.4 reveals some of the students' partial answers ${}_1A_{2.2.1.1}{}^{\heartsuit}$ (cf. Appendix D.3):

Visiting Teacher: When is something proven? What do you think?

Student 1: Well, we said that, uh, when something, like when something works the same way every single time it's used.

Visiting Teacher: And how can we be completely sure that it does? We can move on ... what do you think?

Student 2: When you can prove that, like, an expression is true.

Visiting Teacher: Yes, how do we do that? What methods do you use? Have you ever proven something in social studies? [students shake their heads] Why not? What do you think?

Student 3: You can't really prove anything in social studies because there can be several different...

Visiting Teacher: What about in math? There, we can. How do we do it then?

Student 4: It's about testing. It must be possible to put it to a test, and then it has to pass that test.

Visiting Teacher: So it has to pass every time. So if we, if we try, my hypothesis is that I have this bar of balance. My hypothesis is that it is in equilibrium if these conditions I wrote up before hold. How many times do I need to test it before we are sure it's correct? If I try twice and it works, is that proof? [students shake their heads] How many times do I need to do it then? Is there a number?

Student 5: No, but if there is no doubt anymore [inaudible]

This plenum talk, along with the group notes, reveals that the students' ${}_1A_{2.2.1.1}{}^{\heartsuit}$ vary. In the a priori analysis, we presented four types of expected proof conceptions. As exemplified in the final line of the transcription, some students demonstrate a comprehension of type 2, *based on convictions*. They also understand that proving is specific to mathematics. The transcription also exhibits an unexpected conception of proofs, namely *based on testing*. A student suggests that for something to be proved, it has to be subject to a test and pass. This conception is similar to how hypotheses are tested in fields of science outside of mathematics. In one of the group notes includes the following two criteria for a proof:

- if you use letters in the formula instead of numbers. Numbers are used in examples/demonstrations [11]
- Logically compelling arguments: rules. An accepted rule can imply something else [12]

The second bullet adheres to the identified conception 1, *based on understanding of standards*, however, the way it is phrased suggests that it has been added after an institutionalisation of the notion of mathematical proofs by the teacher. The first bullet is related to the same conception, yet much more vague. In general, it seems that the students have trouble consolidating their thoughts regarding the definition of a proof and articulating it, and we conclude that the students did not have the assumed knowledge of proofs in mathematics.

[11] hvis man bruger bogstaver i formlen i stedet for tal. Tal bruges i eksempler/demonstrationer

[12] Logisk tvingende argumenter: regler. En accepteret regel kan medføre noget andet

After this realisation lesson 2 was changed to start with an institutionalising some proof techniques, including how definitions and and axioms can be used to prove a theorem through logical deductions. The notion of rigour was also institutionalised, as the students had never even heard the word before. At this point, it was clear that the existing knowledge the students had available in the milieu was much weaker than expected.

Archimedes' Method Became a Part of Their Media

At the end of lesson 2 the blue area, i.e. figure 11.2, were shown to the students, and they were asked how they would determine the area. This question was revisited again in the beginning of lesson 3.

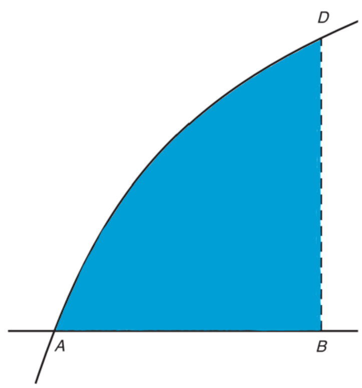

Figure 11.2: Showed to the class when posing Q_2

At this point the students expected available knowledge is both integral calculus and the students new knowledge regarding Archimedes' method in *Proposition 1*.

From a transcription of the students plenum talk in Conference 2.4 about how they would determine the area we can see that they in fact did consider Archimedes' result in *Proposition 1* as an available technique (cf. Appendix D.7).

Student 1: We thought about something with a right-angled triangle, um, inside this one, and then we have at least some of the area, in some way, and then a bit like the other one where we multiplied by $\frac{4}{3}$, so we multiplied by something again.

Visiting Teacher: Yes, can we do it like Archimedes? Can we multiply by $\frac{4}{3}$ here? No no, but um... Well, does anyone have any thoughts, why can we do it, why can't we? What needs to be satisfied for us to use what Archimedes said? Yes?

Student 2: I guess it should be a parabola.

Visiting Teacher: Yes! Is this a parabola?

Student 2: No.

Here we see that one of the groups tried to apply Archimedes' method to the problem. However, the group suggests to inscribe the figure with a right triangle, which is at odds with Archimedes method. It appears that the technique is not accompanied by a strong technology, as they do not consider which assumption one needs to have in order to apply it. The choice of inscribing with a right triangle could be based on the fact that the straight sections on the blue figure form a right angle, so if one wants to inscribe with a triangle, a right one would be an obvious choice. The students change the technique (i.e. type of triangle) slightly for the technique to fit the situation without any justification. It is not until the teacher asks about assumptions necessary for the proofs, a student recognises that it needs to be a parabola.

The plenum discussion of Q_2 is quite brief, because it is so similar to Q_1. However, we had expected that the students would focus more on the assumptions needed to answer Q_2 in light of the reflections they made when answering Q_1, where they were encouraged to reflect explicitly on the assumptions one makes when using a theorem in mathematics. This did not seem to be the case.

In conclusion these two episodes shows two different sides of the how the students action history use came to be in class. The first episode show that because the students did not have the expected prior knowledge, they are missing something in the milieu to validate their answers against. We had designed the milieu in a way where the students should validate their 'observatory' answers using their existing knowledge of proofs within mathematics. In this historical context the students' "decomposing and reconstruction" of the notions rigour and reasoning in historical sources is validated against their existing knowledge regarding rigour and reasoning. However, the students do not have a sufficient amount of prior knowledge. The second episode shows the students are in fact able to identify a technique when reading authentic sources, here they regard Archimedes *Proposition 1* as a 'rule they can apply when they are investigating another area. Furthermore it also seems as they have not corrected their trial and error approach after their initial encounter with Q_1, where they developed answers as to why they could not use their existing knowledge of integral calculus.

15.3 The Assignment

The assignment was constructed as a reflection paper upon the four lessons. It was designed for the students to be able to provide answers after the conclusion of lesson 4, without bringing in new media to the milieu. Therefore, we investigate how the answers $A_i^{\heartsuit}$ formed by the groups in-class have been reconstructed into individual answers, which we will denote $A_i^{\heartsuit}$, in the assignment.

We have selected two of the students' assignments in order to represent different types of comprehension of rigour and reasoning. In the selected assignments, we investigate how the students fluctuate between observer and action uses of history, as to see how the student validates the historical sources against their comprehension of rigour and reasoning.
An interesting observation in the data is that is the students varying conceptions of rigour in modern mathematics. We look more into this in the discussion.

15.3.1 Assignment 1

In assignment 1, the author exhibits some familiarity with the reasoning behind integral calculus as it is taught in upper secondary school, and associates the notion of limits with rigour. Further, they understand rigour in opposition to being abstract.

It appears in the assignment that the author constructed an answer ${}_2A_{2.3.2}{}^{\heartsuit}$ to the question *How did Newton argue for his method?*, as they write:

> In *On Analysis of Infinite Equations* Newton describes, how the area under simple curves can be determined, using what we now know as integrals. He deal with infinitesimal quantities, which are infinitely small quantities that are not equal to zero. (cf. Appendix E.2)

The answer here is brief, but concise. Further, the author understands that Newton does not in fact have the notion of limits available as a tool in his workplace. This is evident later, when the author exhibits an awareness of the intuition behind the Riemann integral, which enables them to compare Newton's method to the notion of limits:

> In comparison to the integral calculus that [regular teacher] has taught us, many similarities can be observed especially with Newton's method, for example with the many terms, whose sum gives the area under the curve. Now we use the limit value to let a parameter approach infinitely close to 0, and see what it does to the other terms. (cf. Appendix E.2).

From the two passages, it appears the author relates Newton's method to integral calculus. However, the author was able to consider the use of infinitesimals related to Θ_{Newt} through an observer history use, and subsequently compare the method to their existing knowledge on the notion of limits $A_i^{\diamondsuit}$ through an action history use. This reflects the fact that the techniques provided by Newton in *Rule 1* have not changed notably, but the reasoning behind the techniques is different today.

In comparison, many of the other assignments reflects that the authors do not understand the distinction between Newton's reasoning with infinitesimals and contemporary reasoning with limits, as an example seen here:

> Compared to the method we use nowadays, which is integral calculus, one could say that Newton's method resembles much more the one we use today. We also use schemes which contain formulas that we can use to calculate the area. Here we also use limits, which makes this method a little less rigorous, because it can never become 0, but very close. (cf. Appendix E.4).

Returning to assignment 1, another observer/action fluctuation is apparent in the authors work with Archimedes. They elaborate in thorough detail on Archimedes' argumentation and application of *Proposition 1*, followed by:

> Archimedes demonstrated his theory, but he did not strictly prove it. We have to assume it is true, although not really rigorous. He also used the law of the lever, so the theory is only true, if this principle is as well. (cf. Appendix E.2).

The author is able to follow Archimedes' argumentation on Archimedes' own terms, reflecting observer history use, and adds his own reflections on when the argument holds - namely, when the *assumptions* hold. However, the author does not associate the notion of indivisibles to the rigour of the argument as we would have expected due to it being treated in-class through question ${}_1Q_{2.2.3.1.1}$. The notion is only apparent in the assignment through a comparison of Newton's and Archimedes' methods:

> The methods resembles each other by breaking figures into smaller parts, like Archimedes does with the parallel lines. The disadvandtage of Archimedes is that it is very abstract. (cf. Appendix E.2).

Here, the author seem to understand Archimedes' use of indivisibles as a partitioning of the figure, and states it as a disadvantage that it makes the argument "abstract". It may be that the student has an understanding of the word "abstract" as meaning "not rigorous", in which case it reflects that the student has a comprehension of "rigour" as being "understandable" or "accessible".

15.3.2 Assignment 2

The author of assignment 2 displayed a comprehension of the rigour of an argument as connected to how understandable and generalisable they find the argument. This is evident from their comment on the rigour of Archimedes' argument:

> In terms of rigour, Archimedes uses too much text without using formulas, which makes it not understandable and harder to read. Furthermore is the text written with many letters, which makes it not understandable without a visual reference. (cf. Appendix E.3).

The author does not regard Archimedes' argument as rigorous because they find it hard to read and because it contains 'a lot of letters'. Furthermore this author does not find Newton's argument rigorous, because Newton does not explain infinitesimals 'properly':

> Newton's argument is not rigorous, as he simply does "something" without explaining what he does. For instance, he does not explain 0=infinitesimal quantities which makes his argument unclear. (cf. Appendix E.3)

The author regards rigour as dependant on how explicit the author of the text is about their considerations and reasoning throughout the argument. In the quote above, they criticise Newton for not disclosing the reasoning behind conventions he makes use of, i.e. for not providing any justification for the use of infinitesimal, rather than criticising the use of infinitesimals because the mathematical foundation for them is questionable. Thus, we get insight into the authors answer ${}_0A_{5.2.2}{}^{\heartsuit}$ (*When would I say that something has been proved?*): To the author, Newton is not rigorous because he is not persuasive. The student has a conception of something being proved *based on convictions*, and the author is not convinced, because they are not familiar with the arguments used. This is reflected in their answer ${}_2A_{3.1.5.1}{}^{\heartsuit}$ to the question *What are infinitesimals?*.

The author exhibits an intuition that "a curve is partitioned in columns of infinitesimals":

> He [Newton] calculated the area by dividing the curve into two, z and ov, where he made o to be infinitesimal (dividing the curve into column of infinitesimal) (cf. Appendix E.3)

However, more details on the notion is not provided in the assignment, and they do not discuss whether they could be rigorous in the context of a mathematical argument, it is just stated that Newton makes no justifications. It is also evident that the author has a misconception on the notion later, when they state that Newton uses the notion of limits in his argument:

> When we are determing area under curves in our teaching, we use the definite integral as well as digital tool such as Nspire. In addition to the above methods, we also typically use three-step rule [3-trins-reglen] (limits) just as Newton did. (cf. Appendix E.3).

As such they are not able to distinguish between infinitesimals and limit values.
Regarding question ${}_0Q_{5.1}$ (*What does it mean for something to be rigorous?*) we can in this assignment see that the author has formed an answer, ${}_1A_{5.1}^{\heartsuit}$, that something is rigorous if it works the same every time as stated here:

> The method we use in the lessons, is rigorous because we can do the same again and again and still obtain the same result. (cf. Appendix E.3).

The student reflects that generalisability is the main indicator for rigour in a mathematical argument. We see the same conception of rigour in several other assignment.

15.3.3 Rigour in General

In both the assignments considered above, the students associates rigour of an argument with it being *understandable*. The first author conceives rigour in relation to the mathematical notions used (e.g. the

notion of limits, which this author accepts as rigorous), and the other is more concerned with whether or not the argument is explicitly layed out.
Other conceptions of rigour can be found throughout the assignments. Several students state that arguments are rigorous if they use 'logically compelling arguments' (DA: logisk tvingende argumenter), as we can see here:

> This method [integral calculus] is rigorous because it is logically compelling arguments which is allowing us to e.g. use integral calculus to determine area under curves, which has also been proven which makes it rigorous. (cf. Appendix E.5)

However, we see no explanation of what the students mean by 'logically compelling arguments', and we have several examples of the phrase being misconceived, for instance:

> Archimedes argument is not rigorous. He used, among other things, the logically compelling argument: a line is a lenght without breadth. (cf. Appendix E.6)

In this case, the author even associates 'logically compelling' to *not* being rigorous, which is at odds with the common conception of logically compelling arguments.

Some assignments have very vague descriptions of what rigour means, as here:

> This method [integral calculus] is rigorous because it is done in a completely correct way. (cf. Appendix E.7)

One student exhibits a conception of rigour very close to our own, and what we wanted to communicate:

> The methods usually taught in lesson for determining areas under curves using definite integrals, can be considered rigorous. This is because they er based on existing mathematical theories and concepts, which is defined and thoroughly proven. (cf. Appendix E.8)

15.4 In Conclusion

From the a posteriori analysis held against the a priori analysis it is evident that some parts of our teaching design was too difficult for the students, especially their work with Newton's *Rule 1*. The difficulties led to a lot fewer question posed by the students than we expected in the a priori analysis. Based on the data we have gathered we do not have sufficient material to create the full SRP paths for any of the four lesson. If we compare our planned lesson plans with the realised it is evident that the realised teaching sequence deviate quite a bit regarding lesson 3 and 4 due to challenges with understanding the text. Lesson 1 and 2 were revised due to an unexpected premature discussion of the law of the lever, which we welcomed.

An essential assumption for the fluctuation between action use of history and observer use of history, was that the students had some prior knowledge regarding proofs within mathematics. We assumed that the students would validate their inquiry of the authentic historical sources against their prior knowledge, in such a way that it would create a reflection upon their existing knowledge of proofs. In some parts this fluctuation did in fact create a reflection upon the rigour and reasoning in contemporary mathematics, however it mostly provided insight into the students misconception of what rigour and reasoning is within mathematics.

Part V

Discussion and Conclusion

15.5 Discussion

15.5.1 What are our Findings?

In the content analysis we found that rigour and reasoning were treated in explicit chapters of the textbook used by the test class, but in the proofs of theorems in the textbook the method was not explicated in details. We considered a proof which followed the axiomatic-deductive method, but the many of the theorems it drew deductions from were implied. Thus, it would require a strong logos-block to justify the steps of this proof. We were able to identify some challenges of how rigour and reasoning is treated in upper secondary school.

Our teaching sequence was designed under the essential assumption that the students had existing knowledge about rigour and reasoning related to area determination. This assumption relied on our content analysis, which led to an expectation that the students were familiar with the axiomatic-deductive method, and would associate logical arguments following this method with a high level of rigour. We knew from the students' regular teacher that they had not worked with rigour and reasoning independently, but that they were familiar with proof techniques. Their text book also followed the structure of the axiomatic-deductive method. However, only few students from the class demonstrated familiarity with the axiomatic-deductive method. Further, it was evident in our a posteriori analysis that the students did not have a strong foundation for discussion of rigour and reasoning. The word "rigour" (DA: stringens) was unfamiliar and had to be institutionalised in-class. The significant deviation of the students' actual knowledge from the expected prior knowledge meant that our designed teaching sequence did not foster a level appropriate learning environment regarding rigour and reasoning.

Through teacher institutionalisation and following group discussions, the students were able to develop partial answers $A_i^{\heartsuit}$ to questions about proof techniques and rigour. In this way part of the initial teaching sequence was changed on the spot to accommodate the students level. In-class the students were able to reflect upon rigour and reasoning regarding the original sources with guidance from the teacher. However, in the assignments, the students' answers $A_i^{\heartsuit}$ still exhibited varying comprehensions of the meaning of mathematical rigour, and most students associated rigour with accessibility, generalisability or applicability.

Our data does not provide information on whether individual students have developed a stronger conception of rigour after participating in the teaching sequence, but by asking the students to comment on the rigour of methods from historical episodes compared to the rigour of contemporary episodes, we have provided a frame in which the students' comprehension of rigour and reasoning is apparent. For example the following quote form an assignment:

> The modern method of calculating the area under curves is rigorous, as it is easy to put numbers into and a lot less complicated to understand. (cf. Appendix E.9)

By working with historical episodes, this student has built a reference to compare modern mathematics against, i.e. as "less complicated". It seems the students had trouble applying the methods from the historical episodes, which helps bring attention to the 'plug and play' approach to problem solving the student exhibits here. This demonstrates a lacking connection between praxis and logos block, and that their comprehension of rigour is connected to applicability rather than logical deduction.

The example above displays an advantage of the inclusion of authentic historical sources, and leads us to the question of whether teaching rigour and reasoning in relation to area determination could just have well been taught without the historical angle. With respect to this, we have found that the students' inquiry and reflection upon the sources created an arena in which their comprehension of rigour and reasoning became apparent. The notions of rigour and reasoning can be difficult to articulate for students, without boiling it down to another 'set of rules' to follow. In a "normal" mathematics teaching setting, there is "one right answer" to the problems that are posed. Since this 'one right answer' has appeared differently in different historical episodes, the work with authentic mathematical sources created an environment in which the students could participate in reflective discussions, which is rare in mathematics. Furthermore we found that introducing students to standards of rigour and reasoning from different historical episodes, they got insight into the fact that it has changed over time, which encouraged the students to reflect upon their own conception.

The fact that inquiry of original sources has indeed fostered a reflection on some aspects of mathematics is evident from some of the students' assignments. For example, one student reflects on how mathematics is not a timeless field:

> Når man i dag skal finde arealet for en valgt figur, tænker vi slet ikke over, hvordan de forskellige teorier og formler er blevet udarbejdet løbet af årene. Det har taget mange år at udvikle de forskellige selve idéen om et areal og det stammer faktisk helt tilbage til omkring 30.000 år f.v.t. (cf. Assignment E.4)

Here, the student exhibits a comprehension that epistemic techniques and objects change over time. Similarly, a student shows that they understand how the mathematics of different historical episodes are dependant on the tools available:

> Her bruger jeg værktøjer som Nspire og formler, til at hjælpe med at udregne det - ting som hverken Archimedes eller Newton havde til rådighed, da de arbejdede med at finde arealer under kurver. (...) Sammenfattende var både Archimedes og Newton banebrydende inden for arealbestemmelse, men deres tilgange var forskellige på grund af forskelle i tid og tilgængelige værktøjer. (cf. Appendix E.7)

By connecting methods of area determination to the epistemic techniques available, the student is able to reflect on how digital tools are useful, but were not available when the theory was developed. We see a potential for students to understand different mathematical techniques in light of how the theory has developed rather than merely through how its associated techniques solves tasks.

15.5.2 What did our Proposed Research Methodology Offer?

Our thesis is based upon our proposed research methodology of DE within the framework of both ATD and historiography, which can be seen as an evolved conception of the established DE within the framework of ATD. Based on our findings, we argue that this extension of DE is necessary when implementing original sources in the classroom within the framework of ATD.

The existing framework of DE within ATD does not offer specific guidelines on selecting, implementing, and utilising historical episodes and authentic mathematical sources in the classroom. Hence, a tool is needed for such inclusion. It was proposed by Johansen and Kjelsen to conduct a theoretical historical analysis (Johansen and Kjeldsen, 2018) and in the light of ATD this can be done by expanding the notion of praxeologies slightly to *historical praxeologies*. With this notion, we developed a tool that can be used to identify knowledge at stake in the selected sources. It could be argued that our methodology needed to address how historical praxeologies, at a historical scholarly level, should be transposed into historical knowledge to be taught. For example it became clear from the confrontation between the a priori and a posteriori analysis that most of the students viewed Newton's reasoning of *Rule 1* with infinitesimals to be equal to the modern notion of limits. This could be due to limits being introduced in upper secondary on a highly intuitive level, and the foundational mathematics that developed in order to form a rigorous notion of limits requires a higher level of abstraction than what is expected of the students. Newtons infinitesimals are more intuitive and more accessible to the students. Our methodology did not account for how some historical praxeologies have not changed as much over time in knowledge to be taught, as it has on a scholarly level. Thus we find that it could be beneficial to more explicitly connect the historical praxeologies identified on a scholarly level to the contemporary praxeologies identified as knowledge to be taught. This could be done in addition to analysing the historical praxeologies, which is still an essential tool to incorporate in the theoretical historical analysis as this is a way to identify passages in the sources that could be used in a teaching sequence, thus orient the design.

The combination of historical praxeologies and "contemporary" praxeologies are both essential in designing a teaching sequence which fosters an observer/action fluctuation, as we have already argued in the thesis. The observer/action fluctuation created a milieu in which the students could validate new historical knowledge against existing knowledge of contemporary mathematics. Further, Epple's notions of epistemic techniques and epistemic objects is used to articulate and identify historical praxeologies that are tied to the workplaces in which authentic mathematical sources have been developed.

To our knowledge, the usefulness of implementing authentic mathematical sources in inquiry-reflective teaching within ATD has not been researched. However, since praxeologies have a historical dimension, as they develop and evolve over time, the inclusion of historiography is not at odds with the basics of ATD.

15.6 Conclusion

To conclude on the impact of using authentic mathematical sources in an inquiry-reflective environment on upper secondary school students' comprehension of rigour and reasoning in relation to area determination RQ_0, we will address the derived questions individually.

First, we consider RQ_1. The selection of original sources relevant to rigour and reasoning in area determination requires a methodological approach that integrates both ATD and historiography. We proposed a research methodology based on a theoretical analyses of historical sources, which involves analysing authentic mathematical texts from the perspective of their mathematical content and historical context. In order to conduct an analysis of the historical sources and the relevant content at stake according to the curriculum, we employed both the notion of praxeologies and the expansion historical praxeologies. Furthermore, we proposed creating an Inquiry-Reflective Learning Environment where students engage with historical sources in a way that encourages both inquiry and reflection. This environment should be based on a fluctuation between the two uses of history: action and observer.

Next, we consider RQ_2. The potential challenges in teaching rigour and reasoning in upper secondary school were determined through an analysis of the didactic transposition of rigour and reasoning in relation to area determination. The assumption that students have prior knowledge about rigour and reasoning was often not met. This gap necessitated the institutionalisation of concepts like "rigour" in class, at the expense of a full implementation of the intended teaching design. Some historical texts, such as Newton's Rule 1, were found to be too difficult for students to comprehend, leading to fewer inquiries than anticipated. The implementation of the teaching sequence did, however, shed light on some misconception the students had regarding rigour and reasoning. We argue, that we could identify the misconceptions due to the fluctuation between action history and observer history approach, as it created an environment in which the students were able to engage in a discussion about mathematics of the past.

Finally, we consider RQ_3. While there was some success in fostering reflection on the rigour and reasoning in contemporary mathematics, our analysis of the didactic transposition did not reflect the students' understanding of rigour and reasoning to a full extend. However, the approach of a fluctuation between observer and action history did help in some instances to highlight the differences between historical and contemporary practices, promoting a deeper understanding of the evolution and importance of rigour in mathematics.

In conclusion, the use of historical mathematical sources in an inquiry-reflective environment has a nuanced impact on students' comprehension of rigour and reasoning in area determination. The effectiveness largely depends on the careful selection of sources, the design of the learning environment, and addressing students' prior knowledge. The fluctuation between action history and observer history can enhance reflection, but requires a more solid foundation of students' understanding of mathematical proof techniques and concepts of rigour. The study illustrates the potential benefits and challenges of integrating historical sources into mathematics education to deepen students' understanding of rigour and reasoning.

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Appendices

A Lesson plans

Lesson 1

Time (min)	Time acc. (min)	Activity
10	10	**In plenum:** Introduction to the project. **Devolution:** Q_1 is posed by the teacher. The students are told to work on this in groups.
5	15	**Groups:** Groups are working
10	25	**Conference 1.1:** The students present answers to Q_1. **Devolution:** The challenges with answering Q_1 leads to the introduction of Archimedes and his work *The Method*. Students are asked *How did Archimedes determine the area, and how did he argue*. Students are encouraged to use compendium, and are told to look at the section concerning Archimedes (pp.2-9).
10	35	**Groups:** The students are investigating how Archimedes determined the area.
10	45	**Conference 1.2:** The groups are asked about their initial encounter with the text and how Archimedes determined the area. **Devolution:** The teacher poses the question: *When did Archimedes believe something to be proved?* The students are guided to read the preface to proposition 1.
10	55	**Groups:** Students work with the compendium and investigate the posed question.
5	60	**Conference 1.3:** The groups share their investigations, and the teacher makes sure to emphasise why Archimedes did not himself regard his text as a proof. **Devolution**: The teacher poses the question: *When would you say that something has been proved?*
5	65	**Groups:** The groups discuss the posed question.
5	70	**Conference 1.4:** The groups share their investigations, and the teacher corrects possible misconceptions.

Table 1: Lesson plan for the first lesson

Lesson 2

Time (min)	Time acc. (min)	Activity
10	10	**In plenum:** Brief recap from Lesson 1. The teacher poses the question: *Which methods do I use today when proving a mathematical theorem?* The students discuss this shortly with the person next to them, and share their initial thoughts on the matter. **Devolution:** The teacher poses the question: *Which methods does Archimedes use in the demonstration of Proposition 1?*.
10	20	**Groups:** Working on the question.
5	25	**Conference 2.1:** Groups share methods they have identified. The teacher ensures that the geometric method is contrasted with the mechanical method. **Devolution**: The teacher poses the question: *What is the law of the lever?*
10	35	**Groups:** The groups discuss the posed question.
10	45	**Conference 2.2:** The groups share their investigations of the law of the lever. **Devolution** The teacher poses the question: *What are the disadvantages of using the law of the lever?*
10	55	**Groups:** The groups discuss the posed question.
5	60	**Conference 2.3:** The students share their thoughts with the class. The teacher institutionalises the notion of indivisibles and associated issues. The groups are asked to investigate indivisibles.
5	65	**Groups/chat:** The groups discuss the posed questions.
5	70	**Conference 2.4:** The students share their investigations.

Table 2: Lesson plan for the second lesson

Modul 3

Time (min)	Time acc. (min)	Activity
5	5	**In plenum:** A brief recap of lesson 2. Now the students should be able to determine the yellow area employing Archimedes method. **Devolution:** The blue area are shown to the students and they are asked to discuss whether or not they are able to determine this area.
5	10	**Chat:** The students are discussing the question with the person next to them.
10	20	**Conference 3.1:** The students share how they would determine the area. The teacher ensures that the students understand what information they need in order to use their already known methods. **Devolution:** The teacher introduces Newton and the fact that he could determine this area. Therefore the students are asked to work with the question about how Newton did this and how he argued.
10	30	**Groups:** The groups are investigating Rule 1.
10	40	**Conference 3.2:** The students share their initial thoughts on the text and are asked if they found somethings noteworthy, strange or weird. **Devolution**: The students are asked to investigate the text focusing on what methods Newton are using.
10	50	**Groups**: The groups are working.
5	55	**Conference 3.3**: The students share what methods they identified in the text. The teacher guides the discussion towards problematic areas. **Devolution** I relation to the problematic areas the students are asked to discuss whether Newton is rigorous in the proof.
10	65	**Groups**: The groups are working
5	70	**Conference 3.4**: Discussion in plenum of rigour in Newtons proof. Teacher makes sure students are aware of Newton's division of *o*.

Table 3: Lesson plan for the third lesson

Modul 4

Time (min)	Time acc. (min)	Activity
5	5	**Plenum** Recap of lesson 3 with focus on how Newton is dividing by zero. **Devolution**: The students should now work on the question of how Newton divides by 0.
10	15	**Groups**: Group work.
5	20	**Conference 4.1**: Plenum discussion of the students work. This leads to the teacher asking about infinitesimals.
10	30	**Groups**: Groups are working.
15	45	**Conference 4.2**: Sharing and discussion of the students work regarding infinitesimals. **Institutionalisering**: Infinitesimals are institutionalised by the teacher. **Devolution & chat** Students are asked to revisit how indivisibles were defined and shortly talk with the person next to them about this matter. This will not be shared immediately with the class as this will be incorporated in the assignment. **Devolution**: The groups are assigned different views on the matter of how to determine an area and how this is argued for (Archimedes, Newton, the method they have been taught in their by their math. teacher.)
10	55	**Groups**: Groups are working.
15	70	**Conference 4.3**: Each groups presents how they would answer this question from their assigned 'perspective'. **Completion**: Introduction to the assignment and a good-bye.

Table 4: Lesson plan for the fourth lesson

B Realised Lesson Plans

Lesson 1

Time (min)	Time of day	Activity
12	9.42-9.54	**Introduction:** We introduce ourselves and the next four modules. **Devolution:** Q_1 is posed by the teacher. The students are told to work on this in groups.
5	9.55-10.00	**Groups:** Groups are working.
10	10.00-10.10	**Conference 1.1:** The students present answers to Q_1. **Devolution:** Students are asked *How did Archimedes determine the area, and how did he argue*. Students are encouraged to use compendium, and are told to look at the section concerning Archimedes (pp.2-9).
7	10.10-10.17	**Groups:** The students are investigating how Archimedes determined the area.
8	10.17-10.25	**Conference 1.2:** The groups are asked about their initial encounter with the text and how Archimedes determined the area. **Devolution:** We pose the question *What is the law of the lever?*.
10	10.25-10.35	**Groups:** Students work with the question.
8	10.35-10.43	**Conference 1.3:** The groups share their investigations on the law of the lever. **Devolution**: The teacher poses the question: *When would you say that something has been proved?*.
5	10.43-10.48	**Groups:** The groups discuss the posed question.
5	70	**Conference 1.4:** The groups share their investigations, and the teacher corrects possible misconceptions.

Table 5: Lesson 1, realised

Lesson 2

Time (min)	Time of day	Activity
7	14.05-14.12	**In plenum:**Brief recap from Lesson 1. **Devolution:** Teacher poses the question *which tools did Archimedes have available*. Students pick 2-3 tools from the tool box to discuss.
10	14.12-14.22	**Groups:** Working on the question.
4	14.22-14.26	**Conference 2.1:** Groups share their investigations. **Devolution**: Students are asked to write 2-3 things answering the question: "Which tools do you have available that Archimedes did not?"
7	14.26-14.33	**Groups:** Discussion of the posed question
14.33	14.36	**Conference 2.2:** Groups share their investigations. **Devolution (14.35):** Teacher poses the question *What possible issues do you identify with the argument in Proposition 1?*
10	14.35-14.45	**Groups:** Discussion of the posed question. Halfway through the teacher hints to look at the definition of lines.
5	14.45-14.50	**Conference 2.3:** Discussion of what is being 'weighed' with the law of the lever, as well as of lines and change of dimensions. **Devolution** Teacher poses the question *What are indivisibles?*
8	14.50-14.58	**Groups/chat:** Discussion of posed question.
4	14.58-15.02	**Conference 2.4:** Groups share their thoughts. **Devolution:** Q_2 is posed: *How would you calculate area of blue figure?*
3	15.02-15.05	**Groups:** Discussion of posed question.
5	15.05-15.10	Groups share their investigations.

Table 6: Lesson 2, realised

Lesson 3

Time (min)	Time of day	Activity
5	14.00-14.05	**In plenum:** A brief recap of lesson 2. **Devolution:** The blue area is shown to the students again and they are asked to discuss whether or not they are able to determine this area.
5	14.05-14.10	**Groups:** Discussion of posed question.
5	14.10-14.15	**Conference 3.1:** Students share what they have discussed. **Devolution:** The teacher introduces Newton and the fact that he could determine this area. Students are asked to orient themselves in the compendium.
9	14.15-14.24	**Groups:** Group investigate the compendium
6	14.24-14.30	**Conference 3.2:** The students share their thoughts and are asked to compare Newton's method to Archimedes'. **Devolution**: Students are asked to follow Newtons argumentation and write down the steps in their own terms.
10	14.30-14.40	**Groups**: Groups investigate *Rule 1*.
10	14.40-14.50	**Conference 3.3:** Plenum talk an walk through of Newton's proof in the example case. **Devolution:** The teacher poses the question *What possible issues do you identify in the proof of Rule 1?*
10	14.50-15.00	**Groups:**: Discussion of posed question.
5	70	**Conference 3.4**: Students shares their initial thoughts. Plenum discussion of rigour in Newtons text.

Table 7: Lesson 3, realised

Lesson 4

Time (min)	Time of day	Activity
5	9.40-9.45	**Plenum** A brief recap of lesson 3. **Devolution**: The students are asked to investigate the proof of the general case, and describe how Newton argues.
10	9.45.9.55	**Groups**: Groups investigate the general case.
10	9.55-10.05	**Conference 4.1**: Plenum talk and walk through of the general case. **Devolution:** The teacher asks what infinitesimals are and to locate the use of them in Newtons argument.
15	10.05-10.20	**Groups**: Group work. **Devolution:** Halfway through the group work the students are encouraged to compare infinitesimals and indivisibles.
5	10.20-10.25	**Conference 4.2:** Infinitesimals are institutionalised by the teacher. **Devolution:** Students are guided to investigate discussions of Newton's work in the compendium.
10	10.25-10.35	**Groups:** Discussion of posed question.
5	10.35-10.40	**Conference 4.3:** Students share their investigations. **Devolution**: The groups are assigned different views on the matter of how to determine an area and how this is argued for (Archimedes, Newton, the method they have been taught by their regular teacher.)
5	10.40-10.45	**Groups**: Groups are working.
5	10-45-10.50	**Conference 4.4**: Each groups presents how they would answer this question from their assigned 'perspective'.

Table 8: Lesson 4, realised

C The Compendium

Historiske ekspeditioner: Arealbestemmelse gennem tiden

Figur 1: Archimedes (venstre) og Sir Isaac Newton (højre)

Indholdsfortegnelse

1 Archimedes

Archimedes (ca. 287-212 f.v.t.) kom fra den græske by Syrakus på Sicilien, hvor han tilbragte det meste af sit liv - dog med undtagelse af et kort studieophold i byen Alexandria i Egypten, hvor oldtidens største lærdomscenter var (*Det Alexandrinske Bibliotek*). Alexandria havde også været Euklids (ca. 325-265 f.v.t) hjemsted og Archimedes studerede på sit ophold hos Euklids efterfølgere.

Archimedes var en af oldtidens største videnskabsmænd, og han arbejdede både inden for matematik, fysik og teknik. Den græske forfatter *Plutarch* (ca. 45-120) skrev om Archimedes værker at:

> Det er umuligt indenfor hele geometriens område at finde mere indviklede og vanskelige opgaver behandlet på en mere almenfattelig og enkel måde.

Vi ved, at Archimedes har skrevet mange værker både om den rene matematik og om den anvendte matematik. Vi har endda en del overleveret. Et værk, som er overleveret næsten ved et tilfælde er *Metoden*. Dette skrift blev først genfundet i 1906 af filologen Johan Ludvig Heiberg (1854-1928) og senere oversat til engelsk af Thomas L. Heath (1861-1940). I Heaths tekst indgår der noget moderne notation, som ikke har været en del af Archimedes' oprindelige værk.

Metoden blev fundet på en såkaldt *palimpsest*, som også indeholdte andre af Archimedes' allerede kendte værker, samt andre skrifter fra andre forfattere. En palimpsest er et manuskript, hvor den originale tekst er blevet fjernet, hvorefter der er skrevet en ny tekst på materialet. I Archimedes' tilfælde var hans værk blevet overskrevet med en religiøs tekst.

Figur 2: Et billede af en af siderne i palimpsesten, hvor Archimedes' tekst *Metoden* er

1.1 Archimedes' *Metoden*

Metoden er skrevet som et brev til Eratosthenes (ca. 276-194 f.v.t). Eratosthenes var på dette tidspunkt en slags øverste bibliotikar på Det Alexandrinske Bibliotek. Archimedes skriver i forordet til *Metoden* følgende tekst, hvor han gør sig overvejelser om sine metoder:

Tekstboks 1: Forord til *Metoden*

[...] Seeing moreover in you [Erastosthenes], as I say, an earnest student, a man of considerable eminence in philosophy, and an admirer [of mathematical inquiry], I thought fit to write out for you and explain in detail in the same book the pecularity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of questions, to supply the proof than it is to fing it without ane previous knowledge.
[...]
I am myself in the position of having first made the discovery of the theorem now to be published [by the method indicated], and I deem it necessary to expound the method partly because I have already spoken of it[a] and I do not want to be thought to have uttered vain words, but equally because I am persuaded that it will be of no little service to mathematics; for I apprehend that some, either of my contemporaries or of my successors, will, by means of the method when once established, be able to discover other theorems in addition, which have not yet occurred to me.
First then I will set out the very first theorem which became known to me by means of mechanics, namely that:

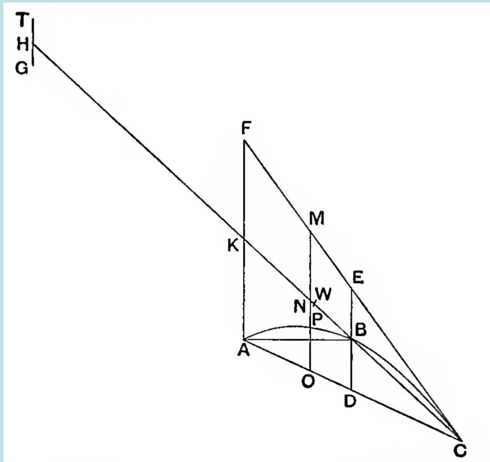
Any segment of a section of a right-angled cone (i.e. a parabola) is four-thirds of the triangle which has the same base and equal height,

and after this I will give each of the of the other theorems investigated by the same method. Then, at the end of the book, I will give the geometrical [proofs of the propositions][b]

[a] Archimedes referer til forordet i sit værk *Quadrature of the Parabola*
[b] De geometriske beviser er ikke med i dette kompendie

Fra dette forord kan vi altså se, at Archimedes bestemte arealet af et parabelsegment. Senere i *Metoden* bliver dette resultat præsenteret som *Proposition 1*, hvor Archimedes viser, hvordan hans metode kan anvendes i praksis. Figur 3 viser konstruktionen af den figur, der omtales i *Proposition 1.* Tekstboks 2 og 3 viser udvalgte dele af Archimedes' argument for *Proposisiton 1.* De dele af teksten, vi har udeladt, har vi opsummeret kort i vores eget sprog mellem tekstboksene.

Figur 3: Konstruktionen i Proposition 1



Tekstboks 2: Proposition 1

Let ABC be a segment of a parabola bounded by the straight line AC and the parabola ABC, and let D be the middle point of AC. Draw the straight line DBE parallel to the axis of the parabola and join AB, BC.
Then shall the segment ABC be $\frac{4}{3}$ of the traingle ABC.
From A draw AKF parallel to DE, and let the tangent to the parabola at C meet DBE in E and AKF in F. Produce CB to meet AF in K, and again produce CK to H, making KH equal to CK.
Consider CH as the bar of balance[a], K being its middle point.
Let MO be any straight line prallel to ED, and let it meeat CF, CK, AC, in M, N, O and the curve in P.

[a]Her benytter Archimedes *vægtstangprincippet, som han selv har udviklet*

Tekststykket, som følger efter det i kan læse i tekstboks 2 og inden tekststykket i tekstboks 3 går ud på, at Archimedes nu har konstrueret en figur, som bruges som udgangspunkt for den videre tekst. Fordi linierne CE og CD er konstrueret på en bestemt måde kan han trække på en sætning, som allerede er vist af Euklid[1] - derfor kan han konkludere at, EB er lig med BD. Og da linierne MA og MO er parallelle med ED, følger det, at også FK er lig med KA, og MN er lig med NO. Herefter gør Archimedes brug af en anden sætning, som han selv har vist i sit værk *Quadrature of Parabola*[2], hvilket leder til:

$$\begin{aligned} MO : OP &= CA : AO \\ &= CK : KN \\ &= HK : KN \end{aligned}$$

Herfra fortsætter Archimedes som i kan læse i tekstboks 3.

[1]Både Euklid og Aristaeus har arbejdet med kegler, og vi har ikke den præcise kilde Archimedes refererer til. For det videre arbejde her kan resultatet blot accepteres som sandt.
[2]Resultatet er 'Proposition 5' i dette værk

Tekstboks 3: Proposition 1, fortsat

Take any straight line TG equal to OP, and place it with its centre of gravity at H, so that TH=HG[a]; then, since N is the centre of gravity of the straight line MO, and

$$MO : TG = HK : KN$$

it follows that TG at H and MO at N will be in equilibrium about K.

[*On the Equilibrium of Planes*, I. 6,7]

Similarly, for all other straight lines parallel to DE and meeting the arc of the parabola, (1) the portion intercepted between FC, AC with its middle point KC and (2) a length equal to the intercept between the curve and AC placed with its center of gravity at H will be in equilibrium about K.

Therefore K is the center of gravity of the whole system consisting (1) of all the straigh lines as MO intercepted between FC, AC and placed as they actually are in the figure and (2) of all the straight lines placed at H equal to the straigh lines as PO intercepted between the curve and AC.

And, since the triangle CFA is made up of all the parallel lines like MO, and the segment CBA is made up of all the straight lines like PO within the curve,
it follows that the traingle, placed where it is in the figure, is in equilibrium about K with the segment CBA placed with its center of gravity at H.

[a] Lighedestegnet som vi kender det i dag, og som det er skrevet her, blev faktisk først introduceret af Robert Recorde i 1500-tallet. Det må således anses for at være en modernisering der er kommet med oversættelsen af værket efter opdagelsen i 1906.

Efter Archimedes har brugt vægtstangsprincippet til at indse, hvordan figurerne forholder sig til hinanden ovenfor, griber han til nogle allerede kendte resultater. Han ved, at hvis han tegner et punkt W på linien CK, som opfylder at CK=3KW, så vil W faktisk være tyngdepunktet for trekanten ACF. Resultat kommer fra Archimedes' værk *On the Equilibrium of Planes* I.15. Dette resultat bruger han sammen med vægtstangsprincippet til at opnå:

$$\begin{aligned} \triangle ACF : (\text{segment } ABC) &= HK : KW \\ &= 3 : 1 \end{aligned}$$

Herfra kommer han med lidt simple udregninger[3] frem til det resultat som han ville demonstrere:

$$\text{segment } ABC = \frac{4}{3}\triangle ABC$$

I Tekstboks 4 kan i læse en kommentar, Archimedes selv knyttede til sin demonstration af *Proposition 1* i værk *Metoden*.

Tekstboks 4: En afsluttende kommentar

Now the fact here stated is not actually demonstrated by the argument used; but that argument has given a sort of indication that the conclusion is true. Seeing then that the theorem is not demonstrated, but at the same time suspecting that the conclusion is true, we shall have recourse to the geometrical demonstration which I myself discovered and have already published.

[3] Hint: start med at vise at: $\triangle ABC$ er $\frac{1}{4}$ af $\triangle ACF$!

1.2 Archimedes' værktøjskasse

Da Archimedes jo levede for utrolig mange år siden, var der mange matematiske resultater, som vi kender i dag, men som endnu ikke var blevet udledt i Archimedes' samtid. Det var altså begrænset hvilke værktøjer han havde til rådighed. Her vil vi præsentere nogle af de sætninger og definitioner, som var tilgængelige for Archimedes.

1.2.1 Euklids *Elementer*

Over 13 bøger lægger Euklid et grundlag for plangeometri og rumgeometri såvel som talteori. Archimedes har været bekendt med Euklids *Elementer* og brugte hyppigt resultater herfra. Vi har samlet nogle relevante resultater for *Proposition 1* i Tekstboks 5.

Tekstboks 5: Vigtige resultater fra Euklid

Udvalgte definitioner fra Euklid's Elementer bog I

1. Et punkt er det, der ikke kan deles
2. En linie er en længde uden en bredde
4. En ret linie er en linie, som ligger lige mellem punkterne på den.
23. Parallelle linier er rette linier, der ligger i samme plan, og som, når de forlænges ubegrænset til begge sider, ikke mødes til nogen af siderne.

'Almindelige Begreber' fra Euklid's Elementer bog I

1. Størrelser som er lig en og samme tredje, er indbyrdes lige store.
2. Hvis lige store størrelser lægges til lige store størrelser, er summerne lige store.
3. Hvis lige store størrelser trækkes fra lige store størrelser, er resterne lige store.
4. Størrelser der kan dække (*epharmozonta*) hinanden, er lige store.
5. Det hele er større end en del af det.

Definitioner fra Euklid's Elementer bog III

2. En ret linie siges at berøre en cirkel hvis den, trukket så den møder cirklen, ikke skærer cirklen.
6. Et segment af en cirkel er figuren afgrænset af en ret linie og cirklens omkreds

1.2.2 Tangent

En tangent var også defineret på en noget anden måde end i dag. Tekstboks 6 viser hvordan Appolonius, som levede ca 262-190 f.v.t., beskrev en ret linje, der kan minde om en tangent - han brugte dog ikke ordet tangent til dette.

Tekstboks 6: Apollonius' definition af en tangent

Proposition 11. **[I. 17, 32.]** If a straight line be drawn through the extremity of the diameter of any conic parallel to the ordinates to that diameter, the straight line will touch the conic, and no other straight line can fall between it and the conic.

1.2.3 Vægtstangsprincippet

I værket *On the Equilibrium of Planes*[4] fremstiller Archimedes ligevægtslæren, som bygger på ideen om, hvornår noget er i ligevægt. Nogle af de nyttige sætninger fra dette værk som Archimedes trækker på i *Proposition 1* kan ses i Tekstboks 7.

Tekstboks 7: Udvalgte sætninger fra *On the Equilibrium of Planes*

1 Byrder som er i ligevægt i lige lange afstande, er lige store.

3 [Hvis] ulige store byrder i ulige lange afstand er i ligevægt, [er] den største byrde i kortest afstand.

6 Kommensurable størrelser er i ligevægt i afstande der er omvendt proportionale med deres vægte.

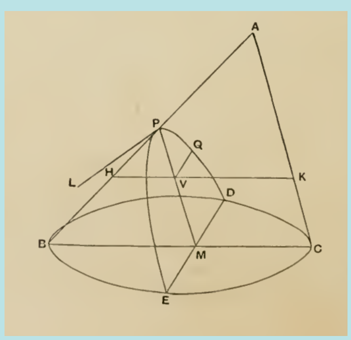
7 Men også hvis størrelserne er inkommensurable, vil de være i ligevægt i afstande der er omvendt proportionale med størrelserne.

I kan også klikke på linket *HER* for at se en animeret illustration af vægtsstangprincippet i *Proposistion 1*

1.2.4 Parabel

En parabel var defineret på en anden måde i Archimedes' samtid. Grækeren Menaechmus (ca. 375-325 f.v.t.) tilskrives opdagelsen 'kegler' og 'keglesnit', som er fundamentet for, hvordan en parabel blev defineret. Selvom overleveringen af figuren herunder er usikker, viser Figur 4 et billede af, hvordan konstruktionen af en parabel blev anset i Archimedes' samtid. Denne figur, eller noget tilsvarende, har indgået i den græske Apollonius (ca. 262-190 f.v.t) værk om kegler (*Conics*).

Figur 4: Keglesnit - her kunne det ligne, at buen EPD svarer til en parabel



[4] *Om plane figures ligevægt* på dansk

1.3 Diskussion af Archimedes' metode

Ved at læse forordet til metoden kan vi se, at Archimedes ikke selv anså metoden som værende et stringent bevis, men vi kan kun spekulere i præcis hvorfor. Måske handler det om, at der i metoden sættes en-dimensionelle liniestykker sammen til to-dimensionelle figurer. Vi har dog ikke en udtalelse fra Archimedes selv, hvor det er klart, hvad han præcis mente.

Vi har dog et citat fra Demokrit (ca. 460-370 f.v.t.), som altså levede et stykke tid før Archimedes. Demokrit skrev følgende, som peger i retning af, at der allerede dengang blev gjort overvejelser om, hvorvidt dette dimensionsskifte var meningsfuldt:

Tekstboks 8: Citat fra Demokrit

> Hvis en kegle skæres af planer parallelle med grundfladen, hvordan skal man da forestille sig lagenes overflader, ens eller uens? Hvis de er uens, så vil de gøre keglen uregelmæssig med mange trappeagtige indskæringer og ujævnheder. Hvis de derimod er ens, så vil alle lagene være ens, og keglen se ud som en cylinder, da den består af ens og ikke uens cirkler; men det er helt meningsløst.

Det dilemma, Demokrit beskriver, har været et velkendt problem gennem tiden, og det omhandler brugen af størrelser kaldet *indivisible*. Indivisible kan beskrives som værende opbygget af geometriske mindstedele. For eksempel, i citatet fra Demokrit, er der tale om, at en kegle bliver betragtet som bestående af uendeligt tynde 'skiver'. Disse skiver har ikke nogen højde, og er således to-dimensionelle, mens selve keglen er tre-dimensionel. I eksemplet her er det skiverne, vi kalder de indivisible, men på samme måde kunne man forestille sig, at en flad to-dimensionel figur består af en-dimensionelle linjer, der ikke har nogen bredde.
Meget tyder på at brugen af indivisble blev anset som et værktøj til at opdage ny matematik, men aldrig i en formel fremstilling af matematikken.

1.4 Omtale af Archimedes i oldtidens litteratur

Archimedes' intellekt har sat varige spor i historien. Vi er så heldige at have nogle tekster fra forfattere, der blot levede ca. 250 år efter hans død, som omtaler Archimedes.

Den romerske retoriker *Marcus Fabius Quintilianus* (ca. 35-98), bedre kendt som Quintilian, nævner Archimedes, i sit værk om *Talerens opdragelse*, som et eksempel på, hvorfor geometri er nyttig for unge mennesker. Quintilian skriver først bredt om geometri og lidt efter nævner han Archimedes, som eksempel på en vis mand indenfor geometrien:

Tekstboks 9: Quintilian om Archimedes

Når det kommer til geometri anerkender vi, at dele af den er nyttig for unge mennesker: den træner nemlig deres hjerner, skærper deres intelligens og gør dem hurtigt opfattende. Men fordelene kommer ikke – som ved de andre kunster – når man har fattet det, men i indlæringsfasen. Dette er den gængse opfattelse. Det er ikke uden god grund, at store mænd har brugt megen tid på denne videnskab.
[. . .]
Lad os ikke her komme ind på det der er nyttigt i krig, og lad os forbigå, at Archimedes ene mand forlængede belejringen af Syrakus. Det følgende er eminent til at illustrere det, vi forsøger at bevise: flere spørgsmål, der er svære at besvare på anden vis, løses ofte ved de 'lineære beviser' om division, om opdeling i det uendelige, samt om vækstshastighed. Så hvis en taler, som jeg skal vise i næste bog, skal tale om alt, kan han på ingen måde være en taler uden viden om geometri.

Plutarchos (ca. 45-120), bedre kendt som *Plutarch*, var en græsk filosof og historieskriver. Han forfattede en lang række værker, herunder en række mere eller mindre historisk velfunderede biografier om både græske og romerske statsmænd. Grundopbygningen i biografierne er en sammenligning af en græker og en romer. I *Marcellus biografien*[5] omtales Archimedes. Her kan vi læse:

Tekstboks 10: Plutarch om Archimedes

[. . .] Archimedes havde et så strålende intellekt, så stor en sjælelig kapacitet og så stor en rigdom af videnskabelig indsigt, at selvom han gennem sine opfindelser havde vundet sig et navn og et ry for at have en ikke menneskelig men guddommelig indsigt, ønskede han ikke at efterlade sig noget værk om disse ting. [4] Han regnede nemlig de mekaniske sager og al anvendt videnskab for ufin og håndværksagtig og havde kun ambitioner om emner, hvis geniale løsning ikke har noget at gøre med den barske nødvendighed. [5] Det er nemlig ikke muligt at finde vanskeligere og mere betydelige spørgsmål i geometrien behandlet på en mere simpel og ren form. Nogle tillægger det hans naturlige begavelse, mens andre mener, at det skyldtes hans utrættelige arbejde, at hvad end han gjorde, fremstod det nemt og let. Hvis nogen leder efter et bevis og ikke kan finde det af sig selv, men lærer det af Archimedes, får man samtidig det indtryk, at man kunne have fundet det selv. Så let og hurtigt leder han én på vej til det, der skulle bevises.

[5]En biografi, som sammenligner den thebanske general Pelopidas (fra 300-tallet f.v.t.) med den romerske imperator Marcus Claudius Marcellus. Marcellus belejrede og indtog Syrakus i 213-212 f.v.t., hvilket giver anledning til at nævne Archimedes, da han var fra Syrakus

2 Newton

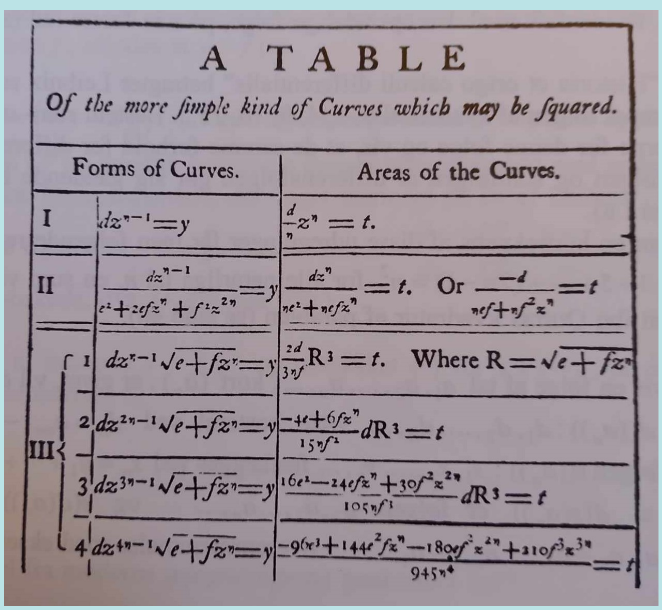
Isaac Newton (1643-1727) var en engelsk matematiker og naturvidenskabsmand. Man kan kalde ham en af de mest betydningsfulde videnskabsmænd i 1600-tallet. Foruden en enorm indflydelse indenfor fysikken, hvad angår bevægelses- og tyngdelove, har Newton også haft stor betydning indenfor arealbestemmelse.

Newton skrev værket *Analyse ved ligninger med uendelig mange led* (Oprindeligt på latin: De analysi per aequationes numero teminorum infinitas) omkring 1969, det blev dog først udgivet i 1711. I dette værk viser Newton, hvordan man kan finde arealer under kurver. Et citat fra dette værk, hvor Newton gør sig nogle betragtninger om sit arbejde kan i se i tekstboksen herunder.

Tekstboks 11: Newtons betragtning om sin egen metode

And whatever the common Analysis [that is, algebra] performs by Means of Equations of a finite number of Terms (provided that can be done) this new method can always perform the same by Means of infinite Equations. So that I have not made any Question of giving this the Name of *Analysis* likewise. For the Reasonings in this are no less certain that in the other; nor the Equations less exact; albeit we Mortals whose reasoning Powers are confined within narrow Limits, can neither express, nor so concieve all the Terms of these Equations as to know exactly from thence Quantities we want. To conclude, we may justly reckon that to belong to the *Analytic Art*, by the help of which the Areas and Lengths, ect. of Curves may be exactly and geometrically determined.

Figur 5: Et udsnit af en tabel over Newtons arealbestemmelser af kurver



A TABLE

Of the more ſimple kind of Curves which may be ſquared.

		Forms of Curves.	Areas of the Curves.
I		$dz^{n-1}=y$	$\frac{d}{n}z^{n}=t.$
II		$\frac{dz^{n-1}}{e^2+2efz^n+f^2z^{2n}}=y$	$\frac{dz^n}{ne^2+nefz^n}=t.$ Or $\frac{-d}{nef+nf^2z^n}=t$
III	1	$dz^{n-1}\sqrt{e+fz^n}=y$	$\frac{2d}{3nf}R^3=t.$ Where $R=\sqrt{e+fz^n}$
	2	$dz^{2n-1}\sqrt{e+fz^n}=y$	$\frac{-4e+6fz^n}{15nf^2}dR^3=t$
	3	$dz^{3n-1}\sqrt{e+fz^n}=y$	$\frac{16e^2-24efz^n+30f^2z^{2n}}{105nf^3}dR^3=t$
	4	$dz^{4n-1}\sqrt{e+fz^n}=y$	$\frac{-96e^3+144e^2fz^n-180ef^2z^{2n}+210f^3z^{3n}}{945n^4}=t$

2.1 Rule 1: Areal under simple kurver

Rule 1 (Regel 1) i Newtons *On Analysis by Equations Unlimited in the Number of Terms*[6] er Newtons regel for, hvordan man beregner arealet under en bestemt type af kurver, nemlig de simple kurver[7]. Herunder kan i læse, hvordan Newton opskriver *Rule 1* og hvordan han viser den for et enkelttilfælde, som er tilfældigt valgt.

Tekstboks 12: Introduktion til *On Analysis by Equations Unlimited in the Number of Terms*

The general method which I had devised some time ago for measuring the quantity of curves by an infinite series of terms you have, in the following, rather briefly explained than narrowly demonstrated
To the base AB of some curve AD, let the ordinate BD be perpendicular and let AB be called x and BD y. Let again a,b,c,... be given quantities and m,n integers. Then:
Rule 1 If $ax^{\frac{m}{n}} = y$, then will $\frac{na}{m+n}x^{\frac{m+n}{n}}$ equal the area ABD.

Efter dette viser Newton en masse eksempler på blandt andet, hvordan man kan anvende *Rule 1*, og først derefter beviser han reglen. Han begynder med at begtragte et særtilfælde, nemlig hvordan man finder arealet under den specifikke kurve $y = x^{\frac{1}{2}}$:

Tekstboks 13: Bevis for *Rule 1*: Et Eksempel

As I look back, two points stand out above all others as needning proof.
1. The quadrature of simple curves in Rule 1. Let then any curve $AD\delta$ have base $AB = x$, perpendicular ordinate $BD = y$ and area $ABD = z$, as before. Likewise take $B\beta = o$, $BK = v$ and the rectangle $B\beta HK(ov)$ equal to the space $B\beta\delta D$. It is, therefore, $A\beta = x + o$ and $A\delta\beta = z + 0v$. With these premisses, from any arbitrarily assumed relationship between x and y, I seek y in the way you see following
Take at will[a] $\frac{2}{3}x^{\frac{3}{2}} = z$ or $\frac{4}{9}x^3 = z^2$. Then, when $x + o$ is substituted for x and $z + ov$ for z, there arises (by the nature of the curve)

$$\frac{4}{9}(x^3 + 3x^2o + 3xo^2 + o^3) = z^2 + 2zov + o^2v^2$$

On taking away equal quantities ($\frac{4}{9}x^3$ and z^2) and dividing the rest by o, there remains $\frac{4}{9}(3x^2 + 3xo + o^2) = 2zv + ov^2$. If we now suppose $B\beta$ to be infinitely small, that is, o to be zero, v and y will be equal and terms multiplied by o will vanish and there will consequently remain $\frac{4}{9} \times 3x^2 = 2zv$ or $\frac{2}{3}x^2(= zy) = \frac{2}{3}x^{\frac{3}{2}}y$, that is, $x^{\frac{1}{2}}(= x^2/x^{\frac{3}{2}}) = y$. Conversely therefore if $x^{\frac{1}{2}} = y$, then will $\frac{2}{3}x^{\frac{3}{2}} = z$. [...]

[a] Læs det som 'Vælg et vilkårligt...'

Derefter går han videre til at vise, at *Rule 1* ikke blot gælder i det udvalgte tilfælde ovenfor, men faktisk gælder for alle *simple kurver*, altså kurver på formen $y = ax^{\frac{m}{n}}$:

[6] *Analyse ved ligninger med uendelig mange led* på dansk
[7] skal vi skrive hvad det er her? eller skal det udelades?

Tekstboks 14: Bevis for *Rule 1*: Det generelle tilfælde

Or in general if $[n/(m+n)]ax^{(m+n)/n} = z$, that is, by setting $na/(m+n) = c$ and $m+n = p$, if $cx^{p/n} = z$ or $c^n x^p = z^n$, then when $x+o$ is substituted for x and $z+ov$ (or, what is equivalent, $z+oy$) for z there arises

$$c^n(x^p + pox^{p-1}...) = z^n + noyz^{n-1}...$$

omitting the other terms, to be precise, which would ultimately vanish.[a] Now, on taking away the equal terms $c^n x^p$ and z^n and dividing the rest by o, there remains $c^n px^{p-1} = nyz^{n-1}(= nyz^n/z) = nyc^n x^p/cx^{p/n}$. That is, on dividing by $c^n x^p$, there will be $px^{-1} = ny/cx^{p/n}$ or $pcx^{(p-n)/n} = y$; in other words, by restoring $na/(m+n)$ for c and $m+n$ for p, that is, m for $p-n$ and na for pc, there will come $ax^{m/n} = y$. Conversely therefore if $ax^{m/n} = y$, then will $[n/(m+n)]ax^{(m+n)/n} = z$. As was to be proved.

[a] Her forudser Newton altså hvilke led han alligevel vil ende med at fjerne, fordi de indeholder o mere end en enkelt gang, og således stadig vil indeholde o efter han har divideret i gennem med et enkelt o.

Herefter beskriver Newton, hvordan hans metode kan benyttes ikke blot til at bestemme arealet under simple kurver, men at metoden også kan benyttes til at opdage kurver som kan kvadreres, ved at antage at vi kender en formel for arealet under kurven (men endnu ikke kender kurven selv). Med metoden kommer altså en høj grad af generaliserbarhed.

2.2 Newtons værktøjskasse

Newton levede altså knap 1700 år efter Archimedes og forventeligt, havde matematikken udviklet sig. Der var kommet nye teorier til, nye måder at argumentere på og nye ting man undersøgte matematisk. Udviklingen gør at Newtons værktøjskasse var anderledes end Archimedes. I dette afsnit bliver der præsenteret de største landvindinger, som spiller en rolle for Newton.

2.2.1 Infinitesimale størrelser

Måske har du hørt, at integral- og differentialregning har en fællesbetegnelse, nemlig 'Infinitesimalregning'. Det stammer fra tilbage før 1800-tallet, hvor man opfattede integral- og differentialregning som regning med uendeligt små størrelser kaldet *infinitesimaler*. Man kan også tænke på det som 'uendelig-dele'.
Det er vigtigt at forstå, at selvom infinitesimale størrelser er uendeligt små, så er de ikke lig nul, og da man ofte tænker på dem som en tilvækst eller en afstand, kan man forestille sig at de er positive (men meget meget små!) størrelser.

2.2.2 Den analytiske geometri

Analytisk geometri, også kendt som koordinatgeometri, er en generel metode, hvor man omformer geometriske spørgsmål til algebra. Denne metode bliver første gang beskrevet i 1637 af Descartes (1596-1650) i værket *La géométrie*. Descartes var en fransk fysiker, filosof og matematiker. En anden fransk matematiker, Pierre de Fermat (1601-1665), udviklede uafhængigt af Descartes også en koordinatgeometri. Den analystiske geometri var altså tilgængelig for Newton.

Descartes kritiserede faktisk det syn på geometri, som vi finder i Euklids *Elementer* og i Apollonios' værker om keglesnit. Descartes mente, at den såkaldte euklidiske geometri var for abstrakt og afhang for meget af figurbetragtninger, idet de enkelte resultater som regel krævede nye idéer.

2.2.3 Hvad er en kurve?

Før 1700-tallet betragtede man kurver som en *geometrisk* størrelse, som det fx ses når man tænker på en parabel som et snit i en kegle. Med den analytiske geometri blev det muligt at beskrive geometriske størrelser med algebraiske udtryk - således også kurver!
Det er eksempelvis det vi gør, når vi beskriver den rette linje med ligningen $y = ax + b$, eller en parabel med ligningen $y = x^2$, og da udtryk af den art kan varieres i det uendelige, åbnede den analytiske geometri op for uendeligt mange kurver, der endnu ikke var blevet arbejdet med.
For Newton var kurver dog ikke blot en geometrisk størrelse beskrevet med et algebraisk udtryk. Han opfattede kurver som et objekt i bevægelse. Newton var nemlig også fysiker - han forestillede sig en partikel i bevægelse, der efterlader et 'spor', som udgør kurven. Således vil placeringen af partiklen være afhængig af hvor lang tid partiklen har været i bevægelse. Derfor kan man tænke at han egentligt aldrig så kurven som geometrisk, men det er måske mest et filosofisk spørgmål, og i praksis endte Newton også med at beskrive sine kurver med algebraiske udtryk, ligesom Descartes og Fermat.

2.2.4 Var der et funktionsbegreb?

I Newtons tid talte man ikke om funktioner på samme måde som vi gør i dag. Gottfried Wilhelm Leibniz, som levede samtidig med Newton, var den første til at bruge ordet *funktion* om en størrelse der afhænger af en anden størrelse. Det minder meget om det funktionsbegreb vi kender i dag, men begrebet var endnu ikke færdigudviklet, og det blev det først over 100 år senere, hvor eksempelvis notationen med $f(x)$, hvor f er en funktion der afhænger af x, blev indført af Leonhard Euler. Derfor kan vi ikke sige, at det moderne funktionsbegreb var tilgængeligt for Newton.

2.3 Diskussion af Newtons metode

Selvom Newtons metode åbnede op for mange muligheder, var der problemer med stringensen, hvilket også er årsagen til at den moderne matematik griber tingene lidt anderledes an. Vi har samlet lidt citater fra mere nutidige[8] matematikere, som peger på nogle af de problemer der var med Newtons matematik.

Tekstboks 15: Senere matematikeres udtalelser om Newtons stringens

Bell (1945): "He had no approach to a limit that would be recognized today."

Kline (1953): "[Newton] ... succeeded in doing [nothing] more with the limit concept than confusing himself..."

Boyer (1959): "This is the clearest statement Newton gave as to the nature of ultimate ratios, but ... it is precisely this lack of arithmetical clarity which led to controversial discussions ... as to what Newton really meant ... The meanings of the terms ... "prime and ultimate ratio" had not been clearly explained by Newton, his answers being equivalent to tautologies ... Such an interpatation of Newton's meaning, which of course results in the ... indeterminate ratio $\frac{0}{0}$, is not unjustified."

Dieudonné (1992): "Newton [spoke] of "ultimate values of vanishing quantities," but this is only to cover up with words the imprecision of the ideas."

Også i Newtons samtid var der personer, der stillede sig skeptiske overfor metoden. Det fremgår i to citater fra Newton selv, hvor han forsøger at svare på kritikken:

Tekstboks 16: Citater fra Newton selv

It can also be contended, that if the ultimate ratios of vanishing quantities [that is, the limits of such ratios] are given, their ultimate magnitudes will also be given; and thus every quantity will consist of indivisibles, contrary to what Euclid has proved.... But this objection is based on a false hypothesis. Those ultimate ratios with which quantities vanish are not actually ratios of ultimate quantities, but limits which ... they can approach so closely that their difference is less than any given quantity...

Og:

to avoid the tedium of working out lengthy proofs by reductio ad absurdum, in the manner of the ancient geometers. ...I preferred to make the proofs of what follows depend on the ultimate sums and ratios of vanishing quantities [instead of the method of indivisibles]... For the same result is obtained by these as by the method of indivisibles, and we shall be on safer ground using principles that have been proved.

Grænseværdibegrebet

Løsningen på de problemer, der fulgte med Newtons metode, viste sig (blandt andet) at være en introduktion af grænseværdibegrebet. I stedet for at sige, at en størrelse nogle gange er nul og nogle gange ikke er nul, siger vi i dag at den bevæger sig mod nul, og undersøger hvad der så sker med andre afhængige størrelser.
Det moderne grænseværdibegrebet blev formaliseret i starten af 1800-tallet, hvor de drivende krafter var Bernard Bolzano, Augustin-Louis Cauchy og Karl Weierstrass. Notationen $\lim_{x\to\infty}$ blev dog først introduceret knap 100 år senere.

[8] Ja, 1945 er måske ikke nutidigt, men den moderne matematik var etableret i 1900-tallet.

D Transcriptions

D.1 Conference 1.1

Student 1: Det bestemte integrale

Visiting Teacher: Det bestemte integrale. Og hvad skal man vide for at kunne lave det bestemte integrale? [stilhed fra elev 1] I andre må også godt. Hvad gør vi hvis vi skal finde det bestemte integrale her. Hvad for nogle informationer har vi brug for?

Student 2: Vi skal kende grænserne

Visiting Teacher: ja, og hvad er grænserne her?

Student 2: Det hvor de der sider mødes

Visiting Teacher: mmh, Ja, og hvad tænker du? [en tredje elev markerer]

Student 3: altså det får vi ikke at vide og vi kender heller ikke forskrifterne

Visiting Teacher: nej [bekræftende]

Student 3: ... så det kan vi faktisk ikke

Visiting Teacher: ja, det er lidt svært at vide præcis hvad forskriften er her ikke? Sååå ... er der andre bud på hvordan vi kan gøre det? Ja dernede bagved [en fjerde elev markerer]

Student 4: Et ubestemt integrale så?

Visiting Teacher: Men der skal vi faktisk stadig vide hvad en funktion er jo. (...) Men hvis vi nu ikke ved hvad det her er en funktion og hvis vi ikke ved, hvor lang, eller hvad grænserne er ift. koordinatsystemet. Hvad kan vi så gøre? Hvad er jeres bedste bud?

Student 5: Tegne den

Visiting Teacher: Tegne den, og hvad så?

Student 5: ... Det ved jeg ikke, måske skal vi bruge sådan et øøh, sådan noget ternet papir? Til at tælle tern

Visiting Teacher: Så kan man tælle tern, ja, præcis. [Elev markerer] Ja hvad tænker du?

(...)

Student 7: Altså man kan jo ikke regne noget når nu man ikke har nogle af målene, så man kan bare selv angive mål

Visiting Teacher: Ja?

Student 7: så er det bare en skitse

Visiting Teacher: Så er det en skitse. Og hvor præcis bliver det så?

Student 7: Ikke så præcist…

Visiting Teacher: Men det er rigtigt, vi kan godt komme langt uden rent faktisk at have et godt værktøj eller uden at have en funktion, ved bare at måle og bare lave nogle tern og prøve på at se om vi kan komme tæt på, ikke?

D.2 Plenum Discussion A

Visiting Teacher: . . . før der snakkede vi om hvad et geometrisk og hvad et mekanisk argument. Er der nogle af jer der er kommer frem til noget her?

Student 1: Altså, et, det her er et mekanisk argument fordi at vi bruger altså andre værktøjer i matematik til i hvert fald at tælle til noget** eller forklare noget

Visiting Teacher: Andre værktøjer end hvad?

Student 1: øh, andre hvad?

Visiting Teacher: du sagde andre værktøjer

Student 1: altså f.eks. sådan noget at vi ved at, ja, altså sådan noget 'a' er lig med 3 gange 'b' .. altså sådan vi bruger matematik til . . . [kan ikke lytte]

Visiting Teacher: Ja det er rigtig nok, det giver god mening. Hvad tænker du bag ved?

Student 2: Jamen også bare sådan noget med. Altså. Han starter med ligesom at forklare hvad han gør og så til sidst så ser vi der her sådan billede, geometrisk, figuren, hvor det ligesom er visualiseret. Jeg tænker det er lidt det, der er forskellen på mekanisk og geometrisk.

Visiting Teacher: ja det kan man godt sige. Men han kommer faktisk aldrig, i den tekst i har. Der kommer han faktisk aldrig til det rigtige geometriske argument. Hvad tænker du?

Student 3: Altså jeg tænker bare at sådan geometrisk er mere i forhold til figure. Altså at han ligesom viser den her, hvordan man laver trekanter inden i og [kan ikke høre der bliver hostet: 1:10]

Visiting Teacher: Ja, præcis så er der faktisk også geometri i det her ikke? Fordi der er nogle figurer og nogle trekanter og sådan noget. Så hvad er det så præcist så det mekaniske her? Hvis vi bare lige.. vi har været lidt inde omkring det men hvad er det for en de præcist, der er mekanisk i det her argument?

Student 4: Altså er det ikke at vi ligesom benytter 'a' og 'b' og reducerer det og [*inddeler ud fra det*? Svært at høre, 1:28]

Visiting Teacher: Jo lidt, men altså det der med at man kan sige at 'a' og 'b' og de her brøker og sådan noget. Det hed ikke brøker den gang. Men det er faktisk også inden for geomtetrien. Så det er mere noget med konteksten.

Student 5: øhm, jeg tænker det mekaniske må være det der ligevægtsprincippet

Visiting Teacher: Lige præcis. Så hvorfor er det mekanisk? Hvad tænker i?

Student 5: det ved jeg ikke

Visiting Teacher: Hvad er mekanik? Altså hvad tænker i når i tænker mekanik

Student 6: altså det tager udgangspunkt i en sammenhæng man kan lave sådan i den virkelige verden

Visiting Teacher: ja, præcis. Så det er her hvor vi forestiller os at det faktisk er nogle der har været ude og undersøge nogle ting. Og sagtn nå men hvis jeg gør sådan her, og hænger noget her og så vejer det og så må man vurderer det. Det er noget mekanik. Det er noget, sådan, vi kan lave et forsøg ligesom i fysik og sådan noget, ikke. Hvor at i geomtrien som er denne her "rene matematik" og meget stringente matematik. Der, der lever vi egentlig kun i vores hoveder. Giver det mening? … Er det stadig lidt svært at forstå forskellen på det her? Okay, det er helt okay. Så øh, Ved at samle op og sige at Archimeds han mener altså at det her, det et geometrisk, nej et mekanisk argument og han har senere lavet et geometrisk argument. Men hvorfor, ikke bare starte med det geometriske? Kan se se hvad der kan være fordelen ved at starte med noget mekanisk? Hvis det ikke er rigtig matematik, hvorfor gør vi det så overhovedet?

Student 7: altså hvis man bruger den mekaniske metode, så kan det give en ide om, sådan, om det faktisk kan bevises

Visiting Teacher: lige præcis. Fordi for at vi kan bevise noget. Så skal vi vide hvad det er vi skal bevise. Det kan være svært at bevise noget man ikke ved endnu. Så her der bruger vi mekanikken til at undersøge, kan det måske passe. Og kan vi så vise det rigtig. Ja. Okay. Jeg tænker at vi lige hurtigt hopper tilbage til denne her figur [den gule]. Så hvad tænker i nu? Er der nogle, der har nogle bud på hvordan i vil regne arealet af. Før der snakkede vi om at vi kunne lave nogle firkanter og så se om hvor mange lige firkanter er her. Kan vi endnu mere præcis bestemme arealet af denne her? Og hvordan gør vi det? .... Okay to minutter med sidemakkeren så prøv lige at se, har Archimedes lært os noget til hvordan hvis vi har en lineal kan vi så regne arealet af denne her?

D.3 Conference 1.4

Visiting Teacher: Hvad har i snakket om? Hvad er i kommet frem til? Og her er der virkelig ikke nogle forkerte svar vel. Det her er virkelig bare hvad synes i. Hvornår er noget bevist? Hvad tænker du?

Student 1: Altså vi sagde at øh sådan når noget man, altså når noget fungere på samme måde hver evig eneste gang det bliver brugt

Visiting Teacher: og hvordan kan vi være helt sikre på at det gør det? Vi kan godt gå videre [til elev som lige har svaret] hvad tænker du?

Student 2: Når man kan bevise at sådan et udtryk er sand.

Visiting Teacher: ja hvordan gør vi det? Hvad for nogle metoder bruger i? Har i nogen sinde bevist noget i samfundsfag? [elever ryster på hovedet] hvorfor ikke? Hvad tænker i?

Student 3: Der kan ikke sådan bevises noget i sådan i samfundsfag for der kan være flere forskellige . . . .

Visiting Teacher: hvad med i matematik? Der kan vi godt. Hvordan gør vi det så?

Student 4: det er noget med at afprøve. Det skal være muligt at man sådan kan sætte det under en test for så skal det sådan bestå det.

Visiting Teacher: så skal det bestå hver gang. Så hvis vi, hvis vi prøver, at vi allerede siger at min hypotese er at jeg har den her vægtstang. Min hypotese er at den er i ligevægt hvis de her forhold, jeg havde skrevet op før, holder. Hvor mange gange skal jeg teste det før at vi er sikre på at det er rigtigt? Hvis jeg prøver to gange om det virker, er det så et bevis? [elever ryster på hovedet] Hvor mange gange skal jeg så gøre det? Er der et tal?

Student 5: Nej men hvis man ikke er i tvivl mere [kan ikke høre] (1:30)

Visiting Teacher: altså faktisk vil jeg bare mene at man aldrig har bevist det. Det er fordi at jeg er sådan en rigtig dum matematiker som siger at, hvis vi bare eksperimenterer og gør noget. Selv hvis vi gør det 1 million gange, så kan det godt være at der er noget der har ændret sig ved 1 million of første gang. Men jeg synes stadig der er noget der er beviser. Hvad er det i kalder beviser? Hvornår har i fået præsenteret noget som et bevis i skolen?

Student 6: Formler

Visiting Teacher: ja, formler. Så det der er med formler. Det er der siger vi at der har vi nogle antagelser. Vi antager f.eks., så er der nogle ting er gør at vi ved 100 procent at 2 plus 2 er 4. Og så når vi har de her regler, så kan vi spille under dem og så kan vi bevise noget. Okay og i må gerne gå og tænke lidt mere over det her til i morgen.

D.4 Conference 2.1

Visiting Teacher: Det lyder som om at I er begyndt at have fået valgt nogle ting og har skrevet lidt ned. Jeg vil gerne lige starte med at I bare fortæller én af de ting I har valgt?

Student 1: Det hele er større end en del af det

Visiting Teacher: Hvad betyder det?

Student 1: Øhm vi snakkede om noget brøk, noget med man har en hel brøk og så har man altså et helt tal og så brøker det er så mindre end det

Visiting Teacher: Mmmh, ja, klart, så hvordan er en... hvad er det der er en del og hvad er det der er det hele når vi kigger på en brøk?

Student 1: Jeg ved det ikke

Visiting Teacher: Hvis I har en brøk så er der noget der er det hele og noget der er en del af det

Student 1: det må så være ... [mumle mumle]

Visiting Teacher: Ja, præcis. Okay, cool. Hvad med jer der sidder her? Er det to grupper eller en?

Student 2: Ja det er to grupper ja, vi har skrevet størrelser der kan dække... så er der et eller andet ord, det forstod vi ikke... hinanden er lige store

Amanda: Kommensurable

Student 2: [mumle mumle]

Visiting Teacher: Ja, lige præcis, og hvad... minder det om noget vi kender?

Student 2: Jamen det er sådan vi tror det er på to papirer jo [??] så kan man vende om på den ene og så kan man se den dækker, jamen så må de være ens.

Visiting Teacher: Ja, præcis, cool! Og hvad med den anden gruppe her?

Student 3: Ja altså der står... Parallelle linjer er rette linjer, der ligger i samme plan, og som, når de forlænges ubegrænset til begge sider, ikke mødes til nogen af siderne.

Visiting Teacher: Ja

Student 3: Det tænker vi er meget det samme som det vi har i dag

Visiting Teacher: Helt klart! Ja det er sådan vi tænker på paralelle linjer i dag også, ikke også? Ah men helt klart! Er der nogen af jer der bare har kigget sådan på definitionen af en linje? Ja?

Student 4: det.. jeg kunne ikke helt høre om det var det du sagde men altså, ej det er jo så en ret linje, ja en ret linje.

Visiting Teacher: Jaja men, ja

Student 4: Ja, en ret linje er en linje som ligger lige mellem punkterne på den

Visiting Teacher: Ja, præcis så hvad er det der bliver sagt dér? Hvis vi [tegner] har to punkter og så prøv lige at sige igen, hvad står der?

Student 4: Ja, en ret linje er en linje som ligger lige mellem punkterne på den

Visiting Teacher: yes, okay, så, 'lige' det er også sådan en ting hvor man tænker det ved man godt hvad betyder ik? Men det kommer jo et eller andet sted fra, det betyder at vi ikke gør sådan her men vi siger det er faktisk den korteste afstand i virkeligheden ikke.. Nu kan jeg ikke tegne lige, men jeg håber I ved hvad jeg mener hehe.
Men det vi også skal vide here ik, det er hvis den ligger lige mellem punkterne på den. Men hvad er et punkt? Ja?

Student 2: Noget der ikke kan deles

Visiting Teacher: Ja! Hvad betyder det?

Student 2: Altså jeg tænker det betyder at altså det er i altså så lille en størrelse at det ikke rigtig er noget...

Visiting Teacher: Ja, præcis, så hvis jeg spørger hvor stort er et punkt, hva...? Hvor stort er et punkt? Ja?

Student 4: Det har vel ikke en størrelse, det kan være mange forskellige størrelser [?]

Visiting Teacher: Ja! Og hvad så med en linje? Hvor stor er en linje?

Student 4: Det kan også være en masse... ja... Det har i hvert fald ikke nogen bredde

Visiting Teacher: Ja, det har ikke nogen bredde, lige præcis, men det har en længde, ikke også? Så vi kan faktisk godt måle en linje. Kan vi tage arealet af en linje? Nej? Der er nogle der ryster på hovedet? Ja?

Student 4: Nej

Visiting Teacher: Hvorfor ikke?

Student 4: Fordi at arealet skal ligesom have et område hvor vi har ikke noget område bare på en enkel linje

Visiting Teacher: Nej, rigtigt, så hvis vi skulle prøve at sige det her i sådan moderne termer, hvis der er noget der ikke har nogen bredde. Ja

Student 2: Altså så er den vel to-dimensionel.

Visiting Teacher: Lige præcis! Og hvad skal den være for at vi kan tage arealet?

Student 2: tre-dimensionel

Visiting Teacher: Ja! Okay, så den her måde Archimedes han gør det på, han siger en linje det er noget der ligger mellem to punkter, er det sådan I tænker på linjer i dag også? Er det også en linje i moderne, altså for Archimedes der var en linje noget der ikke havde nogen bredde. Hvad med i dag? Er det det samme? [stilhed] Hvis I glemmer det her var et historieforløb... hvis [general teacher] bare sagde, hvor bred er en linje? Hvad ville I så svare? Hvis han havde spurgt for en uge siden. Ja?

Student 5: Nej... [mumle mumle] bred

Visiting Teacher: Nej, så er det det samme hos Archimedes? Ja! Lige præcis, så på den måde er det faktisk ikke så meget der har ændret sig vel? Okay? Nå, men det vi så kan se her, det er at hvis vi kigger på fx linjer og også mange af de andre ting så er det det samme vi arbejder med, og de matematikere der har lavet den moderne matematik de har også snakket om Archimedes og at han sagde det her, eller Euklid sagde det her, og det bruger vi stadig i dag. Men der er også nogle ting der har ændret sig siden. Så selvom at linjebegrebet er det samme, så er der noget matematik vi har i dag som Archimedes ikke kendte til. Og det er det som vi skal kigge på nu, det er prøv i grupper, prøv at sætte jer ned og snak om, hvad er det for nogle ting som I kender til, som Archimedes ikke kendte til. Er der noget som I har lært af [general teacher] i undervisningen eller i folkeskolen eller noget, er der noget I har lært til at bestemme arealer eller til andre ting, som Archimedes ikke kunne. Ja?

D.5 Plenum Discussion B

Visiting Teacher: okay så lad og prøve at se om der er nogen der er kommet frem til et eller andet spændende. Der har nogen spørgsmål. Eller et eller andet i er nysgerrige på omkring det her.

Student 1: Vi kom frem til det her med at han mener det, når noget er demonstreret så er det bevist. . . . [kan ikke høre dette]

Visiting Teacher: Hvad betyder det at noget er demonstreret

Student 1: at det sådan er vist

Visiting Teacher: og kan vi komme endnu tættere på hvad det egentlig er hvornår noget er bevist? Er det her et bevis? Det han laver. Fra det i læser her. [absolut stilhed] Okay så prøv at, nu, prøv at kigge lidt videre og prøv at sig, okay, men synes han rent faktisk selv, hvis i kigger i forordet og i må også gerne google. Synes Archimedes selv at det her er et godt bevis.

Student 2: Men er det egentlig et bevis så i forhold til at han siger at han bruger sådan mechanism. Eller mechanics?

Visiting Teacher: Ja han bruger mekaniske argumenter

Student 2: til at løse problemer

Visiting Teacher: Synes Archimedes at. Så spørgsmålet er synes Archimedes at man kan bevise noget med mekaniske argumenter

Student 2: øøøh, ja

Visiting Teacher: ja [skeptisk], nej, var min tone meget ledende, nej ikke rigtig

Student 2: nej det kan man så ikke

Visiting Teacher: [en tredje elev markerer] hvad tænker du?

Student 3: altså han siger at han først bruger det der mekaniske metode og så bagefter demonstrere det med geometri. Altså den mekaniske metode kan sådan ikke sådan bevise det. Altså det er ikke en faktisk demonstration (OBS! Dette var svært at høre)

Visiting Teacher: nej, lige præcis. Han siger nemlig, han har den her mekaniske metode og det er rigtigt og så siger han men bagefter så skal vi vise det med geometri. Så øh hvad er forskellen på en mekanisk metode og en geometrisk metode? Er der nogle der har nogle tanker om det? Hvorfor er det her mekanisk? ….. Ellers så snak lige om det hvad er forskellen på den mekaniske og den geometriske og hvorfor er det mekaniske argument ikke godt nok? Og så snakker vi videre om det. Og husk og skriv nogle noter i dokumentet også undervejs.

D.6 Conference 2.3

Visiting Teacher:	Okay, er der nogle der har nogle bud? Hey, vi samler lige op igen, så jeg kan sige så meget at jeg har hørt nogle af jer sige det rigtige, så øh, måske er det med at sige det man tænker, hvad er det han vejer? Hvad er det vi hænger i vægtstangen? Han har et eller andet som han sætter herud ja
Student 1:	Ja på den der figur han er i gang med at bruge, figur 3, der har han jo noget TGH
Visiting Teacher:	Ja, og hvad er THG?
Student 1:	det er en eller anden form for, altså, ligevægt med trekanten
Visiting Teacher:	Jaaaah... selve THG, hvad er det for en slags ting?
Student 2:	En linje
Visiting Teacher:	En linje, så det er linjer! Det er faktisk bare det, ja, nu fiskede jeg efter noget, heh, så han har noget et punkt her, hvor han hænger linjer ud i, ikke også, hvad er det vi ved om linjer? Hvad vejer en linje? Ja? Hvad vejer en linje?
Student 3:	Den vejer ikke noget
Visiting Teacher:	Nej, så hvordan kan vi hænge den ud i en vægtstang? Det virker lidt mærkeligt ikke? Så det er faktisk her hvor vi tager noget der ikke har en vægt, og så siger at det er i ligevægt omkring et eller andet, men det er måske lidt svært at snakke om på den måde ikke? Det andet han så gør det er han siger, han gør det her med alle der her linjer, ned gennem her, og så siger han nå men hvis vi tager alle mulige linjer, så har vi i princippet taget hele figuren ikke? Hvor mange linjer skal jeg lave i denne her figur før jeg har hele figuren? Okay, vi tager lige, altså 1, 2 minutter igen, prøv at overveje, hvor mange linjer skal jeg lave før jeg kan sige jeg har hele figuren? Hvor mange linjesnit ned gennem parablen skal jeg lave før jeg har hele figuren. ... I har et bud! Ja?
Student 4:	uendelig
Visiting Teacher:	Uendelig mange linjer! Fordi, og hvad så hvis vi sagde vi havde uendelig mange linjer, havde vi så hele figuren
Several Students in Plenum:	[sporadisk] Nej...

Visiting Teacher: Nej, så der er noget her der virker lidt underligt ikke også?

Several Students in Plenum: Ja

Visiting Teacher: Ja, og faktisk så er det så underligt, at selv de gamle grækere de havde et navn for hvad de her linjer var, og det er et lidt underligt navn, men de kaldte dem for indivisible, og det kan I også se i jeres kompendie. Og indivisible det er ting som der, øh, vi forestiller os ligesom at de ikke har nogen bredde, så derfor kan vi heller ikke dele dem. Ligesom vi snakkede om før så et punkt det er det der ikke kan deles. En linje, det kan jo godt deles på den ene led, men det kan ikke, nu kigger vi på linjer der, de kan jo ikke deles i bredden, så derfor så, så har vi sådan i princippet så siger vi at denne her figur er udgjort af uendeligt mange linjer, og det kan der være nogle problemer med, og det var de godt opmærksomme på, de gamle grækere. Og I kan se i kompendiet at der er en lille tekstboks fra, af, Demokrit, er det rigtigt?

Amanda: Jo, det tror jeg, jo

Visiting Teacher: som skriver om, øh, hvad han egentligt mener om de her indivisible. Og jeg, nu, prøv lige at sætte jer ned, og det er lidt svært koncept at forstå, men læs om indivisible, og I må gerne bruge Google også. Og så prøv at se i kompendiet hvad det er som der, denne her anden græker har sagt om indivisible. Og prøv at tænke over hvorfor det er det kan gå galt.

D.7 Conference 2.4

Discussion briefly after Conference 2.4:

Visiting Teacher: Hey, håber der er lidt flere der vil sige noget her. Hvad siger du?

Student 1: Vi tænkte noget med en retvinklet trekant, øh, inde i den her, og så har vi da i hvert fald noget af arealet, på en eller anden måde, og så lidt i stil med det andet hvor vi gangede med $\frac{4}{3}$, så gangede vi med noget igen.

Visiting Teacher: Ja, kan vi gøre det ligesom Archimedes? Kan vi gange med $\frac{4}{3}$ her? Nej nej, men altså... Altså, nogle der har nogle tanker, hvorfor kan vi godt, hvorfor kan vi ikke? Hvad skal der gælde for at vi kan bruge det Archimedes sagde? Ja?

Student 2: (pige forrest med brunt hår) Det skal vel være en parabel

Visiting Teacher: Ja! Er det her en parabel?

Student 2: Nej

Visiting Teacher: Det ved vi i hvert fald ikke nødvendigvis at det er, vel?

Student 2: Nej, i hvert fald ikke en hel parabel.

Visiting Teacher: Nej. Men altså vi kan sige, hvis det er et udsnit af en parabel, så kan vi godt gøre det sådan her. Så laver vi en retvinklet trekant. Okay. Så det her... det er noget, hvad er det, hvad er det der, vi ved ikke hvad det er for en funktion her, vel? Men vi er, nu skal vi snart ind i noget som er, om, lidt mere moderne, vi er ude i sådan noget 1600-tallet, hvor vi faktisk begynder at kigge på de her funktioner, eller noget som minder om de funktioner som vi kender i dag. Og det vi skal snakke om i morgen det er en gut der hedder Newton, som netop kunne regne arealet af sådan nogle figurer som den I så lige før, det er det vi kommer til at fortsætte med, I behøver ikke tænke så meget mere over det lige nu. Men så er I klar over hvad der kommer til at ske. Okay, så resten af modulet her der vil jeg bare gerne lige have jer til at få uploadet jeres nye skærmoptagelser i den samme mappe.. [optagelse afbrudt]

D.8 Plenum Discussion C

Visiting Teacher: Så øh, vi er fuldstændig klar over at det er en svær tekst det her. Men har i fået lidt mere, sådan, en ide om hvordan han, hvordan han skriver om sin matematik. Vi snakkede før om det her at han laver en regel. Men er der mere i strukturen vi kan sige noget om? Hvad han gør i de her bokse. [pause] så først så siger han jeg har denne her regel. Den hedder regel 1. Hvad er så det næste Newton skriver? [pause] i behøver ikke, sådan at kunne sige præcist hvordan han argumenterer for det, men hvordan at han gør i det næste [pause]

Student 2: han laver et bevis for reglen med et eksempel

Visiting Teacher: ja han laver et bevis med et eksempel. Lige præcis. Så lige i starten af denne her time, snakkede vi om det her med at der er forskel på hvornår noget er et eksempel og hvornår noget er et bevis. Men hvad er det så det her? Har i nogle tanker om det er et bevis eller et eksempel han kommer med? [lang pause] når man beviser noget med et eksempel.. [elev markerer]

Student 3: så er det en demonstration. Om hvordan han bruger reglen

Visiting Teacher: ja præcis så det er ikke bare, hvad siger sætningen men også hvordan vi kan bruge denne her regel. Hvad gør han så efter at han har lavet det her bevis for et eksempel. [lang pause] okay så, 1 minut snak lige med sidemakkeren, hvad gør han efter at han har bevist det med et eksempel. Så samler vi op lige om lidt.

D.9 Conference 3.4

Below is the transcription of Conference 3.4, including a follow-up plenum discussion just prior to Conference 3.4.

Visiting Teacher: Okay, hvad er det, der sker her? Er der nogle af jer der kan se hvad han gør, hvad er det for nogle udregninger han laver? Hvis vi bare kigger på linjerne der står. [pause] Er der bare nogle af de her ting, hvor i tænker det her det ved jeg godt hvad betyder. Jeg tror ikke på at i læser det her og ikke forstår noget som helst af det. Er der nogle af de ting han gør som i godt kan forstå? Plusser han med noget.. ganger han med noget.. Hvad gør han? [mumlem]

Student 1: han tager et eller andet og så, og så dividerer han resten med 'o'

Visiting Teacher: ja præcis. Så denne her første sætning kan jeg godt forstå den er lidt mærkelig. Det han siger: 'taking away equal quantities'. Hvad tror i 'taking away' kan oversættes til?

Student 2: minus

Visiting Teacher: minus præcis. Så han, han siger, okay vi ved z i anden er det samme som 4/9x i tredje. Det står heroppe. Så hvis vi sletter det samme på begge sider, så må vi godt det ikke, Det er en regel i godt kender, indenfor ligningslæsning. Og så dividerer han med 'o'. Det er også noget vi godt kan gøre ikke? Vi kan altid bare dividere med det samme over det hele, ikke? Godt. Okay og så har vi et eller andet udtryk tilbage. Så står der her 'if we now suppose bbeta to be infinitely small that is 'o' to be zero'. Så hvad er det vi gør nu?

Student 3: Det er sådan så han sletter det der mellemrum på . . . (jeg har svært ved at høre det præcise ord her – 16:09) og så har vi så kun det resterende

Visiting Teacher: præcis, så vi tager denne her afstand og så forestiller vi os ligesom, hvis den er nul, jamen så vil arealet af det her, altså hvis det her stykke er nul, hvad bliver arealet så af denne her firkant?

Student 4: nul

Visiting Teacher: nul, lige præcis. Så nu gør vi det helt vildt småt. Det her, det er sådan. Lad os sige de her tricks han laver her hvor han dividerer med ’o’ og så gør han det meget meget småt bagefter. Det er noget af det som Newton var meget meget kendt for. Og det var meget nyt på det her tidspunkt, at man gjorde det på den her måde fordi at man kunne vise noget meget generelt. Men, nu her til sidst, så sætter i jer ned og så overvejer i lige i grupperne denne her metode. Er der nogle problemer med det? Er der noget der er mærkeligt? Ville det gå i dag? Og hvis i ikke har nogle bud på det, så bare skriv nogle noter om, hvad det er vi lige har arbejdet med og hvordan det her bevis forløber i jeres dokument. . . . arbejdspause . . .

Visiting Teacher: okay super lad mig høre hvad i tænker her. Er der nogle tanker omkring nogle problemer?

Student 5: han dividerer med nul

Visiting Teacher: gør han?

Student 5: det må man ikke

Visiting Teacher: og hvor gør han det henne? Han siger at han dividerer med ’o’ ikke?

Student 5: ja, men ’o’ det var så nul. Han sætter ’o’ til at være nul

Visiting Teacher: altså det er jo rigtigt, altså men det han jo i virkeligheden siger det er at han siger, vi dividerer med ’o’ og så siger vi at ’o’ var nul. Så han starter faktisk bare med at fividerer med det og så siger han. Her var det ikke nul, det er først nul bagefter. Det er helt rigtigt. Det er her der er nogle problemer med det. Det er i hvert fald her hvor vi er lidt skeptiske overfor hvad der foregår.

D.10 Conference 4.2

Below is the transcription of Conference 4.2, including a follow-up plenum discussion just prior to Conference 4.2.

Visiting Teacher: okay lad mig høre har i et bud på, hvad det er vi kigger på med de her infinitesimaler? Okay jeg prøver lige at spørge på en anden måde. Hvor har i kigget henne? Hvor har i prøvet at finde svar på det her spørgsmål?

Student 1: google

Visiting Teacher: google ja, og hvad dukker der op når man kigger på google

Student 1: at de er så små at man ikke kan måle dem eller se dem.

Visiting Teacher: ja præcis, minder det om, ja hvad siger du?

Student 2: altså det er bare sådan at man betragter dem som 0 men at de ikke helt er det [LIDT SVÆRT AT HØRE]

Visiting Teacher: ja præcis, så der kan man godt se lidt at det er det Newton gør her, ikke? Minder det om noget? Vi har arbejdet med? det her med noget der er så småt at det øhh, ja?

Student 3: De der indivisible [udtalt forkert]

Visiting Teacher: ja indivisible. Jeg tænker at det var det du mente. Præcis, så er det de samme som de indivisible? [pause]

Student 4: Nej for det var vel noget der ikke kunne deles op. Det var sådan noget som altså linjer og sådan noget i bredden . . . . [BLIVER UKLART TIL SIDST]

Visiting Teacher: ja, præcis, så de havde ikke nogen bredde overhovedet, vel? Så hvad er det der er det infinitesimale på den her tegning? Tag den lige igen i grupper og så prøv og find ud af hvad er det der er det infinitesimale, hvis vi skal pege på det. Snak med hinanden om det og så samler vi op. Ny seance 11:05

Visiting Teacher: hvis vi kigger på den her tegning, hvad er det så [afbrudt af støj] hvad er det der er noget infinitesimalt?

Student 5: er det ikke ’o’?

Visiting Teacher: jo det er denne her ikke. Så det han i virkeligheden gør og nu vil jeg lige prøve at forklare jer lidt mere om hvad der foregår i beviset. Og det er helt fair at man ikke lige kan læse det ud fra. Det er bar for at sige, hvad er det egentlig han gør her. Og det han gør, det er at han kigger jo på. Han kigger på det fulde areal her. Og så sammenligner han det med z. og så siger han, okay, det her det er jo to forskellige arealer. Denne her røde, den er jo helt klart større end z. Men hvis vi nu sætter denne her til at være uendelig lille. Denne her afstand. Så er de jo det samme, de to arealer, ikke? Så det er det han kigger på. Så man kan sige, så har han lavet den her firkant, den snakkede vi også lidt om i går ikke? Eller i onsdags. Han har lavet den her firkant. Og så siger han, nå men, den her firkant her, det er sådan en søjle af en art. Den har samme størrelse som denne her tilvækst. Så hvad sker med arealet af denne her firkant når vi sætter 'o' til at være lig med 0? eller uendeligt smat? Hvad bliver arealet så af denne her rektangel. Ja?

Student 6: 0

Visiting Teacher: ja lige præcis. Så både, altså begge arealer det bliver jo uendeligt små. Det bliver ikke helt 0, men de bliver så tæt på 0 man overhovedet kunne forestille sig. Okay, så nu er vi her hvor denne her rektangel den har en eller anden bundlinje som der er næsten 0. Så hvilken dimension er den i? hvor mange dimensioner er denne her firkant i? [lang pause] okay så tag lige et minut med sidemakkeren, hvor mange dimensioner har vi her?

E Assignments

E.1 Posed Assignment

Et besøg i Archimedes' og Newtons arbejdsværelser

Over fire moduler har vi arbejdet med nogle historiske kilder, der begge handler om beregning af arealer. Når vi læser gamle tekster og prøver at sætte os ind i forfatternes tanker og overvejelser, er vi en slags *observatører* i en matematisk kontekst, som er anderledes en den I selv lever i i dag.

Sammen har vi kigget Archimedes og Sir Isaac Newton over skulderen for at undersøge, hvordan deres matematiske arbejdsværelse så ud. Vores hovedfokus i arbejdet med teksterne har været:

> *Hvordan har man bestemt arealet under en kurve, og hvordan har man argumenteret for det?*

De kilder vi har valgt at arbejde med er kun en lille del af den omfattende historie matematikken har. Faktisk er det sådan, at det første vidnesbyrd om, at mennesket har været interesseret i at tælle ting, kan dateres helt tilbage til omkring 30.000 år f.v.t! Det kan vi se på gamle knogler, hvorpå der er lavet markeringer.

Da mennesket langt senere udviklede skrivekunsten, blev matematikken systematiseret. De første matematiske tekster, som vi har overleveret, er fra det gamle Egypten fra omkring 1850 f.v.t. Moskva-papyrussen er en af de ældste bevaret kilder til den Egyptiske matematik. I denne papyrus kan vi se, hvordan man allerede på dette tidspunkt har haft kendskab til arealberegningen. Et billede af denne papyrus ses nedenfor til højre.

Til venstre ses et billede af en knogle, hvor der er ridset 55 streger, hvilket vidner om at nogen har talt noget. Til højre ses et udsnit af Moskva-papyrussen.

Jeres opgave

Med udgangspunkt i de fire moduler skal du skrive en kort rapport, hvor du redegør for Archimedes' og Newtons metoder til at bestemme arealer.

I rapporten skal du besvare følgende spørgsmål:

- Hvordan bestemte Newton og Archimedes arealer? Hvad kunne de bestemme arealet af? I skal her kommentere på stringensen i deres argumenter.
- Hvilke forskelle og ligheder er der mellem Archimedes' og Newtons metoder? Hvilke fordele og ulemper er der ved de forskellige metoder?
- Hvordan relaterer Archimedes' og Newtons metode sig til den metode, du har lært i den normale undervisning, om at bestemme arealer under kurver? Er den metode stringent? *(Hint: Tænk på grænseværdibegrebet!)*

Når I svarer på spørgsmålene vil vi opfordrer jer til at kigge i de noter, I har skrevet undervejs, og I det kompendie, vi har udleveret. I er også meget velkomne til selv at søge videre på Google.

E.2 Assignment 1

Matematikhistorisk rapport – Archimedes og Newtons metoder

Archimedes levede ca. 300 år før vores tid og er kendt for hans banebrydende matematiske tankegang. På trods af de dengang meget begrænsede definerede og beviste matematiske love og sætninger (eksempelvis Euklids elementer) lykkedes det alligevel Archimedes at udvikle metoder til at udregne arealet af komplekse figurer som f.eks. arealet af et parabelsegment. Han benyttede sig af kvadratur og viste, at arealet af et hvilket som helst parabelsegment er 4/3 af trekanten, der kan tegnes mellem parabelsegmentets to snit med parabelkurven og punktet på parabelkurven der ligger over disse to punkters midtpunkt og parallelt med parablens symmetriakse. Archimedes brugte to principper han kendte:

- Trekantens tyngdepunkt, som er skæringspunktet mellem trekantens medianer.
- Vægtstangspricippet, som han selv havde udviklet, der viser, hvordan to objekter med forskellig vægt kan balancere på en vægtstang, hvis de placeres med rette forhold mellem vægt og afstand til balancepunktet.

Han fordoblede arealet af parabelsegmentets trekant ved af fordoble højden og holde grundlinjen konstant (altså trekant AKC, der ses på figur 3 i kompendiet). Det gjorde han igen og fik en trekant med fire gange arealet af parabelsegmentets trekant (altså trekant FAC, der ses på figur 3 i kompendiet.) Så lavede han en linje der fordoblede hypotenusen på trekant AKC og fik linjen CH. Herefter kunne han fylde trekant FAC med parallelle linjer (MO) og parabelsegmentet med parallelle linjer (PO). MO'erne og PO'erne ville kunne balancere med vægtstangsprincippet med linje CH som vægtstang, men kun i trekant FAC's tyngdepunkt (W). WK er 1/3 af KH, hvilket må betyde at MO linjerne "vejede" tre gange så meget som PO linjerne. Siden PO linjerne kom fra trekant FAC, som var fire gange så stor som trekanten i parabelsegmentet, må parabelsegmentets trekant (ABC) være 4/3 af PO linjerne, som jo var parabelsegmentet.

Archimedes har demonstreret sin teori, men sådan set ikke bevist den. Vi må antage den er sand, men måske ikke super stringent. Han brugte også sit eget vægtsstangsprincip, så teorien er kun sandt, hvis dette princip også er.

Sir Issac Newton anses som værende en af 1600-tallets mest betydningsfulde videnskabsmænd særligt indenfor fysikken, men i særdeleshed også indenfor matematikken – nærmere bestemt arealbestemmelse - gennem hans værk: "Analyse med ligninger med uendeligt mange led." I værket beskriver Newton, hvordan arealet under simple kurver kan udregnes, med det vi nu kender som integraler. Han beskæftiger sig med infinitesimale størrelser, som er uendeligt små størrelser, som ikke er lig nul.

Stringensen i newton teori er veldiskuteret både i nutiden og i hans samtid. Blandt andet menes hans forklaringer at være upræcise og ikke ordentligt forklaret.

Metoderne minder om hinanden ved at dele figurerne op i mindre dele, som Archimedes gør med de parallelle linjer. Ulempen ved Archimedes er at det er meget abstrakt.

I forhold til den integralregning, Rune har lært os, ses der mange ligheder især med Newtons metode, for eksempel med de mange led, hvis sum giver arealet under kurven. Nu bruger vi grænseværdien til at lade et parameter gå uendelig tæt på 0, og se hvad det gør ved de andre led.

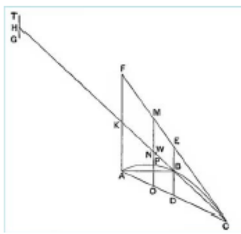
E.3 Assignment 2

Matematikhistorisk opgave

Hvordan bestemte Newton og Archimedes arealer? Hvad kunne de bestemme arealet af? I skal her kommentere på stringensen i deres argumenter.

Archimedes:

Archimedes beskrev linjer med ord, hvor han lavede sine egne formler for at beregne arealet. Det gjorde han, da han ville opdele arealet i mindre dele. Han fandt arealet af ABC, og fortsatte rækken - han lavede altså konstruktioner. Han tegnede trekanter/firkanter, inde i parablen, som han ville finde arealet af, da de er nemmere at finde arealet af. Han tog arealet af en trekant gange 4/3.



Han brugte også vækststangsprincippet, som bygger på, hvornår noget er i ligevægt/balance.

I forhold til stringensen, så bruger Archimedes for meget tekst uden brug af formler, hvilket gør det uforståeligt og sværere at læse. Samtidig er teksten også skrevet med mange bogstaver, som gør det uforståeligt uden et billede af forholde sig til.

Newton:

Newton beskrev linjer ved brug af ligninger og potenser. Han regnede arealet ved at dele kurven op i to, z og ov, hvor han gjorde o til at være infinitesimal (deler kurven op i søjler af infinitesimale).

Newtons regel for, hvordan man beregner arealet under en bestemt type af kurver, nemlig de simple kurver, ses nedenfor:

Rule 1 If $ax^{\frac{m}{n}} = y$, then will $\frac{na}{m+n}x^{\frac{m+n}{n}}$ equal the area ABD.

Newtons argument er ikke stringent, da han blot gør "noget", uden at forklarer hvad han gør. Eksempelvis forklarer han ikke 0=infinitesimale størrelser ordentligt, hvilket gør hans argument uklart.

Hvilke forskelle og ligheder er der mellem Archimedes' og Newtons metoder? Hvilke fordele og ulemper er der ved de forskellige metoder?
Archimedes og Newton bruger forskellige metoder til at finde arealer.

- Newton brugte ligninger og potenser, mens Archimedes anvendte geometri og vækststangsprincippet.

Selvom deres metoder er forskellige, i og med de levede i forskellige årtier, så deler de begge arealet op i mindre dele, for at gøre det nemmere at finde arealet af en parabel. De starter også begge med at se om det kan bevises.

Archimedes

- Fordele:
  - Han bruger ikke præcise tal, men bogstaver, hvilket gør det muligt at generalisere

- Ulemper:
  - Ingen formler eller beregninger
  - Heller ingen ligninger eller funktioner

Newton

- Fordele:
  - Ligninger

- Ulemper:
  - Han dividerer med 0, hvilket man ikke må

Hvordan relaterer Archimedes' og Newtons metode sig til den metode, du har lært i den normale undervisning, om at bestemme arealer under kurver? Er den metode stringent?
Når vi i undervisningen bestemmer arealer under kurver, bruger vi det bestemte integrale samt digitale værktøjer som eksempelvis Nspire.

Udover de ovenstående metoder bruger vi også normalt 3-trins-reglen (grænseværdier) ligesom Newton også gjorde. Newton anvendte også ligninger, som minder om at vi i dag bruger funktioner. Archimedes og moderne matematik er derimod helt forskellige, da han brugte ord hvor både Newton og vi i dag bruger tal.

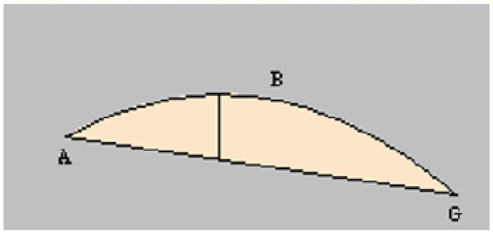
Metoden vi bruger i undervisningen, er stringent, da vi kan gøre det samme om og om igen, og stadig få det samme resultat.

E.4 Assignment A

Matematik historisk rapport

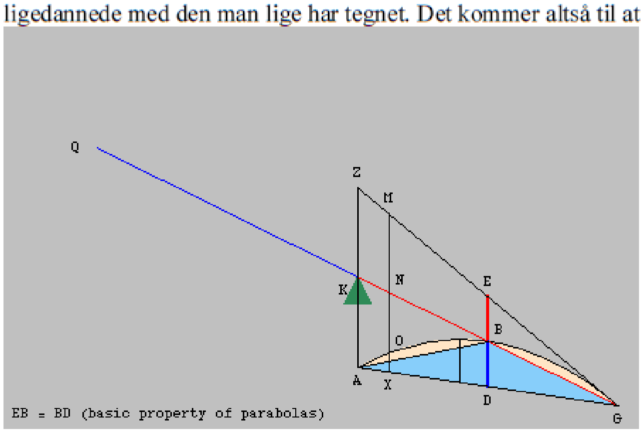
Når man i dag skal finde arealet for en valgt figur, tænker vi slet ikke over, hvordan de forskellige teorier og formler er blevet udarbejdet løbet af årene. Det har taget mange år at udvikle de forskellige selve idéen om et areal og det stammer faktisk helt tilbage til omkring 30.000 år f.v.t.

Da der kan snakkes om mange figurer, bliver der i denne rapport taget udgangspunkt i arealet under en kurve med en bestemt snit. Det kan altså være at figuren kommer til at se ligeledes ud:



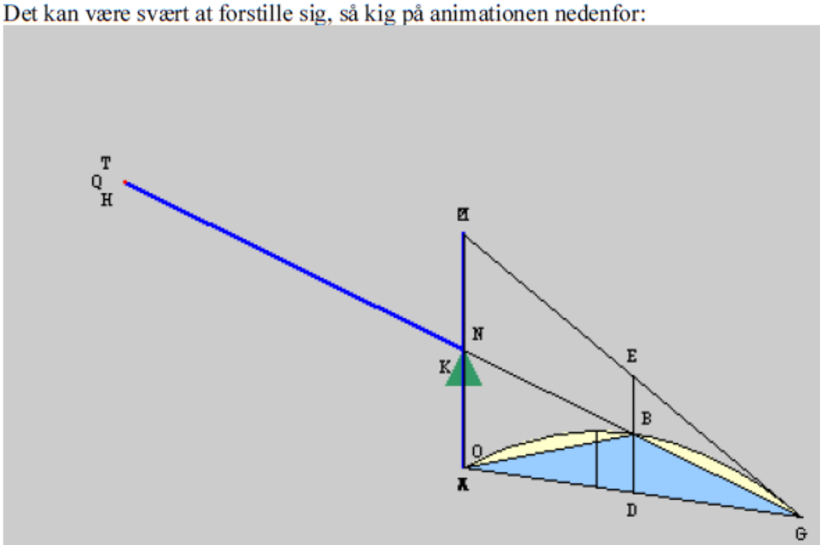
For lige præcis en figur som denne er der blevet udarbejdet mange forskellige teorier og metoder for hvordan man netop regnede figurens areal.

Vi kan starte med at tage udgangspunkt i Archimedes, der mente at figurens areal kunne blive udregnet med hjælp af vægt og tyngdekraft. Dette bliver også kaldt en mekanisk metodefremgang, hvor man tager udgangspunkt i den virkelige verden for at bevise sin teori. Denne teori kaldte han også for vægtstangsprincippet. Det man gjorde var, at man først lavede en trekant inde i parablen, hvor man tog højden for trekant og tilføjede den i forlængelse med den gamle. Ud fra dette kunne man nu tegne en trekant, der løber udenom parablen og dernæst kan man tegne en ny trekant, der er ligedannede med den man lige har tegnet. Det kommer altså til at se sådan ud:

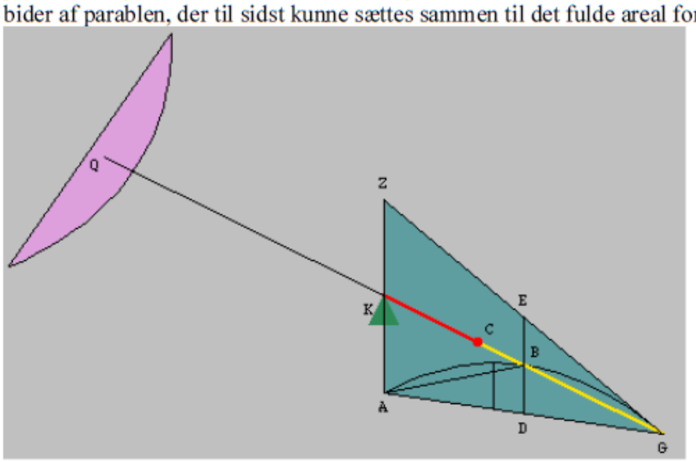


Her tilføjede man så en linje, der har midte i midten af de udenoms liggende trekanter. På billede er det linjen QG som løber igennem følgende linjers midte: ZA, MX og ED.Denne linje er hvad der for Archimedes teori skulle være nøglen til at finde arealet. Det linjen gør er, at den fungerer ligesom en gammeldags vægt. Når man sætter noget ud i den ene ende, skal det der er på den anden ende være magen til ellers så tipper vægten.

Det man gjorde, var altså at tilføje en ny linje for enden af linjen ved Q. Denne linjes længde svarede altså til linjen højde inde i parablen, for ellers ville der ikke være ligevægt.
Det kan være svært at forstille sig, så kig på animationen nedenfor:

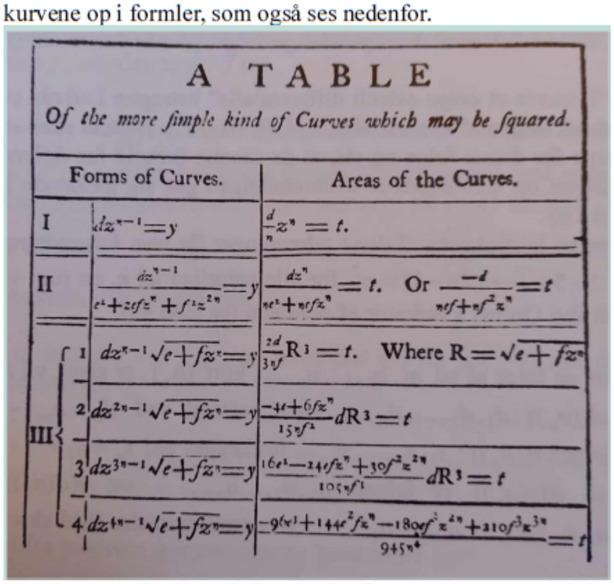


Man kan altså se, at linjen TH først bliver større og derefter mindre, i takt med at linjen MX bevæger sig igennem trekanten og hen over parablen. Linje TH's længde er altså hvad der svarer til parablens højde, som i animationen er Linje OX. Ud fra dette har linje TH altså repræsenteret små bider af parablen, der til sidst kunne sættes sammen til det fulde areal for parablen.



Stringensen ved denne teori er altså at det hele er baseret på noget mekanisk og ikke geometrisk. Derfor kan det ikke præcist vides om beviset derfor vil være gældende, hvilket er en ulempe ved Archimedes teori. Dette vidste han dog godt selv og derfor vente han det om til en fordel, ved at sige at denne metode kan være med til at give en indikation om, at hans tankegang er rigtig.

En anden mand man kan vælge at kigge på, er Newton. Han ville også gerne bevise, hvordan man udregnede arealet under en kurve. Hans metode går lidt anderledes til værk, fordi han har delt kurvene op i formler, som også ses nedenfor.



A TABLE

Of the more simple kind of Curves which may be squared.

	Forms of Curves.	Areas of the Curves.
I	$dz^{\eta-1}=y$	$\frac{d}{\eta}z^{\eta}=t.$
II	$\frac{dz^{\eta-1}}{e^2+2efz^{\eta}+f^2z^{2\eta}}=y$	$\frac{dz^{\eta}}{\eta e^2+\eta efz^{\eta}}=t.$ Or $\frac{-d}{\eta ef+\eta f^2z^{\eta}}=t$
III 1	$dz^{\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{2d}{3\eta f}R^3=t.$ Where $R=\sqrt{e+fz^{\eta}}$
III 2	$dz^{2\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{-4e+6fz^{\eta}}{15\eta f^2}dR^3=t$
III 3	$dz^{3\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{16e^2-24efz^{\eta}+30f^2z^{2\eta}}{105\eta f^3}dR^3=t$
III 4	$dz^{4\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{-96e^3+144e^2fz^{\eta}-180ef^2z^{2\eta}+210f^3z^{3\eta}}{945\eta^4}=t$

Hvis kurvens ligning minder om den første, så kan man se i skemaet, hvordan formlen for den kurves areal ser ud. Her kan man så forsætte ned af listen, indtil man finder den kurve der passer den ligning. Ulempen ved denne måde er at hvis man ikke kan finde en ligning der passer den kurve, så kan man ikke finde arealet. Fordele er at det er meget mere præcist for at regne de individuelle kurvers areal. Dette gør altså hans metode mere stringens

I forhold til den metode vi bruger nu til dags, som er integralregning, så kan man sige at Newtons metode ligner meget mere den vi bruger i dag. Vi bruger også skemaer der indeholder formler, som vi så kan bruge til at regne arealet. Her bruger man blandt andet også grænseværdier, hvilket er med til at gøre denne metode en lille smule mindre stringens, da det aldrig nogensinde helt bliver 0, men meget tæt på.

E.5 Assignment B

Et besøg i Archimedes’ og Newtons arbejdsværelser

Hvordan bestemte Newton og Archimedes arealer? Hvad kunne de bestemme arealet af? I skal her kommentere på stringensen i deres argumenter.

Arealer

- Archimedes anvendte vægtstangsprincippet og geometriske betragtninger til at bestemme arealet under kurver. Han brugte eksempelvis en model baseret på trekanters areal og multiplikerede det med 4/3 for at få et tilnærmelsesvis areal under kurven.
- Newton udviklede en metode baseret på infinitesimalregning, hvor han opfandt begrebet infinitesimaler, der er ekstremt små størrelser, og brugte dem til at manipulere med funktioner og finde arealer under kurver.

Stingent

- Archimedes' metode er baseret på geometriske betragtninger og vægtstangsprincippet, hvilket giver en intuitiv forståelse, men det kan være svært at præcisere og generalisere præcist.
- Newtons metode med infinitesimalregning er mere stringent i den forstand, at den bygger på matematiske principper og definitioner. Dog blev Newtons argumenter mødt med kritik og diskussioner på grund af begrebet infinitesimaler, som ikke var klart defineret på det tidspunkt

Hvilke forskelle og ligheder er der mellem Archimedes’ og Newtons metoder?

- Archimedes' metode er geometrisk og baseret på intuition og observationer af fysiske fænomener som vægtstangsprincippet. Det er mere visuelt og intuitivt forståeligt.
- Newtons metode er matematisk og abstrakt, baseret på infinitesimalregning og differential- og integralregning. Det tillader mere præcise beregninger og generaliseringer.

Hvilke fordele og ulemper er der ved de forskellige metoder?

- Fordelen ved Archimedes' metode er dens intuitive forståelse og visualisering. Ulempen er, at den kan være begrænset i dens anvendelse og præcision.
- Fordelen ved Newtons metode er dens matematiske stringens og mulighed for at håndtere komplekse funktioner. Ulempen er, at den kan være sværere at forstå og kræver en dybere matematisk baggrund.
- -Hvordan relaterer Archimedes' og Newtons metode sig til den metode, du har lært i den normale undervisning om at bestemme arealer under kurver? Er den metode stringent?

Hvordan relaterer Archimedes' og Newtons metode sig til den metode, du har lært i den normale undervisning om at bestemme arealer under kurver? Er den metode stringent?

- I vores normale undervisning har vi lært at bestemme arealer under kurver med integralregning. Denne metode relaterer sig til Newtons tilgang, da den bygger på samme matematiske principper. Integralregning giver en præcis metode til at bestemme arealer under kurver og er baseret på det matematiske fundament af differentialregning. denne metode er stingent, fordi det er logisk tvingende argumenter som gør at vi fx kan bruge intergralregning til at beregne bestemte arealer under kurver, dette er også noget som er bevist som gør at det er med til at være stingent

E.6 Assignment C

5. april 2024

Matematik historie: Archimedes og Newton

Archimedes og newton er begge matematikere der fremstillede metoder til at beregne arealer under kurver. Metoderne er anderledes fra integralregning der benyttes i dag. Jeg vil i denne rapport gennemgå argumenterne og kommentere på dem.

Hvordan bestemmer Newton og Archimedes arealer? Hvad kunne de bestemme arealet af? I skal her kommentere på det stringente i deres argumenter.

Archimedes kunne beregne arealer under kurver. Han beregnede arealet ved at tegne hjælpelinjer omkring og igennem kurven. Hjælpelinjerne bestod især af trekanter, da man på hans tid viste mere om arealet af trekanter. Det centrale i metoden er vægtstangsprincippet, hvor en linje fungerer som en vægtstang eller center og gravity. Det beregnes hvor meget skal tegnes på hver side for at den er i balance. Archimedes argument er ikke stringent. Han benyttede bl.a. der logisk tvingende argument: en linje er en længde uden brede. Dvs. at linjer er indivisible på det ene led; de kan ikke opdeles. Men med vægtstangsprincippet konstruere man kurven med en masse linjer og bruger bredden af dem til at bregne arealet. Disse to ting hænger ikke sammen.

Vægtstangsprincipet er en mekanisk metode, princippet tager udgangspunkt i metoder man kan bruge i den virkelige verden. Archimedes mener ikke at den mekaniske metode var nok til at bevise hans argument. Den mekaniske metode kan bruges til at give en ide om argumentets troværdighed, derefter skal geometrien bruges til at verificere argumentet. Hvis man bruger en geometrisk metode, arbejder man med abstrakt matematik. Det argument vi har arbejdet med, tager udgangs i den mekaniske metode og kan derfor ifølge Archimedes ikke siges at være at bevis. Dette er nødvendigvis ikke en ulempe ved argumentet. Archimedes verificerer senere sin metode via geometrien, men det var pga. af denne mekaniske metode han var i stand til dette.

Newton kunne også beregne arealer under kurver. Jeg kan ikke på samme måde forklarer hvordan han beregnede arealer. Men han benyttede sig også at hjælpe figurer som rektangler og regnede på arealet af dem. Newtons argument er ikke stringent. Newton dividerer i løbet af sine beregninger med bredden O, men hans metoder tager også udgangspunkt i at $O = 0$. I matematikken kan man ikke dividere med 0. Disse to ting hænger ikke sammen. Newton var godt klar over denne svaghed, men afviste det med initesimale størrelser. Initesimale størrelser er positive men meget meget små størrelser. De er altså næsten lig 0. O har i Newtons bevis altså en brede der er en initesimale størrelser og det er argumentet for at det kan lade sig gøre at dividere med O.

Dette princip minder meget om det vi i dag kalder grænseværdibegrebet.

E.7 Assignment D

Matematik/historisk opgave

Metode til at beskrive arealer:

Både Archimedes og Newton gjorde betydelige fremskridt inden for arealbestemmelse, selvom deres tilgange var forskellige på grund af den tid, de levede i og de værktøjer, de havde til rådighed. Den græske matematiker Archimedes, som levede før vore tidsregning, benyttede sig primært af geometriske metoder til at bestemme arealer. Han brugte en teknik, hvor han opdelte komplekse figurer, såsom parabler, i mindre og mere håndterbare dele, som han kunne beregne arealerne af. Hans anvendelse af geometriske figurer og hans vægtstangsprincip er tidlige eksempler på matematisk analyse og anvendelse af abstrakte koncepter til at løse praktiske problemer.

Når det kommer til stringens i argumentationen, er Archimedes kendt for at levere beviser, der er logisk tvingende og baseret på geometri. Archimedes mente at når noget er bevist eller demonstreret, så har vi et bevis.

Isaac Newton, som levede fra 1643-1727, altså lang tid efter Archimedes, var en engelsk matematiker og naturvidenskabsmand. Hans tilgang til arealbestemmelse involverede brugen af infinitesimale størrelser (så tæt på 0 som overhovedet muligt, men ikke helt 0) hvor han behandlede en kurve som en serie af små rektangler og brugte grænser til at finde det endelige areal.

Når man kigger på stringensen af Newtons argumentation, kan man bemærke, at han imens gav sig selv en vis frihed i sine beviser. For eksempel, når han dividerede med hvad han kaldte "0", det vil sige når han arbejdede med infinitesimale størrelser, forklarede han ikke altid nøje de logiske begrundelser bag dette, hvilket kan gøre hans argumentation mindre stringent i visse tilfælde.

Forskelle og ligheder:

Både Archimedes og Newton arbejdede på at løse praktiske problemer ved hjælp af matematik. De havde begge fokus på at bestemme arealer under kurver, selvom deres tilgange var forskellige. Begge matematikere var banebrydende inden for deres tid og bidrog væsentligt til udviklingen af matematikken. Deres metoder og resultater dannede grundlaget for senere matematiske discipliner.

Archimedes benyttede primært geometriske metoder, hvor han opdelte komplekse figurer i mindre og mere håndterbare dele. Newton derimod benyttede sig af infinitesimale størrelser og inddeling af kurver som små rektangler og linjer, skrevet på en anden måde, til at arbejde med kurver. En anden forskel på de to er, at Archimedes ikke havde adgang til moderne matematiske værktøjer som allerede kendte formler og tilgange, mens Newton udviklede og anvendte disse værktøjer i sit arbejde.

Fordelen ved Archimedes metode er at geometriske metoder kan være mere visuelle, hvilket gør dem lettere at forstå for nogle, dog kan de være begrænsede i deres anvendelse på komplekse kurver eller figurer og opdeling af figurer i mindre dele kan være tidskrævende og kræve en omhyggelig håndtering af detaljer.
Fordelen ved Newtons metode at matematikken er udviklet mere, hvilket kan gøre hans trin lettere at forstå, dog kan hans brug af infinitesimale størrelser kræve en præcis definition, for at der ikke opstår misforståelser.

I egen undervisning:
Metoden, jeg har lært i den normale undervisning om at bestemme arealer under kurver, er kendt som bestemt integralregning. Her bruger jeg værktøjer som Nspire og formler, til at hjælpe med at udregne det - ting som hverken Archimedes eller Newton havde til rådighed, da de arbejdede med at finde arealer under kurver. Denne metode er stringent, da det er udført på en helt korrekt måde. Sammenfattende var både Archimedes og Newton banebrydende inden for arealbestemmelse, men deres tilgange var forskellige på grund af forskelle i tid og tilgængelige værktøjer. De har begge bidraget til, hvordan vi i dag finder arealer under kurver ved hjælp af integralregning.

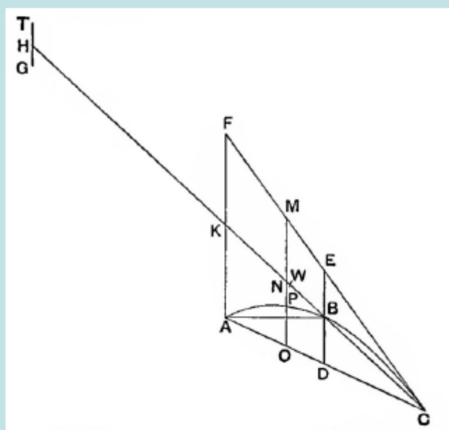
E.8 Assignment E

Matematikhistorisk opgave

Hvordan bestemte Newton og Archimedes arealer? Hvad kunne de bestemme arealet af? I skal kommentere på stringensen i deres argumenter.

Archimedes opdelte arealet i mindre figurer, såsom trekanter og firkanter, inde i den parabel, han ønskede at beregne arealet af. Han brugte også vækststangsprincippet, der bygger på ligevægt, for at demonstrere og bevise sit resultat. Archimedes lavede sine egne formler og konstruktioner ved at finde arealet af ABC og fortsatte rækken.

Figur 3: Konstruktionen i Proposition 1



Newton introducerede konceptet om infinitesimaler, hvilket tillod ham at behandle kurver som en samling af utallige små elementer. Ved at tage grænsen, når størrelsen af disse elementer nærmer sig nul, udviklede Newton integralregning som en metode til at bestemme arealer under kurver. Han regnede arealet ved at dele kurven op i 2 søjler af infinitesimale og udviklede ligninger.

Både Newton og Archimedes var i stand til at bestemme arealet af geometriske figurer.
Newton og Archimedes metoder er begge stringens, da de er baseret på matematiske principper.

Hvilke forskelle og ligheder er der mellem Archimedes og Newtons metoder? Hvilke fordele og ulemper er der ved de forskellige metoder?
Archimedes levede for over 2000 år siden, mens Newton levede i det 17. århundrede, og de har hver især haft forskellige muligheder i forhold til at udvikle matematikken. Archimedes udviklede sine egne formler og brugte geometriske konstruktioner og logiske betragtninger, mens Newton arbejdede med ligninger og udviklede infinitesimalregning.

Både Archimedes og Newton startede med at undersøge, om deres metoder kunne bevises og de begge forsøgte at bestemme arealer.

Fordelene ved Archimedes metode er brugen af geometriske konstruktioner, hvilket gør det visuelt forståeligt og viser essentielle principper som vægtstangsprincippet.
Ulemperne ved Archimedes metode er, at det kan være tidskrævende, og det kan være svært at se de komplekse figurer.

Fordelene ved Newtons metode er brugen af ligninger og formler, da det giver en præcis måde at vise matematiske sammenhænge på.
Ulempen ved Newtons metode er, at det kræver en dybere forståelse af matematik

Hvordan relaterer Archimedes og Newtons metode sig til den metode, du har lært i den normale undervisning, om bestemte arealer under kurven. Er den metode stringent?
Newtons metode er mere relateret til det der normalt læres i undervisningen, da den involverer brugen af bestemte integraler til at beregne arealet under kurven. Newton lavede ligninger og i den normale undervisning beskæftiger vi os med funktioner, hvilket minder meget om hinanden.

Metoderne, der normalt læres i den normale undervisning om bestemmelse af arealer under kurver ved brugen af bestemte integraler, kan betragtes som stringent. Dette skyldes, at de er baseret på eksisterende matematiske teorier og koncepter, der er definerede og grundigt bevist.

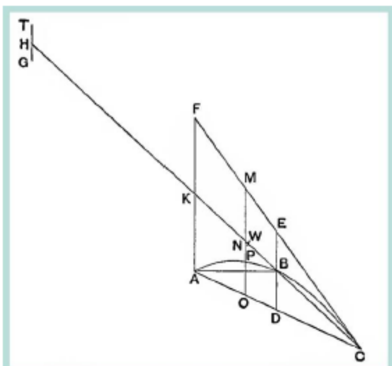
E.9 Assignment F

Historisk matematik moduler afslutning

Vi nu arbejdet i fire moduler med Archimedes' og Newtons metoder til at bestemme arealer. Jeg vil derfor redegøre for dette og besvare følgende spørgsmål.

Hvordan bestemte Newton og Archimedes arealer? Hvad kunne de bestemme arealet af?

- Archimedes brugte vægtstangsprincippet til at bestemme arealet. Vægtstangsprincippet går ud på at finde det punkt hvor der er balance. Hvis der er meget vægt i den ene side og mindre vægt i den anden side, så vil balancepunktet være tættere på den tunge side.
- Udover at bruge vægtstangsprincippet til at beregne arealet af en trekant, så brugte han også følgende formel $ABC = \frac{4}{3}\triangle ABC$.



- Newton

Hvilke forskelle og ligheder er der mellem Archimedes' og Newtons metoder? Hvilke fordele og ulemper er der ved de forskellige metoder?

- Archimedes benytter sig af indivisble, som er lig 0, hvilket vil sige at når de plusses, så vil resultatet aldrig blive større, hvorimod Newton bruger infitesimaler, hvilket er værdi som er meget tæt på 0, men stadig vil blive større hvis man plusser dem.

- Kritikken af newtons metode var at den ikke var stringent nok, og der dermed førte til uvished om hvad han egentlig mente med metoden.

Hvordan relaterer Archimedes' og Newtons metoder sig til den metode, du har lært i den normale undervisning, om at bestemme arealer under kurver? Er den metode stringent?

- Den nutidige måde at regne arealet under kurver er stringent, i og med en er let at sætte tal ind i og en hel del mindre kompliceret at forstå.